

RESEARCH ARTICLE

Removable Isolated Singularities for Solutions of Doubly  
 Nonlinear Anisotropic Parabolic Equations

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It is studied a local behaviour of solutions for a wide class of degenerate quasi-linear second order parabolic equations with the model representative

$$\frac{\partial u}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( |u|^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = 0, \quad 2 \leq p_1 \leq p_2 \leq \dots \leq p_n.$$

We establish the point-wise condition for removable isolated singularity of solutions of such equations.

**Keywords:** Doubly nonlinear anisotropic parabolic equation; removable isolated singularity; point-wise estimates

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1. Introduction

In this paper we study solutions of the following initial problem

$$\frac{\partial u}{\partial t} - \sum_{i=1}^n \frac{d}{dx_i} a_i(x, t, u, \nabla u) = a_0(x, t, u, \nabla u), \quad (x, t) \in \Omega_T \setminus \{(x_0, 0)\}, \quad (1)$$

$$u(x, 0) = 0 \quad \text{for all } x \in \Omega \setminus \{x_0\}, \quad (2)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ ,  $x_0 \in \Omega$ ,  $\Omega_T := \Omega \times (0, T)$ ,  $0 < T < \infty$ .

Throughout the paper we suppose that the functions  $a_i(x, t, u, \xi)$ ,  $i = 0, 1, \dots, n$ , are defined for  $(x, t, u, \xi) \in \Omega_T \times \mathbb{R}^1 \times \mathbb{R}^n$  and satisfy the Caratheodory conditions. We assume also that the following structure conditions are satisfied:

$$\sum_{i=1}^n a_i(x, t, u, \xi) \xi_i \geq \nu_1 \sum_{i=1}^n |u|^{(m_i-1)(p_i-1)} |\xi_i|^{p_i} - g_1(x, t) |u|^{p(m-d)+1} - f_1(x, t),$$

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$$|a_i(x, t, u, \xi)| \leq \nu_2 |u|^{\frac{(m_i-1)(p_i-1)}{p_i}} \left( \sum_{j=1}^n |u|^{(m_j-1)(p_j-1)} |\xi_j|^{p_j} \right)^{1-\frac{1}{p_i}} + g_2(x, t) |u|^{(m_i+p(m-d))\frac{p_i-1}{p_i}} + f_2(x, t), \quad i = 1, \dots, n, \quad (3)$$

$$|a_0(x, t, u, \xi)| \leq \sum_{i=1}^n h_i(x, t) |u|^{m-d-\frac{p-1}{p}} \left( |u|^{(m_i-1)(p_i-1)} |\xi_i|^{p_i} \right)^{1-\frac{1}{p}} + g_3(x, t) |u|^{p(m-d)} + f_3(x, t)$$

with some positive constants  $\nu_1, \nu_2$  and nonnegative functions  $g_i(x, t), f_i(x, t), h_j(x, t), i = 1, 2, 3, j = 1, \dots, n$ , which satisfy the following property:

$$H(x) := 1 + f_1 + f_3 + g_1 + g_3 + \sum_{i=1}^n \left( f_2^{\frac{p_i}{p_i-1}} + g_2^{\frac{p_i}{p_i-1}} + h_i^p \right) \in L_{q_0}(\Omega_T), \quad (4)$$

where  $q_0 = \frac{n+\kappa}{p(1-\delta)}, 0 < \delta < \frac{1}{2}$ .

We assume further on that the following conditions are satisfied:

$$2 \leq p_1 \leq \dots \leq p_n, \quad \min_{1 \leq i \leq n} m_i \geq 1, \quad \max_{1 \leq i \leq n} m_i(p_i - 1) < 1 + \frac{\kappa}{n}, \quad \sum_{i=1}^n \frac{1}{p_i} > 1, \quad (5)$$

where

$$\kappa = n(p(m-d) - 1) + p, \quad (6)$$

$$\frac{1}{p} = \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}, \quad m = \frac{1}{n} \sum_{i=1}^n m_i, \quad d = \frac{1}{n} \sum_{i=1}^n \frac{m_i}{p_i}. \quad (7)$$

The qualitative theory of quasi-linear equations of type (1) in the case  $m_1 = \dots = m_n = 1, p_1 = \dots = p_n = 2$ , is being extensively studied because of various applications in mathematical physics and reach mathematical structures.

We are interested here in precise point-wise conditions on solutions to guarantee the singularity at  $\{(x_0, 0)\}$  is removable.

It is well known that the necessary and sufficient conditions for the harmonic function  $u$  to have a removable singularity at  $\{x_0\}$  is  $u(x) = o(|x - x_0|^{2-n})$  as  $x_0 \rightarrow 0$ . Until recently such a result for quasi-linear elliptic equations was known only since the celebrated paper by Serrin [10]. Serrin's condition on removability of singularity of isolated singularities reduces to

$$u(x) = O(|x - x_0|^{-\frac{n-p}{p-1} + \mu}), \quad 1 < p < n \quad (8)$$

with some  $\mu > 0$ .

The precise condition for removability of isolated singularities for quasi-linear non-anisotropic elliptic equation has the form:

$$u(x) = o(|x - x_0|^{-\frac{n-p}{p-1}}), \quad 1 < p < n, \quad (9)$$

which has been proved in [8]. This result was extended by authors [7] for solutions of quasi-linear anisotropic elliptic equations.

Let us denote by  $\mathcal{D}(r)$ ,  $r > 0$  the following set

$$\mathcal{D}(r) = \left\{ (x, t) \in \Omega_T : \sum_{i=1}^n \left( \frac{|x_i - x_i^{(0)}|}{r^{\kappa_i}} \right)^{p_i} + \frac{t}{r^\kappa} \leq 1 \right\}, \tag{10}$$

where

$$\kappa_i = \frac{p + n(p(m - d) - m_i(p_i - 1))}{p_i}, \quad i = 1, \dots, n. \tag{11}$$

It is easy to see that  $\sum_{i=1}^n \kappa_i = n$ .

We formulate the removability result in the form of behaviour of the function

$$M(r) = \text{ess sup} \{ |u(x, t)| : (x, t) \in \mathcal{D}(R_0) \setminus \mathcal{D}(r) \}, \tag{12}$$

where  $R_0$  is some sufficiently small fixed positive number such that  $\mathcal{D}(R_0) \subset \Omega_T$ . It follows from [5] that  $M(r)$  is a finite number for any  $r > 0$ . In the next sections we prove that the singularity of solution  $u(x, t)$  at the point  $(x_0, 0)$  is removable if

$$\lim_{r \rightarrow 0} M(r) r^n = 0. \tag{13}$$

This result is well-known for the heat equation. For nonnegative solutions of linear and quasi-linear parabolic equations of the type (1) with  $m_1 = \dots = m_n = 1$ ,  $p_1 = \dots = p_n = 2$  analogous result follows from the papers of Aronson [1] and Aronson, Serrin [2]. Problems of removability of isolated singularities for the solutions of special form elliptic and parabolic equations with Laplacian or  $p$ -Laplacian in principle part were studied by many authors, see [12] for a survey of the relevant results. These results were generalized in [9] for the general non-anisotropic parabolic equation with  $m_1 = \dots = m_n = 1$ ,  $p_1 = \dots = p_n > 2$ .

As a particular case of the main result of the present paper we show that if  $u$  is a weak solution to the initial problem (1), (2), then condition (13) suffices for the singularity of  $u$  at the point  $\{(x_0, 0)\}$  to be removable. The proof of our main result is an extension of the method of point-wise and integral estimates of potential type solutions, developed in [11]. This method was applied to the problems of removability of isolated singularities for solutions to general quasi-linear elliptic and parabolic equations [7]–[9].

The paper is organized as follows. The main result is formulated in Section 2. In Section 3 we prove the auxiliary integral estimates for the solutions to the problem (1), (2). We derive the fundamental point-wise estimate of the solution to the problem (1), (2) in Section 4. The boundedness of solutions with isolated singularity under the condition (13) we prove in Section 5.

## 2. Formulation of the main results

We introduce the anisotropic Sobolev space:

$$u \in W^{1, \bar{p}}(\Omega) := \left\{ u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L_{p_i}(\Omega), \quad i = \overline{1, n} \right\}, \tag{14}$$

and define the Banach spaces:

$$V^{2,\bar{p}}(\Omega_T) := C(0, T; L_2(\Omega)) \cap L_p(0, T; W^{1,\bar{p}}(\Omega)),$$

$$V_0^{2,\bar{p}}(\Omega_T) := C(0, T; L_2(\Omega)) \cap L_p(0, T; W_0^{1,\bar{p}}(\Omega)),$$

with the norm  $\|u\|_{V^{2,\bar{p}}(\Omega_T)} = \text{ess sup}_{0 < t < T} \|u(\cdot, t)\|_{L_2(\Omega)} + \|\nabla u\|_{L_{\bar{p}}(\Omega_T)}$ . If  $p_1 = \dots = p_n = 2$  we set  $V_0^{2,2}(\Omega_T) := V_0^2(\Omega_T)$  and  $V^{2,2}(\Omega_T) := V^2(\Omega_T)$ . For the properties of these spaces we refer to [3–6].

Solution to the problem (1), (2) are understood in the following weak sense. Namely, we say that  $u = u(x, t)$  is a weak solution to the problem (1), (2) if:

- (i)  $u\zeta \in V^{2,p}(\Omega_T)$  where the function  $\zeta \in C^\infty(\bar{\Omega}_T)$  vanishes near  $(x_0, 0)$ ;
- (ii) for any function  $\varphi(x, t) = \psi\zeta$ , vanishing near  $\partial\Omega \times (0, T)$  and satisfying

$$\frac{\partial\psi}{\partial t} \in L_2(\Omega_T), \quad \sum_{i=1}^n \iint_{\Omega_T} |\psi|^{(m_i-1)(p_i-1)} \left| \frac{\partial\psi}{\partial x_i} \right|^{p_i} dx dt < \infty,$$

the following integral identity holds:

$$\int_{\Omega} u(x, \tau) \varphi(x, \tau) dx + \int_0^\tau \int_{\Omega} \left\{ -u \frac{\partial\varphi}{\partial t} + \sum_{i=1}^n a_i \left( x, t, u, \frac{\partial u}{\partial x} \right) \frac{\partial\varphi}{\partial x_i} - a_0 \left( x, t, u, \frac{\partial u}{\partial x} \right) \varphi \right\} dx dt = 0 \tag{15}$$

for every  $\tau \in (0, T]$ .

**Remark 1:** Without lost of generality assume that  $\frac{\partial u}{\partial t} \in L_2(\Omega_T)$ , because in the opposite case we can consider the Steklov averages, see for example [3].

Now we are ready to formulate the results of the paper.

**Theorem 2.1:** *Assume that conditions (3)–(5) are fulfilled. Let  $u$  be a weak solution of the problem (1), (2) satisfying equality (13). Then there exist positive constants  $K_1, \gamma$ , depending on known parameters only, such that the following inequality is true*

$$M(\rho) \leq K_1 \rho^{-n+\gamma}, \quad 0 < \rho < R_0. \tag{16}$$

**Theorem 2.2:** *Let the conditions of Theorem 2.1 be fulfilled, and  $g_1(x, t) \equiv f_1(x, t) \equiv 0$ . Then there exists a positive constant  $K_2$ , depending on known parameters only, such that the following inequality is valid*

$$|u(x, t)| \leq K_2 \quad \forall (x, t) \in \mathcal{D} \left( \frac{R_0}{2} \right). \tag{17}$$

Using the usual approach, from Theorem 2.2 we obtain our main result.

**Theorem 2.3:** *Let the conditions of Theorem 2.1 be fulfilled. Then the singularity of solution  $u$  of the problem (1), (2) at the point  $\{(x_0, 0)\}$  is removable.*

**Remark 2:** Since the technique we apply is in part developed in [7]–[9], we believe it is more useful to focus our attention on what is really new. Therefore,

in order to avoid unnecessary technical complications, we prove our main result under simplified assumptions. We assume

$$g_j(x, t) \equiv f_i(x, t) \equiv 0, \quad j = 1, 2, 3, \quad h_i(x, t) \equiv 1, \quad i = 1, \dots, n. \quad (18)$$

The general case is left to reader and it can be recovered by modification of the arguments [7]–[9].

**Remark 3:** In what follows by  $C, K_j, c_j$  we denote generic constants that depend on  $\nu_1, \nu_2, n, T, \|H\|_{L_{q_0}(\Omega_T)}, \alpha_i, p_i, i = 1, \dots, n$ , only.

### 3. The integral estimates of solution

We suppose that

$$\lim_{r \rightarrow 0} M(r) = \infty, \quad (19)$$

and fix the number  $R_0$  so small that  $M(R_0) \geq 1$ . For every  $R \leq R_0$  let us use the following notations:

$$u_R(x, t) = (u(x, t) - M(R))_+, \\ E_R = \{(x, t) \in \mathcal{D}(R) : u(x, t) > M(R)\}.$$

For every  $r : 0 < r < R \leq R_0$  we define also

$$\varepsilon(r) = M(r)r^n + r^p(M(r)r^n)^{p(m-d)} + \sum_{i=1}^n (M(r)r^n)^{m_i(p_i-1)}. \quad (20)$$

We suppose that the numbers  $r, \rho, R$  satisfy the additional conditions

$$0 < r < \rho < R, \quad M(\rho) > 2M(R), \quad (21)$$

and introduce the following notations:

$$\Phi_{\rho, R}(u) = \min\{u_R, M(\rho) - M(R)\}, \quad (22)$$

$$E(\rho, R) = \{(x, t) \in \Omega_T : 0 < u_R(x, t) < M(\rho) - M(R)\}. \quad (23)$$

It is easy to see that  $E(\rho, R) \subset \mathcal{D}(\rho)$ .

Let  $\eta_r \in C^\infty(\Omega_T)$  be the cut-off function such that **(i)**  $0 \leq \eta_r(x, t) \leq 1$  in  $\Omega_T$ , **(ii)**  $\eta_r \equiv 0$  in  $\mathcal{D}(r)$ ,  $\eta_r \equiv 1$  outside  $\mathcal{D}(2r)$ , **(iii)**  $\left| \frac{\partial \eta_r}{\partial t} \right| \leq c_1 r^{-\kappa}$ ,  $\left| \frac{\partial \eta_r}{\partial x_i} \right| \leq c_1 r^{-\kappa_i}$ , where  $\kappa, \kappa_i, i = 1, \dots, n$ , are defined by (6), (11).

First, let us prove the auxiliary integral estimate.

**Lemma 3.1:** *Let the conditions of Theorem 2.1 be fulfilled. Then for the sufficiently large  $k > 0$ , there exists a positive constant  $K_4$ , depending on known*

parameters only, such that the following estimate is valid

$$\begin{aligned} \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} u_R^2(x, t) \eta_r^k(x, t) dx + \sum_{i=1}^n \iint_{E_R} |u|^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \eta_r^k dx dt \leq \\ \leq K_4 \varepsilon(r) M(r) + K_4 R^{n+\kappa} M^{p(m-d)+1}(R), \end{aligned} \quad (24)$$

for every  $r : 0 < r < R_0$ , and  $R_0$  small enough.

**Proof:** Testing (15) by

$$\psi(x, t) = u_R(x, t) \eta_r^{k-1}(x, t), \quad \zeta(x, t) = \eta_r(x, t),$$

using conditions (3), and Young's inequality, we obtain:

$$\begin{aligned} \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} u_R^2(x, t) \eta_r^k(x, t) dx + \sum_{i=1}^n \iint_{E_R} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \eta_r^k dx dt \\ \leq c_2 \iint_{E_R} u_R^2 \left| \frac{\partial \eta_r}{\partial t} \right| dx dt + c_2 \sum_{i=1}^n \iint_{E_R} u^{(m_i-1)(p_i-1)+p_i} \left| \frac{\partial \eta_r}{\partial x_i} \right|^{p_i} dx dt \\ + c_2 \iint_{E_R} u^{p(m-d)+1} \eta_r^k dx dt. \end{aligned} \quad (25)$$

We estimate the first and the second integrals on the right-hand side of (25). Using the definitions of  $M(r)$  and  $\eta_r(x, t)$  we derive:

$$\begin{aligned} \iint_{E_R} u_R^2 \left| \frac{\partial \eta_r}{\partial t} \right| dx dt + \sum_{i=1}^n \iint_{E_R} u^{(m_i-1)(p_i-1)+p_i} \left| \frac{\partial \eta_r}{\partial x_i} \right|^{p_i} dx dt \\ \leq c_3 \left( M^2(r) r^{-\kappa} + \sum_{i=1}^n (M(r))^{(m_i-1)(p_i-1)+p_i} r^{-\kappa_i p_i} \right) \operatorname{meas}\{D(2r) \setminus D(r)\} \\ \leq c_4 M(r) \varepsilon(r). \end{aligned} \quad (26)$$

Applying Lemma 6.1 from the Appendix with  $\alpha_i = (m_i - 1)(p_i - 1)$ ,  $i = 1, \dots, n$ ,  $q = \frac{n}{n-p}(p(m-d) + 1)$ , to the third integral on the right-hand side of (25), we obtain:

$$\begin{aligned} \iint_{E_R} u^{p(m-d)+1} \eta_r^k dx dt \leq c_5 \iint_{E_R \setminus D(2r)} u_R^{p(m-d)+1} dx dt + c_5 \iint_{D(2r) \setminus D(r)} u^{p(m-d)+1} \eta_r^k dx dt \\ + c_5 R^{n+\kappa} M^{p(m-d)+1}(R) \leq c_6 R^p \sum_{i=1}^n \iint_{E_R} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \eta_r^k dx dt \\ + c_6 M(r) \varepsilon(r) + c_5 R^{n+\kappa} M^{p(m-d)+1}(R). \end{aligned} \quad (27)$$

We assume also that the following inequality is true

$$c_2 c_6 R_0^p \leq \frac{1}{2}.$$

Then combining estimates (25)–(27) we derive the required estimate (24) and the proof of Lemma 3.1 is completed.  $\square$

Integral estimate (24) will be used in the proof of the following Lemma.

**Lemma 3.2:** *Let the conditions of Theorem 2.1 be fulfilled. Then there exists a positive constant  $K_5$ , depending on known parameters only, such that the following estimate*

$$\begin{aligned} & \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} \Phi_{\rho, R}^2(u) \eta_r^k dx + \sum_{i=1}^n \iint_{E(\rho, R)} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \eta_r^k dx dt \\ & \leq K_5 (M(\rho) - M(R)) \sum_{i=1}^n \iint_{E_\rho} u^{m-d-\frac{p_i-1}{p}} \left( u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \right)^{1-\frac{1}{p}} \eta_r^k dx dt \\ & \quad + K_5 \varepsilon_1(r) (M(\rho) - M(R)), \quad (28) \end{aligned}$$

holds with  $\varepsilon_1(r) = \varepsilon(r) + \sum_{i=1}^n \varepsilon^{\frac{1}{p_i}}(r)$  and the sufficiently large  $k > 0$ .

**Proof:** Test the integral identity (15) by

$$\psi(x, t) = \Phi_{\rho, R}(u(x, t)) \eta_r^{k-1}(x, t), \quad \zeta(x, t) = \eta_r(x, t).$$

Evaluating the terms on (15) and using conditions (3) we obtain:

$$\begin{aligned} & \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} \Phi_{\rho, R}^2(u(x, t)) \eta_r^k(x, t) dx + \sum_{i=1}^n \iint_{E(\rho, R)} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \eta_r^k dx dt \\ & \leq c_7 (M(\rho) - M(R)) \left( \iint_{E_R} u_R \left| \frac{\partial \eta_r}{\partial t} \right| dx dt \right. \\ & \quad + \sum_{i=1}^n \iint_{E_\rho} u^{m-d-\frac{p_i-1}{p}} \left( u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \right)^{1-\frac{1}{p}} \eta_r^k dx dt \\ & \quad \left. + \sum_{i=1}^n \iint_{E_R} u^{\frac{(m_i-1)(p_i-1)}{p_i}} \left( \sum_{j=1}^n u^{(m_j-1)(p_j-1)} \left| \frac{\partial u}{\partial x_j} \right|^{p_j} \right)^{1-\frac{1}{p_i}} \left| \frac{\partial \eta_r}{\partial x_i} \right| \eta_r^{k-1} dx dt \right), \quad (29) \end{aligned}$$

Analogously to (25) we have:

$$\iint_{E_R} u_R \left| \frac{\partial \eta_r}{\partial t} \right| dx dt \leq c_8 \varepsilon(r). \quad (30)$$

Using Hölder's inequality and Lemma 3.1 we derive:

$$\begin{aligned} & \sum_{i=1}^n \iint_{E_R} u^{\frac{(m_i-1)(p_i-1)}{p_i}} \left( \sum_{j=1}^n u^{(m_j-1)(p_j-1)} \left| \frac{\partial u}{\partial x_j} \right|^{p_j} \right)^{1-\frac{1}{p_i}} \left| \frac{\partial \eta_r}{\partial x_i} \right| \eta_r^{k-1} dx dt \\ & \leq \sum_{i=1}^n \left( \sum_{j=1}^n \iint_{E_R} u^{(m_j-1)(p_j-1)} \left| \frac{\partial u}{\partial x_j} \right|^{p_j} \eta_r^k dx dt \right)^{1-\frac{1}{p_i}} \left( \iint_{E_R} u^{(m_i-1)(p_i-1)} \left| \frac{\partial \eta_r}{\partial x_i} \right|^{p_i} dx dt \right)^{\frac{1}{p_i}} \\ & \leq c_9 \varepsilon(r) + c_9 \sum_{i=1}^n \varepsilon^{\frac{1}{p_i}}(r) (M(R))^{(p(m-d)+1)\frac{p_i-1}{p_i}} R^{(n+\kappa)\frac{p_i-1}{p_i}}. \quad (31) \end{aligned}$$

Combining inequalities (29)–(31), and using condition (13) we obtain estimate (28). Lemma 3.2 is proved.  $\square$

To estimate the last integral on the right-hand side of (28), we need the next Lemma.

**Lemma 3.3:** *Let the conditions of Theorem 2.1 be fulfilled. Then there exists a positive constants  $K_6$  depending on known parameters only, such that the estimate*

$$\begin{aligned} & \sum_{i=1}^n \iint_{E_\rho} u_R^{-q} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \eta_r^k dx dt \\ & \leq K_6 \varepsilon_1(r) (M(\rho) - M(R))^{1-q} + K_6 \rho^{p-n(q-1)(p-1)} (M(\rho) - M(R))^{1-q}, \quad (32) \end{aligned}$$

holds with any  $1 < q < 1 + \frac{p}{n(p-1)}$  and sufficiently large  $k > 0$ .

**Proof:** Test the integral identity (15) by

$$\psi = ([M(\rho) - M(R)]^{1-q} - [\max(u_R, M(\rho) - M(R))]^{1-q}) \eta_r^{k-1}, \quad \zeta = \eta_r.$$

Evaluate all terms in integral identity (15), using conditions (3) and Young's inequality. As a result we obtain:

$$\begin{aligned} & \sum_{i=1}^n \iint_{E_\rho} u_R^{-q} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \eta_r^k dx dt \leq c_{10} (M(\rho) - M(R))^{1-q} \left( \iint_{E_\rho} u \left| \frac{\partial \eta_k}{\partial t} \right| dx dt \right. \\ & + \sum_{i=1}^n \iint_{E_\rho} u^{\frac{(m_i-1)(p_i-1)}{p_i}} \left( \sum_{j=1}^n u^{(m_j-1)(p_j-1)} \left| \frac{\partial u}{\partial x_j} \right|^{p_j} \right)^{1-\frac{1}{p_j}} \left| \frac{\partial \eta_k}{\partial x_i} \right| \eta_r^{k-1} dx dt \\ & \left. + \iint_{E_\rho} u^{p(m-d)-p+1} u_R^{q(p-1)} \eta_r^k dx dt \right). \quad (33) \end{aligned}$$

The first and the second integrals on the right-hand side of (33) were estimated in (30), (31). To estimate the last integral on the right-hand side of (33) we use

condition (13), as result we have

$$\begin{aligned} & \iint_{E_\rho} u^{p(m-d)-p+1} u_R^{q(p-1)} \eta_r^k dx dt \\ & \leq c_{11} \iint_{\bar{D}(\rho) \setminus \bar{D}(r)} \left( \sum_{i=1}^n |x_i - x_i^{(0)}|^{\frac{1}{\kappa_i}} + t^{\frac{1}{\kappa}} \right)^{-np(m-d)-n(q-1)(p-1)} dx dt \leq c_{12} \rho^{p-n(q-1)(p-1)}, \end{aligned} \quad (34)$$

where  $\bar{D}(\rho) = \{(x, t) : \sum_{i=1}^n |x_i - x_i^{(0)}|^{\frac{1}{\kappa_i}} + t^{\frac{1}{\kappa}} \leq \rho\}$ . Now using inequalities (30), (31), (34), from (33) we obtain (32). This completes the proof of Lemma 3.3.  $\square$

Using Lemma 3.2 and Lemma 3.3 we obtain the following estimate.

**Lemma 3.4:** *Let the conditions of Theorem 2.1 be fulfilled. Then there exist a positive constant  $K_7$ , depending on known parameters only, such that the following estimate holds*

$$\begin{aligned} & \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} \Phi_{\rho, R}^2(u(x, t)) \eta_r^k(x, t) dx + \sum_{i=1}^n \iint_{E(\rho, R)} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \eta_r^k dx dt \\ & \leq K_7 \varepsilon_1(r) (M(\rho) - M(R)) + K_7 \rho^{p-n(q-1)(p-1)} (M(\rho) - M(R))^{1-q}, \end{aligned} \quad (35)$$

where  $\varepsilon_1(r)$  is defined in Lemma 3.2 and  $1 < q < 1 + \frac{p}{n(p-1)}$ .

**Proof:** Using Young's inequality, we have:

$$\begin{aligned} & \sum_{i=1}^n \iint_{E_\rho} u^{m-d-\frac{p-1}{p}} \left( u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \right)^{1-\frac{1}{p}} \eta_r^k dx dt \\ & \leq c_{13} \sum_{i=1}^n \iint_{E_\rho} u_R^{-q} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \eta_r^k dx dt \\ & + c_{13} \sum_{i=1}^n \iint_{E_\rho} u_R^{q(p-1)} u^{p(m-d)-p+1} \eta_r^k dx dt. \end{aligned} \quad (36)$$

Inequality (36) together with estimates (28), (32), (34) imply (35). This proves Lemma 3.4.  $\square$

Taking into account condition (13), we can pass to the limit as  $r \rightarrow 0$  in (35) and obtain the following statement.

**Corollary 3.5:** *Let the conditions of Theorem 2.1 be fulfilled. Then there exist a positive constant  $K_8$  depending on known parameters only, and  $q \in (1, 1 + \frac{p}{n(p-1)})$*

such that the following inequality is valid

$$\begin{aligned} \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} \Phi_{\rho,R}^2(u(x,t)) \, dx + \sum_{i=1}^n \iint_{E(\rho,R)} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \, dx \, dt \\ \leq K_8 \rho^{p-n(q-1)(p-1)} (M(\rho) - M(R))^{1-q}. \end{aligned} \quad (37)$$

#### 4. The point-wise estimate of the solution

In this Section we prove Theorem 2.1. First we obtain the additional integral estimate for the solution of problem (1), (2).

Let  $(\tilde{x}, \tilde{t})$  be an arbitrary point in  $\mathcal{D}(R_0) \setminus \mathcal{D}(\rho)$ . For any  $\rho$  satisfying (27) and any positive  $h$  we define the following sequences of numbers

$$\rho_j := \frac{\rho}{2} (1 + 2^{-j}), \quad h_j := 2h(1 - 2^{-j}), \quad j = 1, 2, \dots,$$

and the families of sets:

$$\begin{aligned} Q(\rho_j) := \left\{ (x,t) : \sum_{i=1}^n \left( \frac{|x_i - \tilde{x}_i|}{\rho_j^{\kappa_i}} \right)^{p_i} + \frac{|t - \tilde{t}|}{\rho_j^{\kappa}} < 1 \right\}, \\ A_j := \{(x,t) \in Q(\rho_j) : u_R > h_j\}. \end{aligned}$$

We denote by  $\varphi_j \in C_0^\infty(Q(\frac{\rho_j + \rho_{j+1}}{2}))$  the cut-off function satisfying the following properties: **(i)**  $\varphi_j(x,t) \equiv 1$  for  $(x,t) \in Q(\rho_{j+1})$ ,  $\varphi_j(x,t) \equiv 0$  for  $(x,t) \notin Q(\frac{\rho_j + \rho_{j+1}}{2})$ , **(ii)**  $\left| \frac{\partial \varphi_j}{\partial t} \right| \leq c_{14} 2^{j\kappa} \rho^{-\kappa}$ ,  $\left| \frac{\partial \varphi_j}{\partial x_i} \right| \leq c_{14} 2^{j\kappa_i} \rho^{-\kappa_i}$ ,  $i = 1, \dots, n$ .

Test the integral identity (15) by the following functions

$$\psi = (u_R - h_j)_+ \varphi_j^{p_n - 1}, \quad \zeta = \varphi_j.$$

From conditions (3), Young's inequality, and properties of the cut-off functions  $\varphi_j$ , we obtain:

$$\begin{aligned} \operatorname{ess\,sup}_{0 < t < T} \int_{A_j(t)} (u_R - h_j)_+^2 \varphi_j^{p_n} \, dx + \sum_{i=1}^n \iint_{A_j} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \varphi_j^{p_n} \, dx \, dt \\ \leq c_{15} \iint_{A_j} (u_R - h_j)_+^2 \left| \frac{\partial \varphi_j}{\partial t} \right| \varphi_j^{p_n - 1} \, dx \, dt + c_{15} \iint_{A_j} u^{p(m-d)-p+1} (u_R - h_j)_+^p \varphi_j^{p_n} \, dx \, dt \\ + c_{15} \sum_{i=1}^n \iint_{A_j} (u_R - h_j)_+^{p_i} u^{(m_i-1)(p_i-1)} \left| \frac{\partial \varphi_j}{\partial x_i} \right|^{p_i} \, dx \, dt. \end{aligned} \quad (38)$$

Using Hölder's inequality, condition (13), and the definition of  $M(\rho)$ , we estimate

the integrals on the right-hand side of (38) in the following way:

$$\begin{aligned} & \iint_{A_j} (u_R - h_j)^2 \left| \frac{\partial \varphi_j}{\partial t} \right| \varphi_j^{p_n-1} dx dt \\ & \leq c_{16} 2^{j\kappa} \rho^{-\kappa} M^2 \left( \frac{\rho}{2} \right) \text{meas } A_j \leq c_{17} \rho^{-np(m-d)-n-p} \text{meas } A_j, \end{aligned} \quad (39)$$

$$\begin{aligned} & \sum_{i=1}^n \iint_{A_j} (u_R - h_j)^{p_i} u^{(m_i-1)(p_i-1)} \left| \frac{\partial \varphi_j}{\partial x_i} \right|^{p_i} dx dt \leq c_{18} \sum_{i=1}^n \left( M \left( \frac{\rho}{2} \right) \right)^{(m_i-1)(p_i-1)+p_i} \\ & \times 2^{j\kappa_i} \rho^{-\kappa_i p_i} \text{meas } A_j \leq c_{19} 2^{j c_{20}} \rho^{-np(m-d)-n-p} \text{meas } A_j, \end{aligned} \quad (40)$$

$$\begin{aligned} & \iint_{A_j} u^{p(m-d)-p+1} (u_R - h_j)_+^p \varphi_j^{p_n} dx dt \leq c_{21} \left( M \left( \frac{\rho}{2} \right) \right)^{p(m-d)+1} \text{meas } A_j \\ & \leq c_{22} \rho^{-np(m-d)-n} \text{meas } A_j. \end{aligned} \quad (41)$$

Combining inequalities (39)–(41), we derive the following additional integral estimate.

**Lemma 4.1:** *Let the conditions of Theorem 2.1 be fulfilled. Then there exist positive constants  $K_9, K_{10}$ , depending on known parameters only, such that the following inequality holds*

$$\begin{aligned} & \text{ess sup}_{0 < t < T} \int_{A_j(t)} (u_R(x, t) - h_j)^2 \varphi_j^{p_n}(x, t) dx + \sum_{i=1}^n \iint_{A_j} u^{(m_i-1)(p_i-1)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \varphi_j^{p_n} dx dt \\ & \leq K_9 2^{j K_{10}} \rho^{-np(m-d)-n-p} \text{meas } A_j. \end{aligned} \quad (42)$$

Now we are ready to prove Theorem 2.1. Using inequality (53) from the Appendix with  $\alpha_i = (m_i - 1)(p_i - 1)$ ,  $i = 1, \dots, n$ ,  $\varepsilon = p$ , and Lemma 4.1, we obtain the following estimate

$$\begin{aligned} & \iint_{A_{j+1}} (u_R - h_{j+1})^p dx dt \\ & \leq c_{23} 2^{j K_{11}} h^{-p(1+\frac{p^2}{n\tilde{q}})} (M(\rho))^{(p-2)\frac{p^2}{n\tilde{q}}} \rho^{-\frac{p(n+p)}{n\tilde{q}}(np(m-d)+n+p)} \left( \iint_{A_j} (u_R - h_j)^p dx dt \right)^{1+\frac{p^2}{n\tilde{q}}}, \end{aligned}$$

where  $\tilde{q} = p(m-d) + 1 + \frac{p^2}{n}$  is defined by (52). Finally, by standard way we derive

the following estimation of the solution

$$u(\tilde{x}, \tilde{t}) \leq c_{24} (M(\rho))^{\frac{p(p-2)}{n\tilde{q}+p^2}} \rho^{-\frac{n+p}{n\tilde{q}+p^2}(np(m-d)+n+p)} \left( \iint_{Q(\rho)} u_R^p dx dt \right)^{\frac{p}{n\tilde{q}+p^2}}, \quad (43)$$

see for example [6], [3]. To estimate the integral on the right-hand side of (43) we use Lemma 6.2 from the Appendix with  $\alpha_i = (m_i - 1)(p_i - 1)$ ,  $i = 1, \dots, n$ ,  $\varepsilon = p$ , and points out that

$$u_R(x, t) \leq M(\rho) - M(R), \quad (x, t) \in \mathcal{D}(R_0) \setminus \mathcal{D}(\rho).$$

Indeed, taking into account Corollary 3.5, we have:

$$\begin{aligned} \iint_{Q(\rho)} u_R^p dx dt &\leq c_{25} (M(\rho) - M(R))^{(p-2)\frac{p^2}{n\tilde{q}} + (1-q)\frac{p(p+n)}{\tilde{q}n}} \\ &\quad \times \rho^{\left(1-\frac{p}{\tilde{q}}\right)(np(m-d)+p) + (p-n(q-1)(p-1))\frac{p(p+n)}{n\tilde{q}}}, \end{aligned} \quad (44)$$

under the additional restriction on the parameter  $q$ , that is

$$1 < q < 1 + \min \left\{ \frac{p}{n(p-1)}, \frac{p(p-2)}{n+p} \right\}. \quad (45)$$

Eventually, from inequalities (43), (44), using condition (13), we derive:

$$\begin{aligned} u(\tilde{x}, \tilde{t}) &\leq c_{26} (M(\rho))^{\frac{p(p-2)}{\tilde{q}n} + \frac{p^2(p+n)(1-q)}{\tilde{q}n(\tilde{q}n+p^2)}} \rho^{-\frac{(n+p)}{\tilde{q}n+p^2}(np(m-d)+n+p)} \\ &\quad \times \rho^{\frac{1}{\tilde{q}n+p^2} \left( \frac{p^2(n+p)}{\tilde{q}n}(p-n(q-1)(p-1)) + p^2(n(m-d)+1)\left(1-\frac{p}{\tilde{q}}\right) \right)} \leq c_{27} \rho^{-n\beta}, \end{aligned} \quad (46)$$

where

$$\begin{aligned} \beta &:= \frac{p(p-2)}{\tilde{q}n} + \left( (2-p)(1-q) - \frac{p}{n} \right) \frac{p^2(p+n)}{\tilde{q}n(\tilde{q}n+p^2)} \\ &\quad + \frac{1}{\tilde{q}n+p^2} \left( p \left( 1 + \frac{p^2}{n\tilde{q}} \right) (n(m-d)+1) + n+p \right). \end{aligned}$$

To prove Theorem 2.1 it is sufficient to check that  $\beta < 1$ . This inequality holds if

$$-1 + \frac{p^2}{\tilde{q}n+p^2} \left( (p-2)(q-1) - \frac{p}{n} \right) < 0. \quad (47)$$

Estimate (47) is always satisfied for  $q$  defined by (45). This means that inequality  $\beta < 1$  is fulfilled and this proves Theorem 2.1.

## 5. Proof of Theorem 2.2

Let  $\xi \in C_0^\infty(\mathcal{D}(R_0))$  be such that **(i)**  $0 \leq \xi(x) \leq 1$ , **(ii)**  $\xi(x) \equiv 1$  in  $\mathcal{D}(\frac{R_0}{2})$ ,  $\xi(x) \equiv 0$  outside  $\mathcal{D}(R_0)$ , **(iii)**  $|\frac{\partial \xi}{\partial t}| \leq c_{28} R_0^{-\kappa}$ ,  $|\frac{\partial \xi}{\partial x_i}| \leq c_{28} R_0^{-\kappa_i}$ , where  $\kappa, \kappa_i, i = 1, \dots, n$ , are defined by (6), (11).

We test the integral identity (15) by the following functions:

$$\psi = (u - k)_+^\lambda \xi^l \eta_r^{l-1}, \quad \zeta = \eta_r$$

for every  $k > 0$ , the sufficiently large  $l > 0$ , and  $\lambda < \frac{\gamma}{2(n-\gamma)}$ , with  $\gamma$  from (16). The cut-off function  $\eta_r$  was defined in Section 3. Using conditions (3), (18) we derive:

$$\begin{aligned} & \operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} (u - k)_+^{\lambda+1} \xi^l \eta_r^l dx \\ & + \sum_{i=1}^n \iint_{\Omega_T} u^{(m_i-1)(p_i-1)} (u - k)_+^{\lambda-1} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \xi^l \eta_r^l dx dt \leq c_{29} \sum_{i=1}^5 J_i, \end{aligned} \quad (48)$$

where

$$\begin{aligned} J_1 & := \sum_{i=1}^n \iint_{\Omega_T} |u|^{(m_i-1)(p_i-1)} (u - k)_+^{p_i-1+\lambda} \left| \frac{\partial \eta_r}{\partial x_i} \right|^{p_i} \xi^l dx dt, \\ J_2 & := \iint_{\Omega_T} (u - k)_+^{\lambda+1} \left| \frac{\partial \eta_r}{\partial t} \right| \xi^l dx dt, \\ J_3 & := \sum_{i=1}^n \iint_{\Omega_T} |u|^{(m_i-1)(p_i-1)} (u - k)_+^{p_i-1+\lambda} \left| \frac{\partial \xi}{\partial x_i} \right|^{p_i} \xi^{l-1} \eta_r^l dx dt, \\ J_4 & := \iint_{\Omega_T} (u - k)_+^{\lambda+1} \left| \frac{\partial \xi}{\partial t} \right| \xi^{l-1} \eta_r^l dx dt, \\ J_5 & := \iint_{\Omega_T} |u|^{p(m-d)-p+1} (u - k)_+^{p-1+\lambda} \xi^l \eta_r^l dx dt, \end{aligned}$$

Denote by  $A(k)$  the following set

$$A(k) := \{(x, t) \in \mathcal{D}(R_0) : u(x, t) > k\}.$$

Conditions (4), (16), (18), Theorem 2.1, and Hölder's inequality imply that

$$\begin{aligned} J_1 + J_2 & \leq c_{30} r^{\gamma/2}, \\ J_3 & \leq c_{31} \sum_{i=1}^n \left( \iint_{A(k)} (u - k)^{\tilde{q}} \left( \xi^{l-p_i} \eta_r^l \right)^{\frac{\tilde{q}}{m_i(p_i-1)+\lambda}} dx dt \right)^{\frac{m_i(p_i-1)+\lambda}{\tilde{q}}} (\operatorname{meas} A(k))^{1 - \frac{m_i(p_i-1)+\lambda}{\tilde{q}}} \end{aligned}$$

$$+ c_{31} \sum_{i=1}^n k^{(m_i-1)(p_i-1)} \left( \iint_{A(k)} (u-k)^{\tilde{q}} \left( \xi^{l-p_i} \eta_r^l \right)^{\frac{\tilde{q}}{p_i-1+\lambda}} dx dt \right)^{\frac{p_i-1+\lambda}{\tilde{q}}} (\text{meas } A(k))^{1-\frac{p_i+\lambda-1}{\tilde{q}}}, \tag{49}$$

$$J_4 \leq c_{32} \left( \iint_{A(k)} (u-k)^{\tilde{q}} \left[ \xi^{l-1} \eta_r^l \right]^{\frac{\tilde{q}}{\lambda+1}} dx dt \right)^{\frac{1+\lambda}{\tilde{q}}} (\text{meas } A(k))^{1-\frac{1+\lambda}{\tilde{q}}},$$

$$J_5 \leq c_{33} \left( \iint_{A_k} (u-k)^{\tilde{q}} \left[ \xi^{l-p_i} \eta_r^l \right]^{\frac{\tilde{q}}{p(m-d)+\lambda}} dx dt \right)^{\frac{p(m-d)+\lambda}{\tilde{q}}} \text{meas } A_k^{\frac{p(m-d)+\lambda}{\tilde{q}}},$$

where  $\tilde{q}$  is defined by equality (52) from the Appendix with  $\varepsilon = 1 + \lambda$ ,  $\alpha_i = (m_i - 1)(p_i - 1) + \lambda - 1$ ,  $i = 1, \dots, n$ .

We combine estimates (48), (49). Taking into account the finiteness of the integrals on the right-hand sides of inequalities (49), we can pass to the limit in the resulting inequality as  $r \rightarrow 0$ . Finally, we obtain the boundedness of  $u$  in  $\mathcal{D}(\frac{R_0}{2})$  analogously to the argumentations from Section 4. This proves Theorem 2.2.

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**6. Appendix**

**Lemma 6.1:** ([4]) Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$  be a bounded domain. Let  $v$  be an arbitrary function from the Sobolev space  $W_0^{1,1}(\Omega)$  such that

$$\sum_{i=1}^n \int_{\Omega} |v|^{\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx < \infty,$$

where  $p_i \geq 1$ ,  $1 + \frac{\alpha_i}{p_i} > 0$ ,  $i = 1, \dots, n$ ,  $\sum_{i=1}^n \frac{1}{p_i} < 1$ . Then

$$v \in L_{q_*}(\Omega), \quad q_* := \frac{np}{n-p} \left( 1 + \frac{1}{n} \sum_{i=1}^n \frac{\alpha_i}{p_i} \right) \tag{50}$$

and there exists a positive constant  $\tilde{K}_1$  depending on  $n, \alpha_i, p_i$ ,  $i = 1, \dots, n$  only, such that the following inequality is valid

$$\|v\|_{L_{q_*}(\Omega)} \leq \tilde{K}_1 \prod_{i=1}^n \left( \int_{\Omega} |v|^{\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i n \left( 1 + \frac{1}{n} \sum_{k=1}^n \frac{\alpha_k}{p_k} \right)}}. \tag{51}$$

**Lemma 6.2:** ([5]) Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$  be a bounded domain,  $\Omega_T = \Omega \times (0, T)$ ,  $T > 0$ . Let  $v$  be an arbitrary function belonging to the Sobolev space  $W_0^{1,1}(\Omega)$  for

every fixed  $0 < t < T$ , and satisfying the following condition

$$\operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} v^2(x, t) \, dx + \sum_{i=1}^n \iint_{\Omega_T} |v|^{\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} \, dx \, dt < \infty,$$

where  $p_i \geq 1$ ,  $p_i + \alpha_i > 0$ ,  $i = 1, \dots, n$ ,  $\sum_{i=1}^n \frac{1}{p_i} < 1$ . Then

$$v \in L_{\tilde{q}}(\Omega_T), \quad \tilde{q} = p + \frac{p}{n} \left( \varepsilon + \sum_{k=1}^n \frac{\alpha_k}{p_k} \right). \tag{52}$$

Moreover, there exists a positive constant  $\tilde{K}$  depending on  $n, \alpha_i, p_i$ ,  $i = 1, \dots, n$ , and  $\theta \in (0, 1)$  only, such that

$$\|v\|_{L_{\tilde{q}}(\Omega_T)} \leq \tilde{K} \operatorname{ess\,sup}_{0 < t < T} \|v\|_{L_{\varepsilon}^{\frac{p}{n}}(\Omega)} \sum_{i=1}^n \left( \iint_{\Omega_T} |v|^{\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} \, dx \, dt \right)^{\frac{1}{\tilde{q}}}, \tag{53}$$

where the constant  $\varepsilon$  is defined from the following equality

$$\frac{1}{q} = \frac{\theta}{q_*} + \frac{1 - \theta}{\varepsilon}, \quad \tilde{q} - \varepsilon \frac{p}{n} < q \leq q_*. \tag{54}$$

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