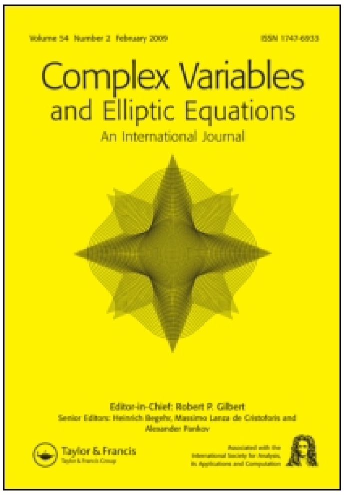

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The Väisälä inequality for mappings with finite length distortion†

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The well-known Väisälä inequality for quasiregular mappings is extended to open discrete mappings with finite length distortion.

Keywords: modulus inequalities; capacity; open discrete mappings; path families; geometric function theory

AMS Subject Classifications: Primary 30C65; Secondary 30C75

1. Introduction

Mappings with finite length distortion form one of the many interesting classes studied in the modern theory of space mappings, in particular, including quasiregular mappings or mappings with bounded distortion (BD), cf. Theorem 4.7 in [1–3]. Mappings with finite length distortion (FLD) is a natural generalization of mappings with bounded length distortion (BLD), see [4]. The interconnections between the mentioned classes can be presented by the next inclusions:

$$\mathrm{BLD} \subset \mathrm{BD} \subset \mathrm{FLD}. \tag{1}$$

Finite length distortion is closely related to the class of mappings with finite distortion (FD), see, e.g. [5,6] and Remark 4.3. In particular, every mapping f of finite length distortion satisfies the inequality

$$M(f\Gamma) \le \int_D K_I(x,f) \cdot \rho^n(x)\mathrm{d}m(x) \tag{2}$$

for every family Γ of paths in D and every admissible function ρ for Γ, see e.g. [7,8] and Theorem 6.10 in [1], where $K_I(x,f)$ stands for the well-known inner dilatation of f at x.

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For $x \in E \subset \mathbb{R}^n$ and a mapping $\varphi : E \to \mathbb{R}^n$, we set

$$L(x, \varphi) = \limsup_{y \to x,\, y \in E} \frac{|\varphi(y) - \varphi(x)|}{|y - x|}, \tag{3}$$

and

$$l(x, \varphi) = \liminf_{y \to x,\, y \in E} \frac{|\varphi(y) - \varphi(x)|}{|y - x|}. \tag{4}$$

Below we assume that D is a domain in $\mathbb{R}^n$, $n \geq 2$, and all mappings $f : D \to \mathbb{R}^n$ are continuous.

A mapping $f : D \to \mathbb{R}^n$ is said to be of finite metric distortion, abbreviated as $f \in \mathrm{FMD}$, if f has the Lusin (N)-property and

$$0 < l(x, f) \leq L(x, f) < \infty \qquad \text{a.e.} \tag{5}$$

Note that $f \in \mathrm{FMD}$ if and only if f is differentiable a.e. and has (N)- and (N^{-1})-property, see Corollary 3.14 in [1]. Recall that a mapping $f : X \to Y$ between spaces with measure (X, Σ, μ) and (X', Σ', μ') is said to have the (N)-property if $\mu'(f(S)) = 0$ whenever $\mu(S) = 0$. Similarly, f has the (N^{-1})-property if $\mu(S) = 0$ whenever $\mu'(f(S)) = 0$.

A path γ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family of paths Γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called admissible for Γ, abbreviated as $\rho \in \operatorname{adm} \Gamma$, if

$$\int_\gamma \rho(x)|\mathrm{d}x| \geq 1$$

for each $\gamma \in \Gamma$. The modulus $M(\Gamma)$ of Γ is defined as

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int_{\mathbb{R}^n} \rho^n(x) \mathrm{d}m(x)$$

interpreted as $+\infty$ if $\operatorname{adm} \Gamma = \varnothing$. Thus, every family Γ which contains a constant path is of infinite modulus. We say that a property P holds for almost every (a.e.) path γ in a family Γ if the subfamily of all paths in Γ for which P fails has modulus zero.

If $\gamma : \Delta \to \mathbb{R}^n$ is a locally rectifiable path, then there is the unique increasing length function l_γ of Δ onto a length interval $\Delta_\gamma \subset \mathbb{R}$ with a prescribed normalization $l_\gamma(t_0) = 0 \in \Delta_\gamma$, $t_0 \in \Delta$, such that $l_\gamma(t)$ is equal to the length of the subpath $\gamma|_{[t_0, t]}$ of γ if $t > t_0$, $t \in \Delta$, and $l_\gamma(t)$ is equal to $-l(\gamma|_{[t, t_0]})$ if $t < t_0$, $t \in \Delta$. Let $g : |\gamma| \to \mathbb{R}^n$ be a continuous mapping, and suppose that the path $\widetilde{\gamma} = g \circ \gamma$ is also locally rectifiable. Then there is a unique increasing function $L_{\gamma, g} : \Delta_\gamma \to \Delta_{\tilde{\gamma}}$ such that

$$L_{\gamma, g}\big(l_\gamma(t)\big) = l_{\tilde{\gamma}}(t) \qquad \forall t \in \Delta. \tag{6}$$

It is said that a mapping $f : D \to \mathbb{R}^n$ has the (L)-property if the following two conditions hold:

(L_1) for a.e. path γ in D, $\widetilde{\gamma} = f \circ \gamma$ is locally rectifiable and the function $L_{\gamma, f}$ has the (N)-property;

(L_2) for a.e. path $\widetilde{\gamma}$ in $f(D)$, each lifting γ of $\widetilde{\gamma}$ is locally rectifiable and the function $L_{\gamma,f}$ has the (N^{-1})-property.

A path γ in D is called here a lifting of a path $\widetilde{\gamma}$ in $\mathbb{R}^n$ under $f\colon D\to\mathbb{R}^n$ if $\widetilde{\gamma}=f\circ\gamma$. Note that the condition (L_2) is applied only to paths $\widetilde{\gamma}$ which have such a lifting.

A mapping $f\colon D\to\mathbb{R}^n$ is said to be of finite length distortion, abbreviated as $f\in$ FLD, if f is FMD and has the (L)-property.

Recall that $f\in$ ACP if and only if $L_{\gamma,f}$ is absolutely continuous on closed subintervals of Δ_γ for a.e. path γ in D. We say that f is absolute continuous on paths in the inverse direction, abbreviated as ACP^{-1}, if $L^{-1}_{\gamma,f}$ is absolutely continuous on closed subintervals of $\Delta_{\tilde{\gamma}}$ for a.e. path $\tilde{\gamma}$ in $f(D)$ and for each lifting γ of $\tilde{\gamma}$. By Proposition 4.3 in [1], (L)-property for a discrete mapping f is equivalent to the condition

$$f\in \mathrm{ACP}\cap \mathrm{ACP}^{-1}.$$

Let a mapping $f\colon D\to\mathbb{R}^n$ be differentiable a.e. in D. Recall that $f'(x)$ denotes the Jacobian matrix of f, $J(x,f)$ is its determinant and $l(f'(x))=\min\{|f'(x)h| : h\in\mathbb{R}^n, |h|=1\}$. The inner dilatation of f at the point x is defined as

$$K_I(x,f)=\begin{cases}\dfrac{|J(x,f)|}{l(f'(x))^n}, & \text{if } J(x,f)\neq 0\\ 1, & \text{if } f'(x)=0\end{cases}\tag{7}$$

and $K_I(x,f)=\infty$ at the rest points.

We remark that mappings of finite metric distortion are differentiable a.e. with $J(x,f)\neq 0$ a.e., see Proposition 3.7 in [1]. In particular, every mapping of finite length distortion is differentiable a.e. with $J(x,f)\neq 0$ a.e. Thus, $K_I(x,f)$ is well-defined for every FLD mapping.

Set $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$. The following definition was proposed in [9]. A homeomorphism $f\colon D\to\overline{\mathbb{R}^n}$ is said to be a Q-homeomorphism if

$$M(f\Gamma)\le\int_D Q(x)\cdot\rho^n(x)\,\mathrm{d}m(x)\tag{8}$$

for every family Γ of paths in D and every admissible function ρ for Γ. The notion of Q-homeomorphism is closely related to the conception of moduli with weights, cf. e.g. [10–13], see also the book [14], where some closely related notions and facts can be found. Every homeomorphism of finite length distortion is Q-homeomorphism with $Q=K_I(x,f)$, see Theorem 6.10 in [1].

Lemma 1.1 *Let $f\colon D\to\mathbb{R}^n$ be a mapping with finite length distortion. Then*

$$M(f\Gamma)\le\int_D K_I(x,f)\cdot\rho^n(x)\mathrm{d}m(x)\tag{9}$$

for every path family Γ in D and $\rho\in\operatorname{adm}\Gamma$.

One of the main goals of this article is to generalize the inequality (9) in the sense of the well-known Väisälä inequality for quasiregular mappings, see e.g. Section 9 Chapter II in [15].

In connection with the topic of our investigation, it is also necessary to refer the results of the papers [16,18], where the method of moduli was applied, the papers [17,18] connected with some investigations of the mappings with finite distortion, and papers [19,20] are connected with the study of some classes of homeomorphisms with finite distortion.

Recall also that a mapping $f\colon D \to \mathbb{R}^n$ is said to be of finite distortion if $f \in W^{1,n}_{\mathrm{loc}}$ and

$$\|f'(x)\|^n \le K(x) \cdot J(x,f) \qquad \text{a.e.,} \tag{10}$$

for some finite-valued function $K(x)\colon D \to [1,\infty)$, cf. [5,6]. Here $\|f'(x)\| = \max\{|f'(x)h| : h \in \mathbb{R}^n,\ |h| = 1\}$. Sometimes they request the conditions $f \in W^{1,1}_{\mathrm{loc}}$ and $J(\cdot,f) \in L^1_{\mathrm{loc}}$ instead of $f \in W^{1,n}_{\mathrm{loc}}$. In the work by Koskela and Onninen [21], some modulus and capacity estimates for classes of mappings of finite distortion were proved. However, these results are proved under very strong conditions on integrability of inner dilatation, as well as conditions on the norm of derivative. The main results there are formulated under such conditions that imply $f \in W^{1,n}_{\mathrm{loc}}$ and, moreover, they assume that the branch set of f has measure zero, cf. Remark 4.10 in [1]. The conditions in the present article do not include any assumptions on the inner dilatation, moreover, we do not suppose that a mapping f belongs to the Sobolev class.

2. Preliminaries

Let D be a domain in $\mathbb{R}^n$, $n \ge 2$. A mapping $f\colon D \to \mathbb{R}^n$ is said to be discrete if the preimage $f^{-1}(y)$ of every point $y \in \mathbb{R}^n$ consists of isolated points, and an open if the image of every open set $U \subseteq D$ is open in $\mathbb{R}^n$. The notation $G \Subset D$ means that $\overline{G}$ is a compact subset of D. We suppose that $f\colon D \to \mathbb{R}^n$ is continuous and sense-preserving, i.e. a topological index $\mu(y, f, G) > 0$ for any $G \Subset D$ and $y \in f(G) \setminus f(\partial G)$. A neighbourhood of a point x or a set A is an open set containing x or A. Suppose that $x \in D$ has a connected neighbourhood G such that $\overline{G} \cap f^{-1}(f(x)) = \{x\}$. Then $\mu(f(x), f, G)$ is well-defined and independent of the choice of G for discrete open f and is denoted by $i(x,f)$. In what follows, $B(x_0,\ r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $m(x)$ denotes a Lebesgue measure in $\mathbb{R}^n$. For $f\colon D \to \mathbb{R}^n$, and $E \subset D$, we use the multiplicity functions

$$N(y,f,E) = \operatorname{card}\{x \in E : f(x) = y\}, \quad N(f,E) = \sup_{y \in \mathbb{R}^n} N(y,f,E).$$

A domain $G \Subset D$ is said to be a normal domain of f, if $\partial f G = f(\partial G)$. If G is a normal domain, then $\mu(y,f,G)$ is a constant for $y \in f(D)$. This constant will be denoted by $\mu(f,G)$. Let $f\colon D \to \mathbb{R}^n$ be a discrete open mapping, then $\mu(f,G) = N(f,G)$ for every normal domain $G \Subset D$, see, e.g. Proposition 4.10 Chapter I in [15].

A map $\varphi\colon X\to Y$ between metric spaces X and Y is said to be Lipschitzian provided

$$\mathrm{dist}(\varphi(x_1),\varphi(x_2))\leq M\cdot\mathrm{dist}(x_1,x_2) \tag{11}$$

for some $M<\infty$ and for all x_1 and $x_2\in X$. The map φ is called bi-Lipschitz if, in addition,

$$M^{*}\mathrm{dist}(x_1,x_2)\leq \mathrm{dist}(\varphi(x_1),\varphi(x_2)) \tag{12}$$

for some $M^{*}>0$ and for all x_1 and $x_2\in X$. Later on X and Y are subsets of $\mathbb{R}^n$ with the Euclidean distance. The following lemma can be found in [1, Lemma 3.20].

LEMMA 2.1 *Let $f\colon D\to\mathbb{R}^n$ be an FMD mapping. Then there is a countable collection of compact sets $C_k^{*}\subset D$ such that* mes $B=0$ *where $B=D\setminus\cup_{k=1}^{\infty}C_k^{*}$ and $f|_{C_k^{*}}$ is one-to-one and bi-Lipschitz for every $k=1,2,\ldots$ and, moreover, f is differentiable at all points C_k^{*} with $J(x,f)\neq 0$.*

Following [2], a condenser is a pair $E=(A,C)$ where $A\subset\mathbb{R}^n$ is open and C is non-empty compact set contained in A. A condenser $E=(A,C)$ is said to be in a domain G if $A\subset G$. For a given condenser $E=(A,C)$, we set

$$\mathrm{cap}\,E=\mathrm{cap}\,(A,\ C)=\inf_{u\in W_0(E)}\int_A|\nabla u|^n\,\mathrm{d}m(x), \tag{13}$$

where $W_0(E)=W_0(A,C)$ is the family of non-negative functions $u\colon A\to R^1$ such that (1) $u\in C_0(A)$, (2) $u(x)\geq 1$ for $x\in C$ and (3) u is ACL. In the above formula $|\nabla u|=(\sum_{i=1}^{n}(\partial_i u)^2)^{1/2}$, and cap E is called the capacity of the condenser E.

Let $f\colon D\to\mathbb{R}^n$ be a discrete open mapping. Let $\beta\colon[a,b)\to\mathbb{R}^n$ be a path and $x\in f^{-1}(\beta(a))$. A path $\alpha\colon[a,c)\to D$ is called a maximal f-lifting of β starting at x, if (1) $\alpha(a)=x$; (2) $f\circ\alpha=\beta|_{[a,c)}$; (3) $c<c'\leq b$, then there does not exist a path $\alpha'\colon[a,c')\to D$ such that $\alpha=\alpha'|_{[a,c)}$ and $f\circ\alpha'=\beta|_{[a,c')}$. Let f be a discrete open mapping, then every path β with $x\in f^{-1}(\beta(a))$ has a maximal f-lifting starting at a point x, see Corollary 3.3 Chapter II in [15]. We need the following statement, see Proposition 10.2, Chapter II in [15].

LEMMA 2.2 *Let $E=(A,C)$ be a condenser in $\mathbb{R}^n$ and Γ_E be the family of all paths of the form $\gamma\colon[a,b)\to A$ with $\gamma(a)\in C$ and $|\gamma|\cap(A\setminus F)\neq\varnothing$ for every compact $F\subset A$. Then* cap $E=M(\Gamma_E)$.

Let $x_1,\ldots,x_k$ be k different points of $f^{-1}(\beta(a))$ and

$$m=\sum_{i=1}^{k}i(x_i,f).$$

We say that the sequence $\alpha_1,\ldots,\alpha_m$ is a maximal sequence of f-lifting of β starting at points $x_1,\ldots,x_k$, if

(a) each α_j is a maximal f-lifting of β,
(b) card $\{j : a_j(a)=x_i\}=i(x_i,f)$, $1\leq i\leq k$,
(c) card $\{j : a_j(t)=x\}\leq i(x,f)$ for all $x\in D$ and for all t.

Let f be a discrete open mapping and $x_1, \ldots, x_k$ distinct points in $f^{-1}(\beta(a))$. Then β has a maximal sequence of f-liftings starting at points $x_1, \ldots, x_k$, see Theorem 3.2 Chapter II in [15].

We further need the following statement, see Corollary 3.4, Chapter II in [15].

Lemma 2.3 *Let* $f\colon G \to \mathbb{R}^n$ *be a discrete open mapping,* G *a normal domain,* $m = N(f, G)$, $\beta\colon [a, b) \to f(G)$ *a path. Then there exist paths* $\alpha_j\colon [a, b) \to G$, $1 \le j \le m$, *such that*:

(1) $f \circ \alpha_j = \beta$,
(2) card $\{j : \alpha_j(t) = x\} = i(x, f)$ *for* $x \in G \cap f^{-1}\,\beta(t)$,
(3) $|\alpha_1| \cup \cdots \cup |\alpha_m| = G \cap f^{-1}|\beta|$.

Given a set E in $\mathbb{R}^n$ and a path $\gamma\colon \Delta \to \mathbb{R}^n$, we identify $\gamma \cap E$ with $\gamma(\Delta) \cap E$. If γ is locally rectifiable, then we set

$$l(\gamma \cap E) = |E_\gamma|,$$

where $E_\gamma = l_\gamma\,(\gamma^{-1}\,(E))$. Here $|A|$ means the length (Lebesgue) measure of a set $A \subset \mathbb{R}$ and $l_\gamma\colon \Delta \to \Delta_\gamma$ as in Section 1. Note that $E_\gamma = \gamma_0^{-1}(E)$, where $\gamma_0\colon \Delta_\gamma \to \mathbb{R}^n$ is the natural parameterization of γ and $l(\gamma \cap E) = \int_{\Delta_\gamma} \chi_{E_\gamma}(s)\mathrm{d}s$.

3. The Väisälä type inequality

Theorem 3.1 *Let* $f\colon D \to \mathbb{R}^n$ *be a discrete open mapping with finite length distortion,* Γ *a path family in* D, Γ' *a path family in* $\mathbb{R}^n$ *and* m *a positive integer such that the following is true. Suppose that for every path* β *in* Γ' *there are paths* $\alpha_1, \ldots, \alpha_m$ *in* Γ *such that* $f \circ \alpha_j \subset \beta$ *for all* $j = 1, \ldots, m$ *and such that for every* $x \in D$ *and all* t *the equality* $\alpha_j(t) = x$ *holds for at most* $i(x, f)$ *indices* j. *Then*

$$M(\Gamma') \le \frac{1}{m}\int_D K_I(x, f) \cdot \rho^n(x)\mathrm{d}m(x) \tag{14}$$

for every $\rho \in \operatorname{adm}\Gamma$.

Proof Let B and C_k^* be as in Lemma 2.1 and B_f be a branch set for f in D. Note that mes $B_f = 0$, see Proposition 3.16 in [1]. Setting by induction $B_0 = B \cup B_f$, $B_1 = C_1^* \setminus B_f$, $B_2 = C_2^* \setminus (B_1 \cup B_f) \ldots$,

$$B_k = C_k^* \setminus \left(\bigcup_{l=1}^{k-1} B_l \cup B_f\right),$$

we obtain the countable covering of D consisting of mutually disjoint Borel sets B_k, $k = 1, 2, \ldots$, with mes $B_0 = 0$. By the construction and (N)-property, mes $f(B_0) = 0$. Thus, by Lemma 2.13 in [1], $l(\overline{\gamma} \cap f(B_0)) = 0$ for a.e. $\overline{\gamma}$ in $f(D)$ and hence by (L_2)-property

$$l(\gamma \cap B_0) = 0 \tag{15}$$

also for a.e. $\overline{\gamma}$ in $f(D)$ and all γ such that $f\circ\gamma=\overline{\gamma}$. Furthermore, we prove (15) for a.e. $\widetilde{\gamma}\in\Gamma'$ and all γ such that $f\circ\gamma\subset\widetilde{\gamma}$.

Let us assume the contrary. Let Γ_1 be a family of all paths $\gamma'\in\Gamma'$ with

$$l(\gamma\cap B_0)>0 \tag{16}$$

such that $f\circ\gamma\subset\gamma'$ for some γ. By the assumption, $M(\Gamma_1)>0$. Let Γ_2 be a family of all subpaths γ'' of Γ_1 which have a whole lifting γ with the property (16). Note that $\Gamma_2<\Gamma_1$ and hence $M(\Gamma_2)\geq M(\Gamma_1)>0$. This contradiction disproves the above assumption.

Let $\rho\in\operatorname{adm}\Gamma$ and

$$\widetilde{\rho}(y)=\frac{1}{m}\cdot\chi_{f(D\setminus B_0)}(y)\sup_{C}\sum_{x\in C}\rho^*(x), \tag{17}$$

where

$$\rho^*(x)=\begin{cases}\rho(x)/l(f'(x)), & x\in D\setminus B_0,\\ 0, & x\in B_0\end{cases} \tag{18}$$

and C runs over all subsets of $f^{-1}(y)$ in $D\setminus B_0$ such that card $C\leq m$. Note that

$$\widetilde{\rho}(y)=\frac{1}{m}\cdot\sup\sum_{i=1}^{s}\rho_{k_i}(y),$$

where sup in (18) is taken over all $\{k_{i_1},\ldots,k_{i_s}\}$ such that $k_i\in\mathbb{N}$, $k_i\neq k_j$ if $i\neq j$ and all $s\leq m$, and

$$\rho_k(y)=\begin{cases}\rho^*\left(f_k^{-1}(y)\right), & y\in f(B_k),\\ 0, & y\notin f(B_k)\end{cases},$$

where $f_k=f|_{B_k}$, $k=1,2,\ldots$ is injective and $f(B_k)$ is Borel. Thus, the function $\widetilde{\rho}(y)$ is Borel, see, e.g. Section 2.3.2 in [22].

Suppose that β is a path in Γ'. There exist paths $\alpha_1,\ldots,\alpha_m$ in Γ such that $f\circ\alpha_j\subset\beta$ and for all $x\in D$ and t the equality $\alpha_j(t)=x$ holds for at most $i(x,f)$ indices j. The above conditions mean that curves α_j are disjoint on B_k because $i(x,f)=1$ for every $x\in B_k$, $k=1,2,\ldots$. By Theorem 3.2.5 for $m=1$ in [22], arguing piecewise on B_k, we have

$$\begin{aligned}\int_{\beta}\widetilde{\rho}\,\mathrm{d}s&\geq\sum_{j=1}^{m}\int_{f\circ\alpha_j}\widetilde{\rho}\,\mathrm{d}s=\sum_{j=1}^{m}\sum_{k=1}^{\infty}\int_{(f\circ\alpha_j)\cap f(B_k)}\widetilde{\rho}\,\mathrm{d}s\\&\geq\frac{1}{m}\cdot\sum_{j=1}^{m}\sum_{k=1}^{\infty}\int_{\alpha_j\cap B_k}\rho(x)\,\mathrm{d}s=\frac{1}{m}\cdot\sum_{j=1}^{m}\int_{\alpha_j}\rho(x)\,\mathrm{d}s\geq\frac{1}{m}\cdot m=1\end{aligned} \tag{19}$$

for a.e. curves $\beta\in\Gamma'$. Consequently, $\widetilde{\rho}\in\operatorname{adm}\Gamma'\setminus\Gamma_0$ where $M(\Gamma_0)=0$ and hence

$$M(\Gamma')\leq\int_{f(D)}\widetilde{\rho}^{\,n}(y)\mathrm{d}m(y). \tag{20}$$

By Theorem 3.2.5 for $m=n$ in [22], we obtain that

$$\int_{B_k} K_I(x,f)\cdot \rho^n(x)\mathrm{d}m(x)=\int_{f(D)}\rho_k^n(y)\mathrm{d}m(y). \tag{21}$$

By Hölder inequality for series,

$$\left(\frac{1}{m}\cdot\sum_{i=1}^{s}\rho_{k_i}(y)\right)^n\le\frac{1}{m}\cdot\sum_{i=1}^{s}\rho_{k_i}^n(y) \tag{22}$$

for each $1\le s\le m$ and every $\{k_1,\dots,k_s\}$, $k_i\in\mathbb{N}$, $k_i\neq k_j$, if $i\neq j$.

Finally, by Lebesgue positive convergence theorem, we conclude from (20) and (22) that

$$\frac{1}{m}\cdot\int_D K_I(x,f)\cdot\rho^n(x)\mathrm{d}m(x)=\frac{1}{m}\cdot\int_{f(D)}\sum_{k=1}^{\infty}\rho_k^n(y)\mathrm{d}m(y) \tag{23}$$

$$\ge\frac{1}{m}\cdot\int_{f(D)}\sup_{\substack{\{k_1,\dots,k_s\},\,k_i\in N,\\ k_i\neq k_j\,\mathrm{if}\,i\neq j}}\sum_{i=1}^{s}\rho_{k_i}^n(y)\mathrm{d}m(y)\ge\int_{f(D)}\widetilde{\rho}^n(y)\mathrm{d}m(y)\ge M(\Gamma').$$

The proof is complete.

4. Applications

We adopt the following conventions. Given a family of paths Γ in $\mathbb{R}^n$, denote

$$M_{K_I(\cdot,f)}(\Gamma)=\inf_{\rho\in\operatorname{adm}\Gamma}\int_{\mathbb{R}^n}\rho^n(x)\,K_I(x,f)\,\mathrm{d}m(x).$$

Let $E=(A,C)$ be a condenser and ω is a non-negative measurable function. We define the ω-weighted capacity of E by setting

$$\operatorname{cap}_\omega E=\operatorname{cap}_\omega(A,C)=\inf\int_A|\nabla u(x)|^n\,\omega(x)\,\mathrm{d}m(x), \tag{24}$$

where inf in (24) is taken over all functions $u\in C_0^\infty(A)$ and $u\ge 1$ on C. Note that, if $\omega\equiv 1$, then $\operatorname{cap}_\omega E$ coincides with cap E in the sense of the definition given in (13), see Lemma 5.5 in [2].

Given a mapping $f\colon D\to\mathbb{R}^n$ and a condenser $E=(A,C)$, we call

$$M(f,C)=\inf_{y\in f(C)}\sum_{x\in f^{-1}(y)\cap C} i(x,f)$$

the minimal multiplicity of f on C.

The following statements generalize the well-known modulus and capacity inequalities for quasiregular mappings, see, e.g. Sections 9 and 10 in [15].

THEOREM 4.1 *Let* $f\colon G\to\mathbb{R}^n$ *be a discrete open mapping of finite length distortion, where* G *is a normal domain for* f, Γ' *be a path family in* $G'=f(G)$, Γ *be a path family* α *in* G *such that* $f\circ\alpha\subset\Gamma'$ *and* $m=N(f,G)$. *Then*

$$M(\Gamma')\leq\frac{1}{N(f,G)}\quad\int_G K_I(x,f)\cdot\rho^n(x)\,\mathrm{d}m(x)$$

for every $\rho\in\operatorname{adm}\Gamma$. *In particular,*

$$M(\Gamma')\leq\frac{1}{N(f,G)}\quad M_{K_I(\cdot,f)}(\Gamma)\,.$$

Proof The proof follows directly from Theorem 3.1 and Lemma 2.3.

THEOREM 4.2 *Let* $f\colon D\to\mathbb{R}^n$ *be a discrete open mapping of finite length distortion,* $E=(A,C)$ *be a condenser in* D. *Then*

$$\operatorname{cap} fE\leq\frac{1}{M(f,C)}\ \operatorname{cap}_{K_I(\cdot,f)}E. \tag{25}$$

Proof Let $E=(A,C)$ be a condenser in D, then $f\,E=(fA,fC)$ is a condenser in $f(D)$. Let Γ_E and Γ_{fE} such as in Lemma 2.2. Set $m=M(f,C)$. Let $\beta\colon[a,b)\to f(A)$ be a path in Γ_{fE}. Then $C\cap f^{-1}(\beta(a))$ contains points $x_1,\dots,x_k$ such that

$$m'=\sum_{l=1}^{k}i(x_l,f)\geq m.$$

By Theorem 3.2, Chapter II in [15], there exists a maximal sequence of $f|A$-liftings $\alpha_j\colon[a,c_j)\to D$ of β, $1\leq j\leq m'$, starting at the points $x_1,\dots,x_k$. Then each α_j belongs to Γ_E. It follows that $\Gamma=\Gamma_E$ and $\Gamma'=\Gamma_{fE}$ satisfy Theorem 3.1. Hence, by Lemma 2.2 and Theorem 3.1,

$$\operatorname{cap} fE\leq\frac{1}{M(f,C)}M_{K_I(\cdot,f)}(\Gamma_E). \tag{26}$$

Finally, (25) follows, because

$$M_{K_I(\cdot,f)}(\Gamma_E)\leq\operatorname{cap}_{K_I(\cdot,f)}E, \tag{27}$$

as is easily seen by considering $\rho(x)=|\nabla u(x)|$ for a given test function u for $\operatorname{cap}_{K_I(\cdot,f)}E$.

Remark 4.3 Let $f\colon D\to\mathbb{R}^n$ be an open discrete mapping of finite distortion with $K(x)\in L^{n-1}_{\mathrm{loc}}$. Then f is of finite length distortion provided that $\operatorname{mes}B_f=0$, see Remark 4.10 in [1]. The above conclusion can also be obtained from the results of [21] because in this case f is absolutely precontinuous on paths. By Remark 4.3, all the results of Sections 1–4 hold for open discrete mappings with finite distortion such that $\operatorname{mes}B_f=0$. In particular, it means that Theorem 4.1 from [21] and consequences from it are proved under more weak conditions in the present article.

Remark 4.4 It is known that every mapping $f: D \to \mathbb{R}^n$ of the class $W^{1,n}_{\mathrm{loc}}$ such that $\|f'(x)\|^n \le K(x) \cdot J(x,f)$ is open and discrete provided that $K(x) \in L^p_{\mathrm{loc}}$ for $p > n-1$, see [23,24] and, consequently, all results of the present article can be applied to this Sobolev class, see also Remark 4.10 in [1].

References

[1] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, *Mappings of finite length distortion*, J. d'Anal. Math. 93 (2004), pp. 215–236.

[2] O. Martio, S. Rickman, and J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I. Math. 448 (1969), pp. 1–40.

[3] Yu.G. Reshetnyak, *Space Mappings with Bounded Distortion*, *Translation of Mathematical Monographs*, Vol. 73, AMS, Providence, RI, 1989.

[4] O. Martio and J. Väisälä, *Elliptic equations and maps of bounded metric distortion*, Math. Ann. 282 (1988), pp. 423–443.

[5] T. Iwaniec and G. Martin, *Geometric Function Theory and Non-linear Analysis*, Clarendon Press, Oxford, 2001.

[6] T. Iwaniec and V. Šverák, *On mappings with integrable dilatation*, Proc. Amer. Math. Soc. 118 (1993), pp. 181–188.

[7] C.J. Bishop, V.Ya. Gutlyanskii, O. Martio, and M. Vuorinen, *On conformal dilatation in space*, Int. J. Math. Math. Sci. 22 (2003), pp. 1397–1420.

[8] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, *Moduli in Modern Mapping Theory*, Springer Science, Business Media, LLC, New York, 2009.

[9] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, *On Q-homeomorphisms*, Ann. Acad. Sci. Fenn. Ser. A1. Math. 30(1) (2005), pp. 49–69.

[10] C. Andreian Cazacu, *Influence of the orientation of the characteristic ellipses of the properties of the quasiconformal mappings*, *Proceedings of the Romanian–Finnish Seminar on Teichmüller Spaces and Riemann Surfaces*, Bucharest, 1971, pp. 65–85.

[11] C. Andreian Cazacu, *Moduli inequalities for quasiregular mappings*, Ann. Acad. Sci. Fenn. Math. 2 (1976), pp. 17–28.

[12] M. Cristea, *Local homeomorphisms having local ACL^n inverses*, Compl. Var. Ellipt. Equat. 53(1) (2008), pp. 77–99.

[13] M. Cristea, *Open discrete mappings having local ACL^n inverses* (2008), Preprint Ser. Inst. Math. of Romanian Acad., Inst. de Math. 'Simion Stoilow' Al Acad. Rom. 2008.6.

[14] V.M. Miklyukov, *Conformal mapping of the non-regular surface and it's applications*, VolGU, Volgograd, 2005.

[15] S. Rickman, *Quasiregular Mappings*, Springer, Berlin, 1993.

[16] M.A. Brakalova and J.A. Jenkins, *On solutions of the Beltrami equation*, J. d'Anal. Math. 76 (1998), pp. 67–92.

[17] T. Iwaniec, P. Koskela, and G. Martin, *Mappings of BMO-distortion and Beltrami-type operators. Dedicated to the memory of Tom Wolff*, J. d'Anal. Math. 88 (2002), pp. 337–381.

[18] K. Astala, T. Iwaniec, P. Koskela, and G. Martin, *Mappings of BMO-bounded distortion*, Math. Ann. 317(4) (2000), pp. 703–726.

[19] K. Astala, T. Iwaniec, G. Martin, and J. Onninen, *Extremal mappings of finite distortion*, Proc. London Math. Soc. 91(3) (2005), pp. 655–702.

[20] S. Hencl, P. Koskela, and J. Onninen, *A note on extremal mappings of finite distortion*, Math. Res. Lett. 12(2–3) (2005), pp. 231–237.

[21] P. Koskela and J. Onninen, *Mappings of finite distortion: Capacity and modulus inequalities*, J. Reine Angew. Math. 599 (2006), pp. 1–26.

[22] H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1969.

[23] J.J. Manfredi and E. Villamor, *Mappings with integrable dilatation in higher dimensions*, Bull. Amer. Math. Soc. 32(2) (1995), pp. 235–240.

[24] J.J. Manfredi and E. Villamor, *An extension of Reshetnyak's theorem*, Indiana Univ. Math. J. 47(3) (1998), pp. 1131–1145.