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ANGULAR VELOCITY OF THE KOVALEVSKAYA TOP

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Images of the Liouville tori and three-dimensional isoenergetic surfaces are constructed in movable space of angular velocities and all possible types of these invariant sets are classified. The characteristic properties of the angular momentum and the angular velocity of the Kovalevskaya top are indicated.

Special Kovalevskaya Edition

1. Introduction

The Euler problem on motion of a heavy rigid body about a fixed point requires to integrate the system of differential equations

$$A\dot{\boldsymbol{\omega}} = A\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{r} \times \boldsymbol{\nu}\,, \qquad \dot{\boldsymbol{\nu}} = \boldsymbol{\nu} \times \boldsymbol{\omega} \tag{1.1}$$

with the three known integrals

$$H = \frac{1}{2}\langle A\boldsymbol{\omega}, \boldsymbol{\omega}\rangle - \langle \mathbf{r}, \boldsymbol{\nu}\rangle = h\,, \qquad G = \langle A\boldsymbol{\omega}, \boldsymbol{\nu}\rangle = g\,, \qquad I = \langle \boldsymbol{\nu}, \boldsymbol{\nu}\rangle = 1\,, \tag{1.2}$$

where $A = \mathrm{diag}(A_1, A_2, A_3)$, $\boldsymbol{\omega}$ is an angular velocity of the body in a movable frame, $\boldsymbol{\nu}$ is a vertical unit vector and $\mathbf{r}$ is a vector directed from the fixed point to the center of gravity of the body. The integrating factor of the system (1.1) is constant, therefore it is enough to have one additional integral non dependent explicitly on time to reduce the problem to quadratures. L. Euler indicated the first case of integrability of the dynamical equations (1.1). Later L. Poinsot found a simple and obvious geometric representation of the motion for the Euler case: the rigid body rotates by inertia around the fixed point so that the inertia ellipsoid, invariable connected with the body, rolls without sliding on a fixed plane which is perpendicular to the angular momentum. The angular velocity of the body and the radius vector of the tangency point of the ellipsoid with the plane of rolling lay on the same straight line and have proportional modules. The set of trajectories of the Euler equations fills the ellipsoid which is a two-dimensional surface of an energy level at a fixed value of the constant h [2].

In general, the common level surface of the first integrals

$$Q^3_{h,g} = \{H = h\,,\ G = g\,,\ I = 1\} \subset \mathbf{R}^6(\boldsymbol{\omega}, \boldsymbol{\nu})$$

is a three-dimensional subset of a phase space which is invariant under a phase flow of the system (1.1). The methods of study of mechanical systems with symmetry developed by S. Smale [18] are effectively applied for description of bifurcations and topology of the surfaces $Q^3_{h,g}$. The topological type of $Q^3_{h,g}$ is uniquely determined by "domain of a possible motion", i. e. by projection of an isoenergetic surface $U_{h,g} = \pi(Q^3_{h,g})$ on the Poisson sphere $S^2 = \{|\boldsymbol{\nu}| = 1\}$. In fact, the manifold $Q^3_{h,g}$ is diffeomorphic to a fiber bundle with the base $U_{h,g}$ and the fiber S^1 at which the fibers above $\partial U_{h,g}$ are identified at points.

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The type of these manifolds varies while passing through a bifurcation set $\Sigma \subset \mathbf{R}^2(h,g)$ consisting of critical values of a map $H \times G\colon\ S^2 \times \mathbf{R}^3 \to \mathbf{R}^2(h,g)$. The topological type of $Q^3_{h,g}$ is the same for all points from one of the connected domains $\mathbf{R}^2(h,g)\backslash\Sigma$. A connected component of a nonsingular surface $Q^3_{h,g}$ is diffeomorphic to one of the following manifolds: $RP^3, S^3, S^1\times S^2, (S^1\times S^2)\#(S^1\times S^2)$.

Critical points of the map $H \times G\colon\ S^2 \times \mathbf{R}^3 \to \mathbf{R}^2(h,g)$ are relative equilibria of the system (1.1):

$$\boldsymbol{\nu} = \text{const}\,, \qquad \boldsymbol{\omega} = \frac{g\boldsymbol{\nu}}{\langle A\boldsymbol{\nu},\boldsymbol{\nu}\rangle} = \text{const}\,, \qquad g^2(A\boldsymbol{\nu}\times\boldsymbol{\nu}) + (\mathbf{r}\times\boldsymbol{\nu})\langle A\boldsymbol{\nu},\boldsymbol{\nu}\rangle^2 = 0\,. \tag{1.3}$$

Bifurcation curves on the plane $\mathbf{R}^2(h,g)$ may be parametrically represented by the equations

$$\begin{aligned} h &= \frac{1}{\mu^2}\left(\frac{\left(\frac{3}{2}A_1-\sigma\right)r_1^2}{(A_1-\sigma)^2} + \frac{\left(\frac{3}{2}A_2-\sigma\right)r_2^2}{(A_2-\sigma)^2} + \frac{\left(\frac{3}{2}A_3-\sigma\right)r_3^2}{(A_3-\sigma)^2}\right), \\ g &= \frac{1}{\mu^3}\left(\frac{A_1r_1^2}{(A_1-\sigma)^2} + \frac{A_2r_2^2}{(A_2-\sigma)^2} + \frac{A_3r_3^2}{(A_3-\sigma)^2}\right), \end{aligned} \tag{1.4}$$

where $\mu = \left(\frac{r_1^2}{(A_1-\sigma)^2} + \frac{r_2^2}{(A_2-\sigma)^2} + \frac{r_3^2}{(A_3-\sigma)^2}\right)^{\frac{1}{4}}$. If $A_1 > A_2 > A_3$, then $\sigma \in (-\infty, A_3)\cup(A_3,A_2)\cup$ $\cup(A_2,A_1)\cup(A_1,\infty)$. The detailed description of the bifurcation diagrams and topology of the manifolds $Q^3_{h,g}$ are presented by A. Iacob [10], S. B. Katok [11], Ya. V. Tatarinov [19] etc.

The additional integral K decomposes $Q^3_{h,g}$ into two-dimensional surfaces of a level

$$J_{h,k,g} = \{H = h\,,\ \ K = k\,,\ \ G = g\,,\ \ I = 1\} \subset \mathbf{R}^6(\boldsymbol{\omega},\boldsymbol{\nu})\,.$$

It is well known due to Liouville [2] that nonsingular compact surfaces $J_{h,k,g}$ are a union of two-dimensional tori filled with almost-periodic trajectories. There exists a problem of a qualitative description of a topological arrangement and bifurcations of Liouville tori on the surface $Q^3_{h,g}$. M. P. Kharlamov [12] studied a phase topology of the known integrable cases of rigid body dynamics. He constructed a bifurcation set $B \subset \mathbf{R}^3(h,k,g)$ consisting of critical values of the map $H\times K\times G\colon\ S^2\times\mathbf{R}^3 \to \mathbf{R}^3(h,k,g)$ and found a structure of critical and regular surfaces $J_{h,k,g}$ for each known case. A. T. Fomenko and his students studied the Liouville foliation of isoenergetic surfaces $Q^3_{h,g}$ up to continuous and smooth trajectorial equivalences. They classified and described the topology of many integrable problems of classical and modern Hamiltonian mechanics [3, 17] using molecules coding a structure of foliations on $Q_{h,g}$. These topological results can be confirmed to demonstrate an evolution of invariant tori in a phase space of the system by immediate computer simulation [4, 5].

In this work one of the basic kinematic characteristics of a rotating rigid body, i. e. its angular velocity is studied. For this purpose the invariant sets, in particular images of the Liouville tori and surfaces $Q^3_{h,g}$, are constructed in space $\mathbf{R}^3(\boldsymbol{\omega})$ and all possible types of these sets are classified. The self-intersections of trajectories (hodographs of the angular velocity and the angular momentum) are investigated. The family of curves on the Poisson sphere, which illustrates the Liouville foliation on $Q^3_{h,g}$ for the Kovalevskaya top [15], is constructed. We also note papers [6, 7, 14] where the angular velocity of other integrable tops is investigated.

2. Integrable case of S. V. Kovalevskaya

We subject parameters describing a mass distribution in a body to the following conditions: $A_1 = A_2 = 2A_3$, $r_2 = r_3 = 0$. Then the Euler–Poisson equations admit the Kovalevskaya integral

$$K = (\omega_1^2 - \omega_2^2 + \nu_1)^2 + (2\omega_1\omega_2 + \nu_2)^2 = k\,. \tag{2.1}$$

Without loss of generality it is possible to assume $g \geqslant 0$ since the equations (1.1) and their first integrals are invariant under transformation $(\omega_1,\omega_2,\nu_3,g) \mapsto (-\omega_1,-\omega_2,-\nu_3,-g)$.

The integrals (1.2), (2.1) and auxiliary real-valued variables

$$s_1 = 2\omega_1^2 - \frac{c_1 - \sqrt{c_1^2 + c_2^2}}{4\omega_2^2}\,, \qquad s_2 = 2\omega_1^2 - \frac{c_1 + \sqrt{c_1^2 + c_2^2}}{4\omega_2^2}\,, \tag{2.2}$$

where

$$c_1 = \operatorname{Re} R(\omega_1 + i\omega_2)\,, \qquad c_2 = \operatorname{Im} R(\omega_1 + i\omega_2)\,, \qquad R(x) = -x^4 + 2hx^2 + 2gx + 1 - k\,,$$

allowed S. V. Kovalevskaya [15] to reduce the equations (1.1) to quadratures. N. E. Zhukovskii constructed level lines of two functions s_1, s_2 on a plane $\mathbf{R}^2(\omega_1, \omega_2)$ in his geometric research of this integrable case [20]. The set of curves $s_i = \text{const}$ forms a system of curvilinear orthogonal coordinates on the plane $\mathbf{R}^2(\omega_1, \omega_2)$, which simplifies the study of an image of a projection of the vector $\boldsymbol{\omega}$ on an equatorial plane of an inertia ellipsoid. The image of a surface $J_{h,k,g}$ on $\mathbf{R}^2(\omega_1, \omega_2)$ is the domain bounded by curves

$$s_1 = e_5\,, \qquad s_1 = e_4\,, \qquad s_2 = e_4\,, \tag{2.3}$$

where $e_5 = h + \sqrt{k}$, $e_4 = h - \sqrt{k}$. The analysis of areas filled with values of the variables ω_1, ω_2 was carried out by G. G. Appelrot [1]. It allowed him to study modifications of the considered domains and select periodic solutions which can be noted as $s_i = \text{const}$ in the Kovalevskaya variables, and to find conditions of existence of asymptotic solutions of the equations (1.1). This work served as the base for studying phase topology of this problem [12]. The critical values of the integrals (1.2), (2.1) belong to a bifurcation set

$$\begin{gathered} B = \bigcup_{i=1}^{4} B_i \subset \mathbf{R}^3(h, k, g)\,, \quad B_1 = \{h, k, g \colon k = 0\}\,, \quad B_2 = \{h, k, g \colon g^2 = 2(h + \sqrt{k})\}\,, \\ B_3 = \{h, k, g \colon g^2 = 2(h - \sqrt{k})\}\,, \quad B_4 = \Big\{h, k, g \colon \frac{27}{4} g^2 = h(9 - 9k + h^2) \pm (3k - 3 + h^2)^{\frac{3}{2}}\Big\}\,, \end{gathered} \tag{2.4}$$

which represents a part of a surface of the multiple roots of the fifth degree polynomial included in the Kovalevskaya's quadrature formulas.

We use (2.2) to write the equations (2.3) in the form

$$\omega_2^2 = \sqrt{k} - \omega_1^2 - g e_5^{-1} \omega_1 \pm e_5^{-1} \sqrt{\Big(e_5 - \frac{g^2}{2}\Big)(e_5 - 2\omega_1^2)}\,, \tag{2.5}$$

$$\omega_2^2 = -\sqrt{k} - \omega_1^2 - g e_4^{-1} \omega_1 \pm e_4^{-1} \sqrt{\Big(e_4 - \frac{g^2}{2}\Big)(e_4 - 2\omega_1^2)}\,. \tag{2.6}$$

For fixed values (h, k, g) the relations (2.5), (2.6) allow us to build domains on the plane $\mathbf{R}^2(\omega_1, \omega_2)$ where real motions evolves. In a typical case a connected component of an image of a manifold $J_{h,k,g}$ is diffeomorphic to a ring or rectangle in dependence on position of the triple (h, k, g) in a subspace $\Omega_i \subset \mathbf{R}^3 \backslash B$ [1, 12] (see Fig. 1).

Four distinct points of an invariant manifold $J_{h,k,g}$ are projected, at each interior point of domains shaded in Fig. 1. The appropriate points of a surface $P_{h,k,g} = p(J_{h,k,g})$, where $p \colon (\boldsymbol{\omega}, \boldsymbol{\nu}) \mapsto \boldsymbol{\omega}$, are not always distinct and can pairwise coincide.

3. Singular surfaces $P_{h,k,g}$

By eliminating the components ν_i in the integrals (1.2), (2.1) we obtain an equation

$$F = (c_1^2 + c_2^2)\omega_3^4 + c_3 \omega_3^2 + c_4^2 = 0\,, \tag{3.1}$$

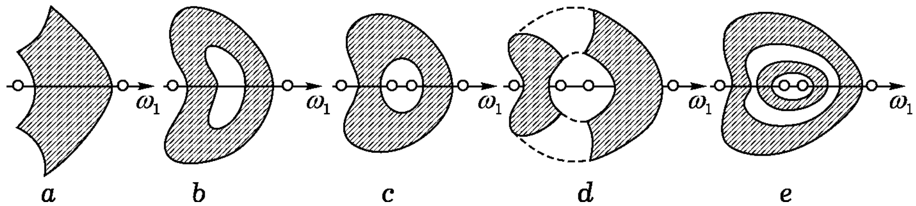

Fig. 1

where again $c_1 = \operatorname{Re} R(\omega_1 + i\omega_2), c_2 = \operatorname{Im} R(\omega_1 + i\omega_2)$ and multipliers c_3, c_4 are

$$c_3 = -\frac{1}{2}c_1c_2^2\omega_2^{-2} + 4c_2^2(2\omega_1^2 - h) + 8c_1\omega_2^2[(2\omega_1^2 - h)^2 - k]\,, \qquad c_4 = \frac{1}{4}\omega_2^{-2}c_2^2 + 4\omega_2^2[(2\omega_1^2 - h)^2 - k]\,.$$

The two-dimensional real surface $P_{h,k,g} = p(J_{h,k,g}) \subset \mathbf{R}^3(\boldsymbol{\omega})$ is determined as a point set which coordinates satisfy (3.1). Let us note that the equation (3.1) was first obtained by V. M. Starzhinskii. The points of the surface $P_{h,k,g}$ at which equalities

$$F(\boldsymbol{\omega}) = 0\,, \qquad \operatorname{grad} F(\boldsymbol{\omega}) = 0 \tag{3.2}$$

are fulfilled will be singular. One can immediately obtain from (3.2) a set

$$N = \bigcup_{i=1}^{3} N_i \subset \mathbf{R}^3(\boldsymbol{\omega})\,, \qquad N_1 = \{\boldsymbol{\omega}\colon c_2 = 0\,, \quad \omega_2 \neq 0\,, \quad c_1\omega_3^2 + c_4 = 0\}\,,$$
$$N_2 = \{\boldsymbol{\omega}\colon \omega_2 = 0\,, \quad c_1\omega_3^2 - c_4 = 0\}\,, \qquad N_3 = \{\boldsymbol{\omega}\colon \omega_3 = 0\,, \quad c_4 = 0\}\,,$$

consisted of singular points of the surface $P_{h,k,g}$. At some points of the set N the intersection of distinct components of $P_{h,k,g}$ takes place, on the other hand the self-intersection takes place also on N. The second case is more interesting since it leads to self-intersections of trajectories in $\mathbf{R}^3(\boldsymbol{\omega})$. It is possible to study a structure and mutual position of the curves N_i by its projections on the plane $\mathbf{R}^2(\omega_1, \omega_2)$. The possible variants of planar curves $c_2 = 0$ and $c_4 = 0$ for $\omega_2 \geqslant 0$ are shown in Fig. 2. We can obtain a case $\omega_2 < 0$ using the mirror map of Fig. 2 to the lower half-plane.

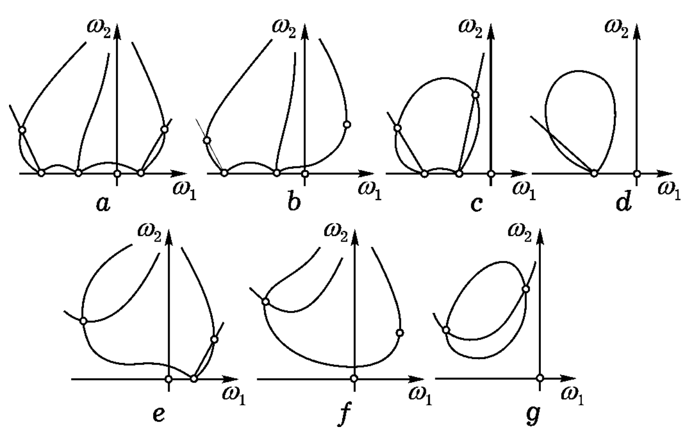

Fig. 2

4. Classification of singular points

The points of the set $N \subset \mathbf{R}^3(\boldsymbol{\omega})$ can be divided into four basic types. We refer points of self-intersection of the surface $P_{h,k,g}$ to the first type. All of them are joined in double lines.

At each such point the tangents to the surface $P_{h,k,g}$ form two real planes. We refer the simple branching point, is the top of a “cross cup” (see Fig. 3 a), to the second type. Images of the considered branching points belong to the curve (2.5), (2.6) in the plane $\mathbf{R}^2(\omega_1, \omega_2)$. A singular point of the third type is represented in Fig. 3 b. Two sheets of the surface $P_{h,k,g}$ intersect with each other along two double lines at this point. Such points belong to a set $N_1 \cap N_3$ or (provided $g = 0$) to the intersection of two different branches of the curve N_1. Six double lines and four sheets of the surface $P_{h,k,g}$ intersect at singular points of the fourth type. All these points are located on the axis $O\omega_1$ since they belong to a set $N_1 \cap N_2 \cap N_3$. The singular set of the surface $P_{h,k,g}$ can be schematically presented as a graph, tops of which correspond to the points of the second, third and fourth types, and the double lines are represented by edges of the graph between the appropriate tops.

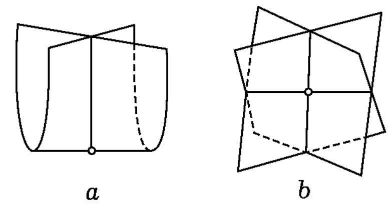

Fig. 3

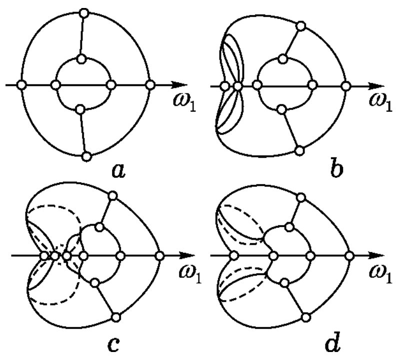

Fig. 4

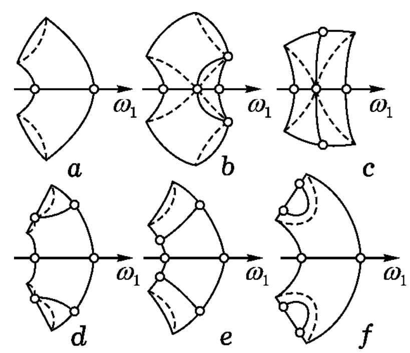

Fig. 5

The equations of planar curves (2.5), (2.6) and $c_2 = 0, c_4 = 0$ allow us to study and to classify possible positions of projections of the components N_i and images of the manifolds $J_{h,k,g}$ on $\mathbf{R}^2(\omega_1, \omega_2)$. The equations of surfaces in the space $\mathbf{R}^3(h, k, g)$, obtained by such a classification, have the following form:

$$
\begin{aligned}
&C_1 = \left\{h, k, g \colon g^2 = \frac{1}{2}(h^2 - k)(h - \sqrt{k})\right\},\\
&C_2 = \left\{h, k, g \colon g^2 = \frac{1}{2}(h^2 - k)(h + \sqrt{k})\right\},\\
&C_3 = \left\{h, k, g \colon g^2 = \frac{16}{27}h^3\right\},\\
&C_4 = \left\{h, k, g \colon g = \pm[2\sqrt{k} - \sqrt{1 + 2\sqrt{k}(h + \sqrt{k})}]\sqrt{2(h + \sqrt{k})}\right\},\\
&C_5 = \left\{h, k, g \colon g = [2\sqrt{k} \pm \sqrt{1 - 2\sqrt{k}(h - \sqrt{k})}]\sqrt{2(h - \sqrt{k})}\right\},\\
&C_6 = \left\{h, k, g \colon h = \sqrt{k} + 3\omega_1^2 - \sqrt{1 - 3\omega_1^4}, g = -4\omega_1^3, \omega_1 \in \left[-\frac{1}{\sqrt{2}}, 0\right]\right\}.
\end{aligned}
\tag{4.1}
$$

On the plane $\mathbf{R}^2(\omega_1, \omega_2)$ the images of the surfaces $P_{h,k,g}$, which are described by the equation (3.1), can look like types, represented in Fig. 4, 5 (dashed lines correspond to solutions of the equation $c_4 = 0$ and solid lines to those of $c_2 = 0$). If a connected component of an image is diffeomorphic to a rectangle, there are more than twenty various variants of projections and only six of them are shown in Fig. 5.

Theorem 1. *Each connected component of a surface* $P_{h,k,g} = p(J_{h,k,g})$ *possesses a nonempty subset of singular points* N. *The qualitative reorganizations of* $P_{h,k,g}$ *are connected with a modification of a structure of the set* N. *The topological structure of* N *varies only on the surfaces* (2.4), (4.1).

5. Portraits of invariant tori

Some types of the surfaces $P_{h,k,g}$ are shown in Fig. 6. Explicit expressions [15] of the components ω_i via the auxiliary variables s_1, s_2 were used for their construction. The more detailed qualitative representation of a behaviour of solutions can be obtained by analysis of rotation numbers of tangent vector fields on invariant tori of the Kovalevskaya problem equal a ratio of periods of a hyperelliptic integral [16]. In a nonresonance case the closure of a trajectory, i. e. a solution of system (1.1) in $\mathbf{R}^3(\boldsymbol{\omega})$ coincides with one of components of the surface $P_{h,k,g}$.

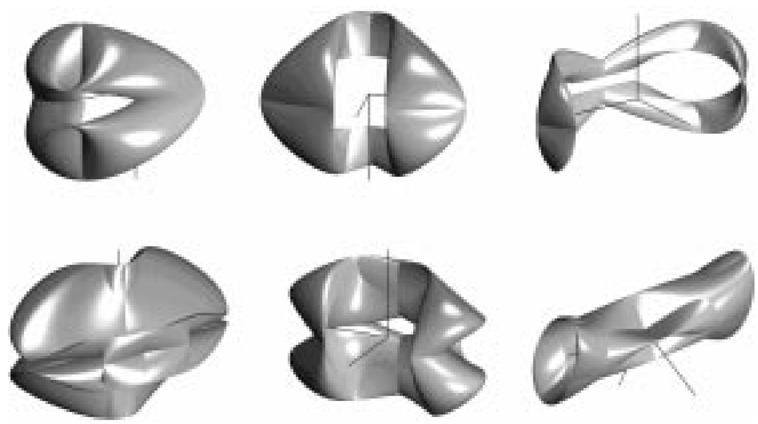

Fig. 6

Let the constants h, g be fixed. The equation (3.1) yields the dependence on k for a one-parameter family of the surfaces $P_{h,k,g}$. Is there exist a surface which touches some surface $P_{h,k,g}$ in each its point? A necessary criterion for the enveloping surface is the system of equalities $F = \dfrac{\partial F}{\partial k} = 0$, from which we really can obtain the equation of a two-dimensional surface in the space $\mathbf{R}^3(\boldsymbol{\omega})$. We shall not limit the deduction of the equation of such a surface only by the Kovalevskaya case and we shall consider a general case.

6. The Hess equations

Let us assume that $A\boldsymbol{\omega}$ and $\mathbf{r}$ are noncollinear vectors and let us present the vector $\boldsymbol{\nu}$ as a sum

$$\boldsymbol{\nu} = a_1 A\boldsymbol{\omega} + a_2\mathbf{r} + a_3(A\boldsymbol{\omega} \times \mathbf{r})\,. \tag{6.1}$$

Then we can derive the following coefficients from the integrals H, G, I

$$a_1 = \frac{\Big[|\mathbf{r}|^2 g + (h-T)\langle A\boldsymbol{\omega}, \mathbf{r}\rangle\Big]}{|A\boldsymbol{\omega} \times \mathbf{r}|^2}\,, \qquad a_2 = \frac{\Big[(T-h)|A\boldsymbol{\omega}|^2 - \langle A\boldsymbol{\omega}, \mathbf{r}\rangle g\Big]}{|A\boldsymbol{\omega} \times \mathbf{r}|^2}\,, \qquad a_3 = \frac{\sqrt{f}}{|A\boldsymbol{\omega} \times \mathbf{r}|^2}\,,$$

where

$$T = \frac{1}{2}\langle A\boldsymbol{\omega}, \boldsymbol{\omega}\rangle\,, \qquad f = |A\boldsymbol{\omega} \times \mathbf{r}|^2 - |(h-T)A\boldsymbol{\omega} + g\mathbf{r}|^2\,.$$

Elementary transformations allow us to write f as a polynomial of the sixth degree in three variables, which are components of the angular velocity vector:

$$\begin{aligned} f = (|A\boldsymbol{\omega}|^2 - g^2)(|\mathbf{r}|^2 - (T-h)^2) - [\langle A\boldsymbol{\omega}, \mathbf{r}\rangle - (T-h)g]^2 = \\ = -T^2|A\boldsymbol{\omega}|^2 + 2hT|A\boldsymbol{\omega}|^2 + 2gT\langle A\boldsymbol{\omega}, \mathbf{r}\rangle + |A\boldsymbol{\omega}|^2(|\mathbf{r}|^2 - h^2) - \\ - \langle A\boldsymbol{\omega}, \mathbf{r}\rangle^2 - 2hg\langle A\boldsymbol{\omega}, \mathbf{r}\rangle - g^2|\mathbf{r}|^2\,. \end{aligned} \tag{6.2}$$

Substituting (6.1) into the first equation of (1.1), we obtain the vector form of the Hess equations [9]

$$A\dot{\boldsymbol{\omega}} = A\boldsymbol{\omega} \times \boldsymbol{\omega} + a_1(\mathbf{r} \times A\boldsymbol{\omega}) + a_3(\mathbf{r} \times (A\boldsymbol{\omega} \times \mathbf{r}))\,. \tag{6.3}$$

The equation (6.3) does not contain $\boldsymbol{\nu}$ and describes dynamics of the vectors $\boldsymbol{\omega}$, $A\boldsymbol{\omega}$ in a frame rigidly connected with the body. It is easy to show that the function $f(\boldsymbol{\omega})^{-\frac{1}{2}}$ serves as an integrating factor of the equation (6.3).

7. Enveloping surface $\partial V_{h,g}$

We again consider a projection $p\colon (\boldsymbol{\omega}, \boldsymbol{\nu}) \mapsto \boldsymbol{\omega}$. The point $(\omega_1, \omega_2, \omega_3)$ belongs to the image $V_{h,g} = p(Q^3_{h,g})$ provided that there exists a real solution (ν_1, ν_2, ν_3) satisfying the three first integrals. We exclude ν_i from a condition

$$\frac{D(H,\ G,\ I)}{D(\nu_1,\ \nu_2,\ \nu_3)} = 0 \tag{7.1}$$

and obtain an equation of a two-dimensional surface bounding the domain $V_{h,g}$. Further we name $\partial V_{h,g} = \{f(\boldsymbol{\omega}) = 0\} \subset \mathbf{R}^3(\boldsymbol{\omega})$ as *the enveloping surface.* The images of phase trajectories of the system (1.1), corresponding to fixed constants (h, g), fill in the closed domain $V_{h,g} = \{f(\boldsymbol{\omega}) \geqslant 0\} \subset \mathbf{R}^3(\boldsymbol{\omega})$. In accordance with (6.1) each interior point of the set $V_{h,g} \subset \mathbf{R}^3(\boldsymbol{\omega})$ has two pre-images on the surface $Q^3_{h,g}$ which differ only by sign of coefficient a_3 in the expression (6.1). The right member of the equation (6.3) can vanish only on the boundary $\partial V_{h,g}$. The equation $f(\boldsymbol{\omega}) = 0$ becomes essentially simpler for $\mathbf{r} = 0$ and reduces to the equation of the ellipsoid $T = h$. In the Euler case this ellipsoid is filled with phase trajectories on a fixed level of an energy. Let us note the known equality

$$\frac{1}{2}\frac{d}{dt}|A\boldsymbol{\omega}|^2 = \sqrt{f(\boldsymbol{\omega})}\,, \tag{7.2}$$

which follows from (6.3). Any trajectory (hodograph of an angular velocity in the body axes), belonging to the set $V_{h,g}$, tangents the enveloping surface $\partial V_{h,g}$ in that instant, when $|A\boldsymbol{\omega}|$ attains its local extremum along the trajectory. If the trajectory, i. e. hodograph, entirely belongs to the surface $\partial V_{h,g}$, then according to the equality (7.2) the module of the vector of angular momentum preserves the initial value. In the problem under consideration all cases when $|A\boldsymbol{\omega}| = \mathrm{const}$ are known. Besides the Euler solution and the steady rotations (1.3), the module of angular momentum is constant only in some special cases of solutions of Lagrange and Hess, where the body performs precession motions around a vertical axis [8].

8. Classification of $\partial V_{h,g}$

The topology of the enveloping surfaces $\partial V_{h,g}$ depends on values of the parameters (h, g). There exist the surfaces $\partial V_{h,g}$ without singular points (for example, diffeomorphic to sphere S^2 or torus T^2). If there are singular points on the real surface $\partial V_{h,g}$, then we can obtain them from the conditions

$$f(\boldsymbol{\omega}) = 0\,, \qquad \operatorname{grad} f(\boldsymbol{\omega}) = 0\,.$$

On the bifurcation set $\Sigma \subset \mathbf{R}^2(h, g)$ the enveloping surface $\partial V_{h,g}$ always has singular points corresponded to steady rotations of the body (i.e. the relative equilibria (1.3) of the system (1.1)). The modification of a topological type of $Q^3_{h,g}$ at passage of the bifurcation set (1.4) reduces to a modification of a type of $V_{h,g}$. If the surface $Q^3_{h,g}$ consists of several connected components, then also $V_{h,g}$ is a union of nonintersecting sets.

Moreover, the enveloping surface $\partial V_{h,g}$ has a singular point, if the vectors $A\boldsymbol{\omega}$ and $\mathbf{r}$ become collinear during the motion of the body. Then the equality $A\boldsymbol{\omega} = \lambda\mathbf{r}$ is fulfilled. It can be derived from the first integrals that the factor λ is the real solution of a cubic equation

$$\frac{1}{2}\langle A^{-1}\mathbf{r}, \mathbf{r}\rangle\lambda^3 - h\lambda - g = 0\,, \tag{8.1}$$

and the vector $\boldsymbol{\nu}$ satisfies to the equations $\lambda\langle\mathbf{r}, \boldsymbol{\nu}\rangle = g$, $|\boldsymbol{\nu}| = 1$. The important property is that in the space $\mathbf{R}^3(\boldsymbol{\omega})$ there exists an axis with a unit vector $\mathbf{l} = \dfrac{A^{-1}\mathbf{r}}{|A^{-1}\mathbf{r}|}$ which intersects the closed domain $V_{h,g}$ no more than at three points. These points $\boldsymbol{\omega} = \lambda A^{-1}\mathbf{r} = \text{const}$ are singular points of the enveloping surface $\partial V_{h,g}$. P. V. Kharlamov [13] introduced a special frame, the first axis of which passes through a center of gravity of the body. He obtained a suitable differential equations for components of angular momentum in that base. It follows from the above mentioned that at the fixed constant (h, g) the trajectories, i. e. the hodograph of angular momentum, can repeatedly intersect the axis carrying a center of gravity, but no more than at three fixed points of this axis. All points of self-intersection of trajectories in the open domain $V_{h,g}\backslash\partial V_{h,g}$ can be only double.

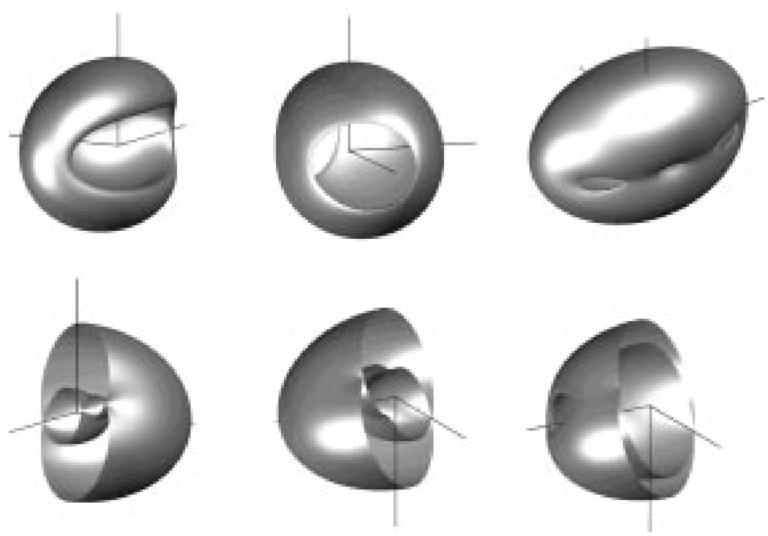

Fig. 7

Let us obtain curves by dividing the plane $\mathbf{R}^2(h, g)$ into domains, inside which a number of singular points of the enveloping surface $\partial V_{h,g}$ preserves. We write the curve of multiple solutions of the equation (8.1) in the form

$$8h^3 - 27g^2\langle A^{-1}\mathbf{r}, \mathbf{r}\rangle = 0\,. \tag{8.2}$$

Using the inequalities $-|\mathbf{r}| \leqslant \langle\mathbf{r}, \boldsymbol{\nu}\rangle \leqslant |\mathbf{r}|$ one can obtain the following restrictions

$$-|\mathbf{r}| \leqslant \left[\frac{1}{2}\langle A^{-1}\mathbf{r}, \mathbf{r}\rangle\lambda^2 - h\right] \leqslant |\mathbf{r}|\,, \tag{8.3}$$

for the parameter λ. From (8.1), (8.3) we obtain the equations of dividing curves in the form

$$g^2\langle A^{-1}\mathbf{r}, \mathbf{r}\rangle - 2|\mathbf{r}|^2(h \pm |\mathbf{r}|) = 0\,. \tag{8.4}$$

Thus, the curves (1.4), (8.2), (8.4) divide the plane $\mathbf{R}^2(h, g)$ into subregions with qualitative various types of the surfaces $\partial V_{h,g}$. Some of these surfaces are shown in Fig. 7. Here as well as in the Kovalevskaya case the center of gravity lays on one of principal axis of inertia, and the singular points of a surface $\partial V_{h,g}$ are located on the same axis.

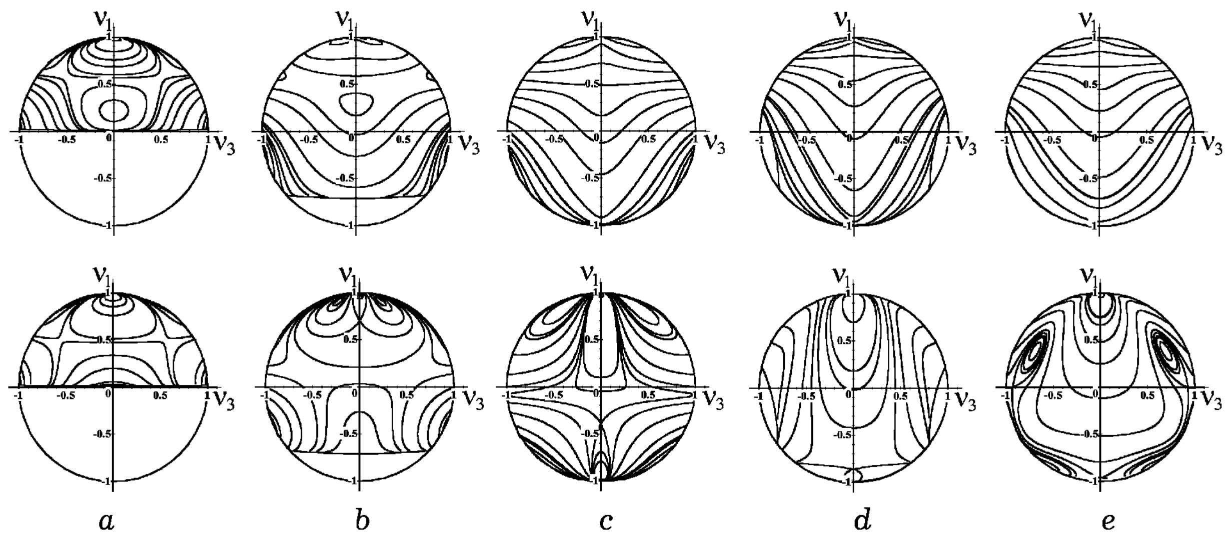

Fig. 8

Theorem 2. *All integral trajectories, which for the fixed constants* (h, g) *correspond to solutions* $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ *of the vector equation* (6.3), *belong to the three-dimensional surface* $V_{h,g} = p(Q^3_{h,g})$. *An arbitrary point of the set* $V_{h,g} \backslash \partial V_{h,g}$ *has two pre-images on the manifold* $Q^3_{h,g}$. *Singularities of the real surface* $\partial V_{h,g}$ *either correspond to the relative equilibria* (1.3), *or belong to the axis with the unit vector* $\mathbf{l}$. *The classification of possible types of the surfaces* $\partial V_{h,g}$ *is determined by the curves* (8.2), (8.4) *and the bifurcation set* $\Sigma \subset \mathbf{R}^2(h, g)$.

9. Set of characteristics on $\partial V_{h,g}$

The completely integrable dynamical system on $Q^3_{h,g}$ admits an additional integral $K(\boldsymbol{\omega}, \boldsymbol{\nu}) = k$ which decomposes this invariant manifold $Q^3_{h,g}$ on connected two-dimensional components $J_{h,k,g}$. The enveloping surface $\partial V_{h,g}$ tangents some component of the set $P_{h,k,g} = p(J_{h,k,g})$ dependent on a parameter k at each point. The curve, along which $\partial V_{h,g}$ comes into contact with a fixed surface $P_{h,k,g}$, is known as *a characteristic* of the given set. The characteristics form on $\partial V_{h,g}$ a set of lines dependent on k. We obtain the equation of characteristics on $\partial V_{h,g}$ by substitution (6.1) and $a_3 = 0$ into an integral:

$$\widetilde{K}(\boldsymbol{\omega}) = K(\boldsymbol{\omega}, \boldsymbol{\nu})|_{\boldsymbol{\nu}=\boldsymbol{\nu}^0(\boldsymbol{\omega})} = k\,, \qquad \boldsymbol{\nu}^0 = a_1 A\boldsymbol{\omega} + a_2\mathbf{r}\,. \tag{9.1}$$

For fixed constants (h, k, g) the points of sequential tangencies of a trajectory of the equation (6.3) with the surface $\partial V_{h,g}$ is placed on closed curves in the Euclidean space $\mathbf{R}^3(\boldsymbol{\omega})$, formed with joint solutions of equations $f(\boldsymbol{\omega}) = 0$, $\widetilde{K}(\boldsymbol{\omega}) = k$. The one-parameter family of characteristics is also convenient to study and to classify on the Poisson sphere. We can write an equation like (9.1) for an sphere $S^2 = \{|\boldsymbol{\nu}| = 1\}$:

$$\widehat{K}(\boldsymbol{\nu}) = K(\boldsymbol{\omega}, \boldsymbol{\nu})|_{\boldsymbol{\omega}=\boldsymbol{\omega}^0(\boldsymbol{\nu})} = k\,, \qquad A\boldsymbol{\omega}^0 = b_1\boldsymbol{\nu} + b_2\mathbf{r}\,, \tag{9.2}$$

where multipliers $b_{1,2}$ satisfy the following system of algebraic equations:

$$\begin{gathered} b_1 + b_2\langle \boldsymbol{\nu}, \mathbf{r}\rangle = g\,, \\ b_2^2\left[\langle \boldsymbol{\nu}, \mathbf{r}\rangle^2 \langle A^{-1}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle + \langle A^{-1}\mathbf{r}, \mathbf{r}\rangle - 2\langle \boldsymbol{\nu}, \mathbf{r}\rangle\langle A^{-1}\boldsymbol{\nu}, \mathbf{r}\rangle\right] + \\ +2gb_2\left[\langle A^{-1}\boldsymbol{\nu}, \mathbf{r}\rangle - \langle \boldsymbol{\nu}, \mathbf{r}\rangle\langle A^{-1}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle\right] + g^2\langle A^{-1}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle - 2\langle \boldsymbol{\nu}, \mathbf{r}\rangle - 2h = 0\,. \end{gathered} \tag{9.3}$$

From (9.3) it follows, that an arbitrary point of the Poisson sphere not lying on an axis carrying the center of gravity of a body, has no more than two pre-images $\boldsymbol{\omega}^0 = b_1 A^{-1}\boldsymbol{\nu} + b_2 A^{-1}\mathbf{r}$ on $\partial V_{h,g}$. We take two copies of the sphere S^2 and construct level lines $\widehat{K}(\boldsymbol{\nu}) = k$ on each of them. The results of computations are shown in Fig. 8 for five various pairs (h, g) of the Kovalevskaya case. Curves on the sphere illustrate a structure of the Liouville foliation of the isoenergetic surface $V_{h,g}$.

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