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On the stability in periodic and almost periodic difference systems

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Abstract

In this paper we consider the system of difference equations

$$x_{n+1} = f_n(x_n), \qquad f_n(0) = 0, \quad n = 0, 1, 2, \ldots.$$

This system has the trivial (zero) solution $x_n = 0$. Sufficient conditions of its asymptotic stability are obtained in the cases when functions $f_n(x)$ are periodic and almost periodic in n.

Keywords: Difference equations; Stability; Lyapunov's direct method

1. Introduction

Difference equations have been studied in various branches of mathematics for a long time. First results in qualitative theory of such systems were obtained by Poincaré and Perron in the end of nineteenth and the beginning of twentieth centuries. The systematic description of the theory of difference equations one can find in books [2,7,16]. Difference

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equations are a convenient model for discrete dynamic systems description and for mathematical simulation of systems with impulse effect [8,12,14,15,19]. One of directions arising from applications of difference equations is linked with qualitative investigation of their solutions (stability, boundedness, controllability, observability, oscillation, robustness...) [1,3–5,9,10,13,17].

Consider the discrete system of the form

$$x_{n+1} = f_n(x_n), \qquad f_n(0) = 0 \tag{1.1}$$

where $n = 0, 1, 2, \ldots$ is the discrete time, $x_n = (x_n^1, x_n^2, \ldots, x_n^p) \in \mathbb{R}^p$, $f_n = (f_n^1, f_n^2, \ldots, f_n^p) \in \mathbb{R}^p$, f_n satisfy Lipschitz conditions uniformly in n: $\|f_n(x) - f_n(y)\| \leqslant L_r\|x - y\|$ for $\|x\| \leqslant r$, $\|y\| \leqslant r$. System (1.1) has the trivial (zero) solution

$$x_n \equiv 0. \tag{1.2}$$

Denote $x_n(n_0, u)$ the solution of system (1.1) coinciding with u under $n = n_0$. We also denote $B_r = \{x \in \mathbb{R}^p\colon\ \|x\| \leqslant r\}$. Suppose functions $f_n(x)$ to be defined in B_H where $H > 0$ is some fixed number. According to [16] we denote $\mathbb{Z}_+$ the set of nonnegative integers.

2. Main definitions and preliminaries

By analogy to ordinary differential equations [11,18,20], let us introduce the following definitions.

Definition 2.1. Solution (1.2) of system (1.1) is said to be stable if for any $\varepsilon > 0$, $n_0 \in \mathbb{Z}_+$ there exists $\delta = \delta(\varepsilon, n_0) > 0$ such that $\|x_{n_0}\| \leqslant \delta$ implies $\|x_n\| \leqslant \varepsilon$ for each $n > n_0$.

Definition 2.2. The trivial solution of system (1.1) is said to be uniformly stable if δ in Definition 2.1 can be chosen independent on n_0, i.e., $\delta = \delta(\varepsilon)$.

Definition 2.3. Solution (1.2) of system (1.1) is called attractive if for every $n_0 \in \mathbb{Z}_+$ there exists $\eta = \eta(n_0) > 0$ and for every $\varepsilon > 0$ and $x_{n_0} \in B_\eta$ there exists $\sigma = \sigma(\varepsilon, n_0, x_{n_0}) \in \mathbb{N}$ such that $\|x_n\| < \varepsilon$ for any $n \geqslant n_0 + \sigma$. Here $\mathbb{N}$ is the set of natural numbers.

In other words, solution (1.2) of system (1.1) is attractive if

$$\lim_{n\to\infty} \big\|x_n(n_0, x_{n_0})\big\| = 0. \tag{2.1}$$

Definition 2.4. The zero solution of system (1.1) is called equiattractive if for every $n_0 \in \mathbb{Z}_+$ there exists $\eta = \eta(n_0) > 0$, and for any $\varepsilon > 0$ there is $\sigma = \sigma(\varepsilon, n_0) \in \mathbb{N}$ such that $\|x_n(n_0, x_{n_0})\| < \varepsilon$ for all $x_{n_0} \in B_\eta$ and $n \geqslant n_0 + \sigma$.

In other words, the zero solution of (1.1) is equiattractive if limit relation (2.1) holds uniformly in $x_{n_0} \in B_\eta$.

Definition 2.5. Solution (1.2) of system (1.1) is said to be uniformly attractive, if for some $\eta > 0$ and any $\varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon) \in \mathbb{N}$ such that $\|x_n(n_0, x_{n_0})\| < \varepsilon$ for all $n_0 \in Z_+$, $x_{n_0} \in B_\eta$, and $n \geqslant n_0 + \sigma$.

In other words, solution (1.2) of system (1.1) is uniformly attractive, if limit relation (2.1) holds uniformly in $n_0 \in Z_+,\ x_{n_0} \in B_\eta$.

Definition 2.6. The trivial solution (1.2) of system (1.1) is called:

– asymptotically stable if it is stable and attractive;
– equiasymptotically stable if it is stable and equiattractive;
– uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

Definition 2.7 [11,18]. Function $r : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class of Hahn functions $\mathcal{K}$ ($r \in \mathcal{K}$) if r is continuous increasing function, and $r(0) = 0$.

A. Halanay and D. Wexler [12] proved the following theorems.

Theorem 2.1. *Solution* (1.2) *of system* (1.1) *is uniformly stable if there exists a sequence of functions* $\{V_n(x)\}$, *with the following properties*:

$$a(\|x\|) \leqslant V_n(x) \leqslant b(\|x\|), \quad a \in \mathcal{K},\ b \in \mathcal{K},\ n \in \mathbb{Z}_+, \tag{2.2}$$

$$V_n(x_n) \geqslant V_{n+1}(x_{n+1}) \quad \textit{for every solution } x_n. \tag{2.3}$$

Theorem 2.2. *Suppose that there exists a sequence of functions* $\{V_n(x)\}$, *with properties* (2.2) *and*

$$V_{n+1}(x_{n+1}) - V_n(x_n) \leqslant -c(\|x_n\|), \quad c \in \mathcal{K}, \tag{2.4}$$

$$\bigl|V_n(x) - V_n(y)\bigr| \leqslant L\|x - y\|, \quad n \in \mathbb{Z}_+,\ x \in B_H,\ y \in B_H,\ L > 0. \tag{2.5}$$

Then the zero solution of system (1.1) *is uniformly asymptotically stable.*

In particular case, when system (1.1) is autonomous, i.e., $f_n(x) = f(x)$, the following theorem is valid [12, p. 34]:

Theorem 2.3. *If there exists a continuous function* $V(x)$ *such that* $a(\|x\|) \leqslant V(x) \leqslant b(\|x\|),\ a \in \mathcal{K}, b \in \mathcal{K}$, *and*

$$V(x_{n+1}) - V(x_n) \leqslant 0 \tag{2.6}$$

for every solution x_n *of system* (1.1), *and equality sign in* (2.6) *holds in some set which does not contain entire semitrajectories, then solution* (1.2) *of system* (1.1) *is asymptotically stable.*

The purpose of this paper is to obtain conditions of asymptotic stability of solution (1.2) of system (1.1) assuming that sequences $\{f_n(x)\}$ are periodic or almost periodic.

3. Stability in periodic systems

Definition 3.1. System (1.1) is said to be periodic with the period q if

$$f_n(x) \equiv f_{n+q}(x) \quad \text{for each } n \in \mathbb{Z}_+,\ x \in B_H. \tag{3.1}$$

Throughout this section we shall assume that system (1.1) is periodic with period q.

Theorem 3.1. *If solution* (1.2) *of system* (1.1) *is stable, then it is uniformly stable.*

Proof. Conditions (3.1) imply

$$x_{n+q}(n_0+q, x) \equiv x_n(n_0, x), \tag{3.2}$$

whence it is sufficiently to show that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $n_0 = 0, 1, \ldots, q-1$, $x_{n_0} \in B_\delta$ the inequality $\|x_n(n_0, x_{n_0})\| \leqslant \varepsilon$ holds for $n \geqslant n_0$. According the assumption, for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that if $x_q = x_q(0, x_{n_0})$ satisfies condition $x_q \in B_{\delta_1}$, then $x_n(q, x_q) \in B_\varepsilon$ for $n \geqslant q$. Functions f_n satisfy Lipschitz condition with Lipschitz constant L uniformly in n. Let us choose $\delta = L^{-q}\delta_1$. If for any $0 \leqslant n_0 \leqslant q-1$ the condition $x_{n_0} \in B_\delta$ holds, then $x_n(n_0, x_{n_0}) \in B_\varepsilon$. This completes the proof. □

Theorem 3.2. *If the zero solution of system* (1.1) *is asymptotically stable, then it is uniformly asymptotically stable.*

Proof. Since solution (1.2) of system (1.1) is asymptotically stable, then in the set

$$n_0 \in \mathbb{Z}_+, \quad x_{n_0} \in B_\lambda \tag{3.3}$$

where λ is positive number small enough, limit relation (2.1) holds. Since the periodicity of system (1.1), we assume that n_0 satisfies the condition $0 \leqslant n_0 \leqslant q-1$. First we define the number $\eta = \eta(\varepsilon)$ from the condition

$$\left\|x_n(n_0, x_{n_0})\right\| \leqslant \varepsilon \quad \text{for } x_{n_0} \in B_\eta,\ n > n_0. \tag{3.4}$$

This is always possible because of uniform stability of the zero solution. Let us show that limit relation (2.1) holds uniformly in n_0, x_{n_0} from set (3.3), i.e., let us show that for every $\varepsilon > 0$ there is $\sigma = \sigma(\varepsilon) \in \mathbb{N}$ such that the inequality $\|x_n(n_0, x_{n_0})\| \leqslant \varepsilon$ holds for all $n \geqslant n_0 + \sigma$. Suppose the contrary: there is not such $\sigma = \sigma(\varepsilon)$. Then for any large natural number m, there is $n_m \in \mathbb{N}$ such that $n_m > mq$ and initial data $(n_{0m}, x_{n_{0m}})$ such that $0 \leqslant n_{0m} \leqslant q-1$, $x_{n_{0m}} \in B_\lambda$, and

$$\left\|x_{n_m}(n_{0m}, x_{n_{0m}})\right\| > \varepsilon. \tag{3.5}$$

Since the sequence $\{n_{0m}\}$ is finite and $\{x_{n_{0m}}\}$ lies in the compact set, the sequence $\{n_{0m}, x_{0m}\}$ contains a subsequence which converges to (n_*, x_*) where $0 \leqslant n_* \leqslant q-1$, $x_* \in B_\lambda$. Without loss of generality we can suppose that the sequence $\{n_{0m}\}$ coincides with n_* and $\{x_{n_{0m}}\}$ itself converges to x_*. Hence for values $n_0 = n_*$, $x_{n_0} = x_*$ limit relation (2.1) is valid, whence it follows that there exists sufficiently large $k = k(\varepsilon) \in \mathbb{N}$ such that the inequality

$$\left\|x_{n_*+kq}(n_*, x_*)\right\| < \frac{1}{2}\eta(\varepsilon) \tag{3.6}$$

holds. Then, by virtue of continuous dependence of solutions on initial data, there exist large enough values of m for which the inequality

$$\|x^{(k)}\| < \eta(\varepsilon) \tag{3.7}$$

holds where $x^{(k)} = x_{n_{0m}+kq}(n_{0m}, x_{0m})$. Inequalities (3.7) and (3.4) imply that $\|x_n(n_{0m}, x^{(k)})\| \leqslant \varepsilon$ for all $n > n_{0m}$. According to (3.2) and the property of uniqueness of the solution, this implies

$$\varepsilon \geqslant \left\|x_n(n_{0m}, x^{(k)})\right\| \equiv \left\|x_{n+kq}(n_{0m} + kq, x^{(k)})\right\| \equiv \left\|x_{n+kq}(n_{0m}, x_{n_{0m}})\right\|.$$

This inequality contradicts to assumption (3.5) because there exists n_m such that $n_m > kq$. The obtained contradiction proves that limit relation (2.1) holds uniformly in n_0, x_{n_0}. This completes the proof. □

Definition 3.2. The sequence of numbers $\{u_k\}_{k=1}^{\infty}$ is called finally nonzero if for any natural number M there exists $k > M$ such that $u_k \neq 0$.

Theorem 3.3. *Suppose that there exists a periodic sequence of functions* $\{V_n(x)\}$ *with period* q *each term of which satisfies* (2.2), (2.3), *and Lipschitz condition*; *the sequence* $\{V_n(x_n) - V_{n+1}(x_{n+1})\}$ *is finally nonzero for each nonzero solution of system* (1.1). *Then the zero solution of system* (1.1) *is uniformly asymptotically stable.*

Proof. Theorem 2.1 implies that solution (1.2) of system (1.1) is uniformly stable, i.e. for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $n_0 \in \mathbb{Z}_+$, $x_{n_0} \in B_\delta$, $n > n_0$ the inequality $\|x_n(n_0, x_{n_0})\| \leqslant \varepsilon$ holds. Let us show that each trajectory $x_n = x_n(n_0, x_{n_0})$ with such initial conditions has property (2.1).

Consider the sequence $\{v_n\}$ where $v_n = V_n(x_n(n_0, x_{n_0}))$. This sequence does not increase and is bounded from below, therefore there exists $\lim_{n\to\infty} v_n = \eta \geqslant 0$. Let us show that $\eta = 0$. Assume the opposite: let

$$\eta = \lim_{n\to\infty} V_n\big(x_n(n_0, x_{n_0})\big) > 0. \tag{3.8}$$

Consider the sequence $\{x^{(k)}\}$ where $x^{(k)} = x_{n_0+kq}(n_0, x_{n_0})$. From $\|x^{(k)}\| \leqslant \varepsilon < H$ we can conclude that there exists its subsequence which converges to $x_* \in B_\varepsilon$. Without loss of generality we suppose that the sequence $\{x^{(k)}\}$ itself converges to $x_* \neq 0$. Since $V_n(x)$ are periodic in n, and each $V_n(x)$ is continuous in x, the equality $V_{n_0}(x_*) = \eta$ is valid. Consider the semitrajectory $x_n(n_0, x_*)$ of system (1.1) for $n \geqslant n_0$ and the sequence $\{v_n^*\}$ where $v_n^* = V_n(x_n(n_0, x_*))$. This sequence does not increase, and $\{v_n^* - v_{n+1}^*\}$ is finally nonzero. It means that there exist $n_* \in \mathbb{N}$, $n_* > n_0$ such that

$$V_{n_*}\big(x_{n_*}(n_0, x_*)\big) = \eta_1 < \eta.$$

Since $\{x^{(k)}\}$ tends to x_* as $k \to \infty$ and continuous dependence solutions on initial conditions,

$$\left\|x_{n_*}(n_0, x_*) - x_{n_*}(n_0, x^{(k)})\right\| < \gamma$$

for all $k > M(\gamma) \in \mathbb{N}$, for any small $\gamma > 0$. Hence,

$$\lim_{k\to\infty} V_{n_*}\big(x_{n_*}(n_0, x^{(k)})\big) = \eta_1. \tag{3.9}$$

Taking into account the periodicity of system (1.1), we can write

$$x_{n_*}(n_0, x^{(k)}) = x_{n_*}\big(n_0, x_{n_0+kq}(n_0, x_{n_0})\big) = x_{n_*+kq}(n_0, x_{n_0}). \tag{3.10}$$

In fact, trajectories I and II of system (1.1) with initial conditions $(n_0, x^{(k)})$ and $(n_0 + kq, x^{(k)})$ for the discrete time $\Delta n = n_* - n_0$ pass to $x_{n_*}(n_0, x^{(k)})$ and $x_{n_*+kq}(n_0, x_{n_0})$, respectively. This proves (3.10). The periodicity of V_n in n implies $V_{n_*}(x) = V_{n_*+kq}(x)$, hence condition (3.9), in view of (3.10), can be written as follows:

$$\lim_{k\to\infty} V_{n_*+kq}\big(x_{n_*+kq}(n_0, x_{n_0})\big) = \eta_1. \tag{3.11}$$

But limit relation (3.11) is in contradiction to the inequality $V_n(x_n(n_0, x_{n_0})) \geqslant \eta_1$, because $\eta_1 < \eta$. The obtained contradiction proves that assumption (3.8) was incorrect, and this proves that $\eta = 0$. Condition (2.2) implies limit relation (2.1). Using Theorem 3.2, we derive that the zero solution of system (1.1) is uniformly asymptotically stable. □

4. Stability in almost periodic systems

Definition 4.1. A sequence $\{u_n\}_{-\infty}^{+\infty}$ is said to be almost periodic if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) \in \mathbb{N}$ such that each segment $[sl, (s + 1)l]$, $s \in \mathbb{Z}$ contains an integer m such that $\|u_{n+m} - u_n\| < \varepsilon$ for all $n \in \mathbb{Z}$. Here $\mathbb{Z}$ is the set of integers. Numbers m with such properties are called ε-almost periods of the sequence $\{u_n\}$.

Definition 4.2. A sequence of functions $\{f_n(x)\}$ is called uniformly almost periodic if for every $\varepsilon > 0$ there exists $l = l(\varepsilon, r) \in \mathbb{N}$ such that each segment of the form $[sl, (s + 1)l]$, $s \in \mathbb{Z}$ contains an integer m such that $\|f_{n+m}(x) - f_n(x)\| < \varepsilon$ for all $n \in \mathbb{Z}$, $\|x\| < r$.

Lemma 4.1 [12, p. 125]. *Let sequences $\{u_n^1\}, \{u_n^2\}, \ldots, \{u_n^M\}$ be almost periodic. Then for every $\varepsilon > 0$ there exists $l = l(\varepsilon) \in \mathbb{N}$ such that each segment of the form $[sl, (s + 1)l]$, $s \in \mathbb{Z}$, contains at least one ε-almost period, common for all these sequences.*

Lemma 4.2. *If for every $x \in B_H$, a sequence $\{F_n(x)\}$ is almost periodic, and each function $F_n(x)$ satisfies Lipschitz condition uniformly in $n \in \mathbb{Z}$, $x \in B_H$, then this sequence is uniformly almost periodic.*

Proof. Functions $F_n(x)$ satisfy Lipschitz condition, hence

$$\big\|F_n(x) - F_n(y)\big\| \leqslant L_1\|x - y\|, \tag{4.1}$$

where L_1 is Lipschitz constant. Let ε be any positive number. B_H is bounded and closed, therefore it is a compact. It means that there exists a finite set of points $z_1, \ldots, z_M$ such that $z_j \in B_H$ $(j = 1, \ldots, M)$, and for any $x \in B_H$ there exists a natural number i $(1 \leqslant i \leqslant M)$ such that

$$\|x - z_i\| < \frac{\varepsilon}{3L_1}. \tag{4.2}$$

From Lemma 4.1 it follows that there exists $l = l(\varepsilon) \in \mathbb{N}$ such that every segment $[sl, (s + 1)l]$, $s \in \mathbb{Z}$, contains a number $m \in \mathbb{Z}$ such that

$$\big\|F_n(z_i) - F_{n+m}(z_i)\big\| < \frac{\varepsilon}{3} \tag{4.3}$$

for all $1 \leqslant i \leqslant M$, $n \in \mathbb{Z}$.

Now let us show that for every $x \in B_H$, any integer m satisfying inequality (4.3) is ε-almost period of the sequence $\{F_n(x)\}$. Let z_k be the same element of the set $z_1, \ldots, z_M$, for which $\|x - z_k\| < \varepsilon/(3L_1)$. Then (4.1)–(4.3) imply

$$\begin{aligned} &\left\| F_{n+m}(x) - F_n(x) \right\| \\ &\quad \leqslant \left\| F_{n+m}(x) - F_{n+m}(z_k) \right\| + \left\| F_{n+m}(z_k) - F_n(z_k) \right\| + \left\| F_n(z_k) - F_n(x) \right\| \\ &\quad \leqslant \frac{\varepsilon}{3} + 2L_1 \cdot \frac{\varepsilon}{3L_1} = \varepsilon. \end{aligned} \tag{4.4}$$

Inequality (4.4) completes the proof of Lemma 4.2. □

Theorem 4.1. *Let a sequence of continuous functions $\{V_n(x)\}$ satisfy conditions*

$$a\big(\|x\|\big) \leqslant V_n(x), \quad a \in \mathcal{K},\ x \in B_H,\ V_n(0) = 0, \tag{4.5}$$

and for every $n_0 \in \mathbb{Z}_+$ there exists $\Delta(n_0) > 0$ such that $\|x_{n_0}\| < \Delta$ implies that the sequence $\{V_n(x_n(n_0, x_{n_0}))\}$ monotonically does not increase and tends to zero. Then the zero solution of system (1.1) *is equiattractive.*

Proof. Pick arbitrary $\delta = \delta(n_0) \in (0, \Delta)$. According to conditions of the theorem, for any $\varepsilon > 0$, $n_0 \in \mathbb{Z}_+$, and $x_{n_0} \in B_\delta$ there exists $\sigma = \sigma(\varepsilon, n_0, x_{n_0}) \in \mathbb{N}$ such that

$$V_{n_0+\sigma}\big(x_{n_0+\sigma}(n_0, x_{n_0})\big) < \frac{1}{2}\varepsilon.$$

Because of the continuity of functions $V_n(x)$ and continuous dependence of solutions on initial data, there exists a neighbourhood $Q(x_{n_0})$ of the point x_{n_0} in which the inequality

$$V_{n_0+\sigma}\big(x_{n_0+\sigma}(n_0, y)\big) < \varepsilon \quad \text{for } y \in Q(x_{n_0}) \tag{4.6}$$

is valid. Since the sequence $\{V_n\}$ monotonically does not increase along solutions of system (1.1), from (4.6) it follows

$$V_n\big(x_n(n_0, y)\big) < \varepsilon \quad \text{for } n \geqslant n_0 + \sigma(\varepsilon, n_0, x_{n_0}),\ y \in Q(x_{n_0}).$$

So the compact set B_δ is covered by the system of neighbourhoods $\{Q(x_{n_0})\}$ from which, by Heine–Borel's lemma, it is possible to select the finite subcovering $Q_1, \ldots, Q_j$ with corresponding numbers $\sigma_1, \ldots, \sigma_j$. Let

$$\sigma(\varepsilon, n_0) = \max\{\sigma_1, \ldots, \sigma_j\}$$

(σ depends only on ε and n_0). Then $V_n(x_n(n_0, x_{n_0})) < \varepsilon$ for all $n \geqslant n_0 + \sigma(\varepsilon, n_0)$ if $\|x_{n_0}\| \leqslant \delta(n_0)$. This inequality proves that solution (1.2) of system (1.1) is equiattractive. □

Later on throughout this section, we shall assume that the sequence $\{f_n(x)\}$ in the right-hand side of system (1.1) is almost periodic for every fixed $x \in B_H$, and functions $f_n(x)$ satisfy Lipschitz condition uniformly in n.

Lemma 4.3. *Consider the solution $x_n(n_0, x_{n_0})$ of system* (1.1)*. We suppose that $x_n(n_0, x_{n_0})$ belongs to B_r $(0 < r < H)$ for $n \geqslant n_0$. Let $\{\varepsilon_k\}$ be a monotonically approaching zero* *sequence of positive numbers, and* $\{m_k\}$ *a sequence of* ε_k*-almost periods of* $\{f_n(x)\}$ (*for every* ε_k *there corresponds an* ε_k*-almost period* m_k). *Then the limit relation*

$$\lim_{k\to\infty} \left\| x_{n_*}(n_0, x^{(k)}) - x_{n_*+m_k}(n_0, x_{n_0}) \right\| = 0, \tag{4.7}$$

holds where $x^{(k)} = x_{n_0+m_k}(n_0, x_{n_0})$, *and* n_* *is a fixed natural number which is more than* n_0 $(n_* > n_0)$.

Proof. Consider solutions

$$x_n(n_0, x^{(k)}) \tag{4.8}$$

and

$$x_n(n_0 + m_k, x^{(k)}) \tag{4.9}$$

of system (1.1). After $\Delta n = n_* - n_0$ steps the point $x^{(k)}$ passes to $x_{n_*}(n_0, x_{n_0})$ along solution (4.8), and $x^{(k)}$ passes to the point $x_{n_*+m_k}(n_0 + m_k, x^{(k)}) = x_{n_*+m_k}(n_0, x_{n_0})$ along solution (4.9). Solution (4.9) of system (1.1) with initial condition $(n_0 + m_k, x^{(k)})$ can be interpreted as the solution of the system

$$x_{n+1} = f_{n+m_k}(x_n) \tag{4.10}$$

with initial data $(n_0, x^{(k)})$. The sequence $\{f_n(x)\}$ is almost periodic, and every function $f_n(x)$ satisfies Lipschitz condition, hence right-hand sides of (1.1) and (4.10) differ arbitrary small from each other for k large enough. This implies limit relation (4.7). □

Theorem 4.2. *Suppose that there exists a sequence of functions* $\{V_n(x)\}$ *such that*

(a) *for every fixed* $x \in B_H$, *the sequence* $\{V_n(x)\}$ *is almost periodic*;
(b) *each member* $V_n(x)$ *satisfies condition* (4.5) *and Lipschitz condition uniformly in* n;
(c) $V_n(x_n) \geqslant V_{n+1}(x_{n+1})$ *along any solution of* (1.1);
(d) *the sequence* $\{V_n(x_n)\}$ *is finally nonzero along any nonzero solution of* (1.1).

Then the zero solution of system (1.1) *is equiasymptotically stable.*

Proof. First let us show that solution (1.2) of system (1.1) is stable. Pick arbitrary $\varepsilon \in (0, H)$ and $n_0 \in \mathbb{Z}_+$. Let $\delta = \delta(\varepsilon, n_0) > 0$ be such that $V_{n_0}(x) < a(\varepsilon)$ for $x \in B_\delta$. Then

$$a\big(\|x_n\|\big) \leqslant V_n(x_n) \leqslant V_{n_0}(x_{n_0}) < a(\varepsilon)$$

whence we have $\|x_n\| < \varepsilon$ for $n > n_0$.

Now let us show that solution (1.2) is equiattractive. Take arbitrary $x_{n_0} \in B_\delta$. The sequence $\{V_n(x_n(n_0, x_{n_0}))\}$ monotonically does not increase, therefore there is the limit

$$\lim_{n\to\infty} V_n\big(x_n(n_0, x_{n_0})\big) = \eta \geqslant 0,$$

and $V_n(x_n(n_0, x_{n_0})) \geqslant \eta$ for $n \geqslant n_0$. Let us show that $\eta = 0$. Suppose the opposite: $\eta > 0$. Consider a monotonically approaching zero sequence of positive numbers $\{\varepsilon_k\}$ where ε_1 is sufficiently small. By Lemmas 4.2 and 4.1, for every ε_i there exists a sequence of

ε_i-almost periods $m_{i,1}, m_{i,2}, \dots, m_{i,k}, \dots$ ($m_{i,k} < m_{i,k+1}$, $\lim_{k\to+\infty} m_{i,k} = +\infty$) for sequences $\{f_n(x)\}$ and $\{V_n(x)\}$ such that inequalities

$$\left|V_{n+m_{i,k}}(x) - V_n(x)\right| < \varepsilon_i, \qquad \left\|f_{n+m_{i,k}}(x) - f_n(x)\right\| < \varepsilon_i$$

hold for any $n \in \mathbb{Z}$, $x \in B_\varepsilon$. Without loss of generality one can suppose $m_{i,k} < m_{i+1,k}$ for all $i \in \mathbb{N}$, $k \in \mathbb{N}$. Designate $m_k = m_{k,k}$.

Consider the sequence $\{x^{(k)}\}$ where $x^{(k)} = x_{n_0+m_k}(n_0, x_{n_0})$ $(k = 1, 2, \dots)$. This sequence is bounded, therefore there exists its subsequence which converges to some point x^*. Without loss of generality we suppose that the sequence $\{x^{(k)}\}$ itself converges to x^*. The sequence $\{V_n(x)\}$ is almost periodic for every fixed $x \in B_H$, and each function $V_n(x)$ is continuous, hence

$$\begin{aligned} V_{n_0}(x_*) &= \lim_{n\to\infty} V_{n_0}(x_n) = \lim_{k\to\infty}\lim_{n\to\infty} V_{n_0+m_k}(x_n) \\ &= \lim_{n\to\infty} V_{n_0+m_n}(x_n) = \lim_{n\to\infty} V_{n_0+m_n}\big(x_{n_0+m_n}(n_0, x_{n_0})\big) = \eta. \end{aligned}$$

Consider the sequence $\{x_n(n_0, x^*)\}$. From conditions of the theorem, it follows that there exists $n_* > n_0$ $(n_* \in \mathbb{N})$ such that the inequality

$$V_{n_*}\big(x_{n_*}(n_0, x^*)\big) = \eta_1 < \eta$$

holds. Functions $f_n(x)$ satisfy Lipschitz condition, hence

$$\lim_{k\to\infty} \left\|x_{n_*}(n_0, x^{(k)}) - x_{n_*}(n_0, x^*)\right\| = 0$$

because

$$\lim_{k\to\infty} \|x^{(k)} - x^*\| = 0.$$

This implies

$$\lim_{k\to\infty} V_{n_*}\big(x_{n_*}(n_0, x^{(k)})\big) = \eta_1. \tag{4.11}$$

The almost periodicity of the sequence $\{f_n(x)\}$ and limit relation (4.7) imply

$$\left\|x_{n_*}(n_0, x^{(k)}) - x_{n_*+m_k}(n_0, x_{n_0})\right\| \leqslant \gamma_k, \tag{4.12}$$

where $\gamma_k \to 0$ as $k \to \infty$. Since the sequence $\{V_n\}$ is almost periodic, we have

$$\left|V_{n_*}(x) - V_{n_*+m_k}(x)\right| < \varepsilon_k \tag{4.13}$$

for every $x \in B_H$, and conditions (4.11), (4.12) imply

$$\left|V_{n_*}\big(x_{n_*+m_k}(n_0, x_{n_0})\big) - \eta_1\right| < \xi_k, \tag{4.14}$$

where $\xi_k \to 0$ as $k \to \infty$. From (4.13) it follows

$$\left|V_{n_*}\big(x_{n_*+m_k}(n_0, x_{n_0})\big) - V_{n_*+m_k}\big(x_{n_*+m_k}(n_0, x_{n_0})\big)\right| < \varepsilon_k. \tag{4.15}$$

Inequalities (4.14), (4.15) imply

$$\left|V_{n_*+m_k}\big(x_{n_*+m_k}(n_0, x_{n_0})\big) - \eta_1\right| < \xi_k + \varepsilon_k, \tag{4.16}$$

where $\xi_k + \varepsilon_k \to 0$ as $k \to \infty$.

On the other hand,

$$\lim_{k\to\infty} V_{n_*+m_k}\big(x_{n_*+m_k}(n_0, x_{n_0})\big) = \eta. \tag{4.17}$$

Inequality (4.16) and limit relation (4.17) are in contradiction to the inequality $\eta_1 < \eta$. This contradiction proves that $\eta = 0$, hence, according to Theorem 4.1, we derive that solution (1.2) of system (1.1) is equiasymptotically stable. □

Theorem 4.3. *Suppose that there exists a sequence of functions $\{V_n(x)\}$ such that for every $x \in B_H$, the sequence $\{V_n(x)\}$ is almost periodic, and each function $V_n(x)$ satisfies Lipschitz condition uniformly in n and next conditions*:

- $|V_n(x)| \leqslant b(\|x\|)$, $b \in \mathcal{K}$, $n \in \mathbb{Z}_+$, *for* $x \in B_H$;
- *for any* $n \in \mathbb{Z}_+$ *and* $\delta > 0$ *there is* $x \in B_\delta$ *such that* $V_n(x) > 0$;
- $V_{n+1}(x_{n+1}) \geqslant V_n(x_n)$ *along any solution* x_n.

Then solution (1.2) *of system* (1.1) *is unstable.*

Proof. Let $\varepsilon \in (0, H)$ be an arbitrary number. Take any $n_0 \in \mathbb{Z}_+$ and sufficiently small $\delta > 0$. Choose $x_0 \in B_\delta$ by such way that $V_{n_0}(x_{n_0}) > 0$. From conditions of the theorem, it follows that there exists $\eta > 0$ such that $|V_n(x)| < V_{n_0}(x_{n_0})$ for every $x \in B_\eta$. Consider the sequence $\{V_n\}$ where $V_n = V_n(x_n(n_0, x_{n_0}))$. This sequence does not decrease i.e., $V_n(x_n(n_0, x_{n_0})) \geqslant V_{n_0}(x_{n_0})$ for $n \geqslant n_0$. This means that $\|x_n(n_0, x_{n_0})\| \geqslant \eta$ for every $n \geqslant n_0$. Let us show that there is $N_0 \in \mathbb{N}$ $(N_0 > n_0)$ such that $\|x_{N_0}(n_0, x_{n_0})\| > \varepsilon$. Assume the opposite:

$$\eta \leqslant \big\|x_n(n_0, x_{n_0})\big\| \leqslant \varepsilon \tag{4.18}$$

for all $n > n_0$. Using the conditions of the theorem and inequality (4.18), we obtain the contradiction by the same way as in the proof of Theorem 4.2. We pass the literal repetition of these reasonings. The contradiction shows that the solution $x_n(n_0, x_{n_0})$ leaves B_ε. The proof is complete. □

Example 4.1. Consider the system

$$x_{n+1} = -y_n \sin(\sqrt{3}n), \qquad y_{n+1} = x_n \sin n \tag{4.19}$$

and the function $V_n(x_n, y_n) = x_n^2 + y_n^2$.

$$V_{n+1}(x_{n+1}, y_{n+1}) - V_n(x_n, y_n) = -(\cos^2 n)\, x_n^2 - (\cos^2 \sqrt{3}n)\, y_n^2. \tag{4.20}$$

According to Corduneanu [6], for any sufficiently small $\varepsilon > 0$ there exists a sequence $n_1, n_2, \ldots, n_k, \ldots \to \infty$ such that

$$0 < \cos^2 n_k < \varepsilon, \qquad 0 < \cos^2(\sqrt{3}n_k) < \varepsilon \quad (k = 1, 2, \ldots).$$

This means that there is not a function $c \in \mathcal{K}$ such that the left-hand side of (4.20) satisfies inequality (2.4), so Theorem 2.2 cannot be applied to this system. System (4.19) is not an autonomous one, therefore Theorem 2.3 cannot be applied to the study of the stability property of its zero solution. But this system is almost periodic, and right-hand side of (4.20) is negative for each nonzero solution of system (4.19). Hence, according to Theorem 4.2, the zero solution of system (4.19) is equiasymptotically stable.

Example 4.2. Consider the system

$$x_{n+1} = y_n - x_n^2 y_n (2 - \sin^2 n - \cos^2 \sqrt{2} n),$$
$$y_{n+1} = x_n + x_n y_n^2 (2 - \sin^2 n - \cos^2 \sqrt{2} n). \tag{4.21}$$

If we choose $V_n(x_n, y_n) = x_n^2 + y_n^2$, then

$$V_{n+1}(x_{n+1}, y_{n+1}) - V_n(x_n, y_n) = x_n^2 y_n^2 \left(x_n^2 + y_n^2\right)(2 - \sin^2 n - \cos^2 \sqrt{2} n)^2.$$

By Theorem 4.3, we can state that the zero solution of system (4.21) is unstable.

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