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ON THE NORMALITY OF FAMILIES OF SPACE MAPPINGS WITH BRANCHING

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We study space mappings with branching that satisfy modulus inequalities. For classes of these mappings, we obtain several sufficient conditions for the normality of families.

1. Introduction

This paper is a continuation of [1], where analogous theorems for homeomorphisms were obtained. To a large extent, the method used for the investigation of mappings with branching differs from that used in [1].

We now give the main definitions and notation used in the present paper. In what follows, D is a domain in $\mathbb{R}^n$, $n \geq 2$. A mapping $f\colon D \to \mathbb{R}^n$ is called *discrete* if the preimage $f^{-1}(y)$ of every point $y \in \mathbb{R}^n$ consists of isolated points and *open* if the preimage of any open set $U \subseteq D$ is an open set in $\mathbb{R}^n$. In what follows, writing $f\colon D \to \mathbb{R}^n$, we assume that the mapping f is continuous. We also assume that a mapping f preserves orientation, i.e., the topological index $\mu(y, f, G)$ is strictly positive for an arbitrary domain $G \subset D$ and an arbitrary $y \in f(G) \setminus f(\partial G)$. We also use the following notation:

$$B(x_0, \varepsilon) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}, \qquad B^n(r) = \{x \in \mathbb{R}^n : |x| < r\}, \qquad \mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\},$$

ω_{n-1} is the area of the unit hypersphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, and $dm(x)$ is the n-dimensional Lebesgue measure.

Recall that a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for a family Γ of curves γ in $\mathbb{R}^n$ if

$$\int_\gamma \rho(x)|dx| \geq 1$$

for all curves $\gamma \in \Gamma$. In this case, we write $\rho \in \operatorname{adm}\Gamma$. The quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int_D \rho^n(x)\, dm(x)$$

is called the *modulus* of the family of curves Γ.

In 2001, Martio proposed the following definition (see [2]): Let $Q : D \to [1, \infty]$ be a Lebesgue-measurable function. A homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is called a *Q-homeomorphism* if

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$$M(f\Gamma) \le \int\limits_D Q(x)\rho^n(x)\,dm(x) \tag{1}$$

for any family Γ of curves γ in D and every admissible function $\rho \in \operatorname{adm}\Gamma$. Here and in what follows, $\overline{\mathbb{R}^n} = \mathbb{R}^n \bigcup \{\infty\}$ is a one-point compactification of $\mathbb{R}^n$.

The definition of Q-homeomorphism is closely related to the investigation of weight moduli (see [3]) because, on the right-hand side of (1), one can pass to the infimum over all admissible functions ρ. Note that estimates of the type (1) characterize conformal mappings in the case $Q(x) \equiv 1$ and quasiconformal mappings for $Q(x) \le q$ (see, e.g., Secs. 8.1, 13.1, and 34.3 in [4]). In the case where the nonconstant mapping f is not a homeomorphism, estimates of the type (1) for a bounded function Q are, in fact, a part of the definition of quasiregular mappings (mappings with bounded distortion). In the present paper, we consider mappings with branching that satisfy the modulus relation (1) with, generally speaking, unbounded function $Q(x)$.

We call a mapping $g\colon D \to \overline{\mathbb{R}^n}$, $n \ge 2$, a *Q-mapping* if condition (1) with $g = f$ is satisfied for any family Γ of curves γ in D and for every admissible function $\rho \in \operatorname{adm}\Gamma$. This definition somewhat differs from the traditional one (see [5]).

Let (X, d) and (X', d') be metric spaces with metrics d and d', respectively. A family $\mathfrak{F}$ of continuous mappings $f\colon X \to X'$ is called *normal* if, for any sequence of mappings $f_m \in \mathfrak{F}$, one can choose a subsequence f_{m_k} locally uniformly convergent in X to a continuous function $f_0\colon X \to X'$.

The notion introduced is closely related to the following: A family $\mathfrak{F}$ of mappings $f\colon X \to X'$ is called *equicontinuous at a point* $x_0 \in X$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all x with $d(x, x_0) < \delta$ and all $f \in \mathfrak{F}$. We say that $\mathfrak{F}$ *is equicontinuous* if $\mathfrak{F}$ is equicontinuous at every point of X. Note that, according to one version of the Arzelà–Ascoli theorem, if (X, d) is a separable metric space and (X', d') is a compact metric space, then a family $\mathfrak{F}$ of mappings $f\colon X \to X'$ is normal if and only if $\mathfrak{F}$ is equicontinuous (see, e.g., Sec. 20.4 in [4]). For this reason, the problem of the normality of a family of mappings is often reduced to finding estimates for the distortion of a distance that guarantee the equicontinuity of the family at every point.

The problem of distortion estimates and the normality of families for quasiconformal mappings and their generalizations was studied by many authors, such as Ahlfors, Belinskii, Vuorinen, Väisälä, Gehring, Gutlyanskii, Krushkal', Lavrent'ev, Lehto, Miklyukov, Mori, Ovchinnikov, Pesin, Reshetnyak, Rickman, Ryazanov, Suvorov, Shabat, etc. One should also note the contribution of Gol'berg, Gutlyanskii, Zorii, Ignat'ev, Martio, Ryazanov, Tamrazov, Srebro, and Yakubov to the development of the theory of Q-homeomorphisms and their generalizations and the modulus-and-capacity technique (see, e.g., [2, 3, 5–10]). In particular, in the works indicated, one can find estimates for distortion under Q-homeomorphisms obtained on the basis of properties of functions of bounded and finite mean oscillation.

2. Preliminary Information

Recall some definitions necessary for what follows. We need the notions of condenser and condenser capacity (see, e.g., Sec. 5 in [11] and Part 10, Chap. 2 in [12]).

A condenser is a pair $E = (A, C)$, where A is an open set in $\mathbb{R}^n$ and C is a compact subset of A. A condenser E is called a *ring condenser* if $A \setminus C$ is a ring.

Recall that a *ring* in $\overline{\mathbb{R}^n}$ is a doubly connected domain $R \subset \mathbb{R}^n$, i.e., a domain the complement with respect to which in $\overline{\mathbb{R}^n}$ consists of two connected components, say, C_1 and C_2. This is briefly written as follows: $R = R(C_1, C_2)$. Let $E = (A, C)$ be a condenser.

The *capacity* of a condenser E is defined as follows:

$$\operatorname{cap} E = \operatorname{cap}(A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^n \, dm(x), \tag{2}$$

where $W_0(E) = W_0(A, C)$ is a family of nonnegative continuous functions $u: A \to \mathbb{R}^1$ with compact support in A such that $u(x) \geq 1$ for $x \in C$, $u \in ACL$, and

$$|\nabla u| = \left(\sum_{i=1}^{n} (\partial_i u)^2 \right)^{1/2}.$$

Recall that a mapping $f: D \to \mathbb{R}^n$ is called *absolutely continuous on lines* ($f \in ACL$) if, in any n-dimensional parallelepiped P such that its edges are parallel to coordinate axes and $\overline{P} \subset D$, all coordinate functions $f = (f_1, \ldots, f_n)$ are absolutely continuous on almost all lines parallel to the coordinate axes.

Remark 1. According to Lemma 5.5 in [11], we have

$$\operatorname{cap} E = \operatorname{cap}(A, C) = \inf_{u \in W_0^{\infty}(E)} \int_A |\nabla u|^n \, dm(x), \tag{3}$$

where $W_0^{\infty}(E) = W_0(E) \cap C_0^{\infty}(A)$ and $C_0^{\infty}(A)$ is the set of real-valued infinitely differentiable functions with compact support in A.

Moreover, the condition "$u(x) \geq 1$ for every $x \in C$" can be replaced by the condition "$u(x) = 1$ for every $x \in C$" or even by the condition "$u(x) = 1$ in a certain neighborhood C" (see Remark 2.2 in [13] or [14, p. 502]).

In what follows, in $\overline{\mathbb{R}^n}$ we use the *spherical (chordal)* metric $h(x, y) = |\pi(x) - \pi(y)|$, where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $\mathbb{S}^n\left(\frac{1}{2} e_{n+1}, \frac{1}{2}\right)$ in $\mathbb{R}^{n+1}$; we have

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}, \qquad h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \qquad x \neq \infty \neq y.$$

The *chordal diameter* of a set $B \subseteq \overline{\mathbb{R}^n}$ is defined as follows:

$$h(B) = \sup_{x, y \in B} h(x, y).$$

The *Teichmüller ring* is defined as follows:

$$R_T(t) = R([-1,0],\,[t,\infty]), \qquad t > 1.$$

We now formulate a very important result of Gehring (see [15] or Sec. 7.37 in [16]).

Lemma 1. *Let $R(B,F)$ be an arbitrary ring such that the continua B and F are nondegenerate. Then*

$$\operatorname{cap}(R(B,F)) \ge \operatorname{cap}\left(R_T\left(\frac{1}{h(B)h(F)}\right)\right). \tag{4}$$

The *Grötsch ring* is defined as follows:

$$R_G(t) = R([t,\infty],\,\overline{\mathbb{B}^n}), \qquad t > 1.$$

Let a function $\Psi(t)$ be defined by the relation

$$\operatorname{cap} R_G(t) = \omega_{n-1}(\log\Psi(t))^{1-n}.$$

It is known [16] that the function $\log\Psi(t) - \log t$ increases. We set

$$\log\lambda_n = \lim_{t\to\infty}(\log\Psi(t) - \log t).$$

It is also known that $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$, and $\lambda_n^{1/n} \to e$ as $n \to \infty$ (see, e.g., [15, pp. 225, 226] and relations (7.19) and (7.22) in [16]). According to Lemma 7.22 in [16], we have

$$\operatorname{cap}(R_T(t)) = \frac{\omega_{n-1}}{\{\log\Phi(t)\}^{n-1}},$$

where the function Φ satisfies the conditions

$$t + 1 \le \Phi(t) \le \lambda_n^2(t+1) < 2\lambda_n^2 t, \qquad t > 1.$$

In what follows, we use λ_n in the sense of the definition presented above without special explanations if this does not lead to misunderstanding.

Using relation (4), we obtain the following estimate for the capacity:

Lemma 2. *For any disjoint nondegenerate continua B and F in $\overline{\mathbb{R}^n}$, the following relation is true:*

$$\operatorname{cap}(R(B,F)) \ge \frac{\omega_{n-1}}{\left[\log\dfrac{2\lambda_n^2}{h(B)h(F)}\right]^{n-1}}.$$

An analog of the lemma presented below was proved in [13] (see Lemma 2.9).

Lemma 3. *Let* $E = (A, C)$ *be a condenser such that* $A \subset B^n(r)$ *and the set* C *is connected. Then the following relation is true:*

$$\operatorname{cap} E \geq \frac{\omega_{n-1}}{\left\{\log \dfrac{2\lambda_n^2}{h(C)h\left(\overline{\mathbb{R}^n} \setminus B^n(r)\right)}\right\}^{n-1}}.$$

Proof. By virtue of the monotonicity of capacities, the required statement follows from Lemma 2.

3. Main Lemma on Distortion Estimate

We introduce several definitions. For the notion presented below, see [12, p. 32] (Part 3, Chap. II). Let β : $[a, b) \to \mathbb{R}^n$ be a certain curve and let $x \in f^{-1}(\beta(a))$. A curve $\alpha : [a, c) \to D$ is called a *maximal lifting* of the curve β under the mapping f with origin at the point x if the following conditions are satisfied:

(i) $\alpha(a) = x$;

(ii) $f \circ \alpha = \beta|_{[a,c)}$;

(iii) if $c < c' \leq b$, then a curve $\alpha' : [a, c') \to D$ such that $\alpha = \alpha'|_{[a,c)}$ and $f \circ \alpha = \beta|_{[a,c')}$ does not exist.

Let f be an open discrete mapping and let $x \in f^{-1}(\beta(a))$. Then the curve β has a maximal lifting under the mapping f with origin at the point x (see Corollary 3.3, Chap. 2, in [12]).

We need the following statement (see Proposition 10.2, Chap. 2, in [12]):

Lemma 4. *Let* $E = (A, C)$ *be an arbitrary condenser and let* Γ_E *be the family of all curves of the form* $\gamma : [a, b) \to A$ *with* $\gamma(a) \in C$ *and* $|\gamma| \cap (A \setminus F) \neq \varnothing$ *for an arbitrary compact set* $F \subset A$, *where* $|\gamma| = \gamma([a, b))$. *Then* $\operatorname{cap} E = M(\Gamma_E)$.

We say that a family of curves Γ_1 is *minorized* by a family Γ_2 (we write $\Gamma_1 > \Gamma_2$) if, for every curve $\gamma \in \Gamma_1$, there is a subcurve that belongs to the family Γ_2. It is known that if $\Gamma_1 > \Gamma_2$, then $M(\Gamma_1) \leq M(\Gamma_2)$ (see, e.g., Theorem 6.4 in [4]).

Analogs of the lemma presented below were proved in [1] for homeomorphisms.

Lemma 5. *Let* $f : D \to \mathbb{R}^n$, $n \geq 2$, *be an open discrete* Q*-mapping such that* $D' = f(D) \subset B^n(r)$ *with* $h\left(\overline{\mathbb{R}^n} \setminus B^n(r)\right) \geq \delta > 0$. *Suppose that, for a certain* $x_0 \in D$ *and for* $0 < \varepsilon_0 < \operatorname{dist}(x_0, \partial D)$, *there exist a number* p, $p \leq n$, *and a family of functions* $\{\psi_\varepsilon(t), \varepsilon \in (0, \varepsilon_0)\}$ *nonnegative on* $(0, \infty)$ *that satisfy the following conditions:*

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x)\psi_\varepsilon^n(|x-x_0|)\,dm(x) \;\le\; K I^p(\varepsilon), \qquad 0<I(\varepsilon)<\infty, \tag{5}$$

where

$$I(\varepsilon) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_\varepsilon(t)\,dt$$

and K is a finite positive constant. Then

$$h(f(x), f(x_0)) \;\le\; \frac{\alpha_n}{\delta}\exp\left\{-\beta_n I^{\gamma_{n,p}}(|x-x_0|)\right\} \quad \textit{for all} \quad x\in B(x_0,\varepsilon_0),$$

where

$$\alpha_n = 2\lambda_n^2, \qquad \beta_n = \left(\frac{\omega_{n-1}}{K}\right)^{1/(n-1)}, \qquad \gamma_{n,p} = 1-\frac{p-1}{n-1}. \tag{6}$$

Proof. Consider a condenser $E=(A,C)$, where $A=B(x_0,\varepsilon_0)$, $C=B(x_0,\varepsilon)$, and $\varepsilon<\varepsilon_0$. Since the mapping f is open and continuous, fC is a compact subset of the open set fA. Therefore, the pair of sets $E'=fE=(fA,fC)$ is also a condenser. For the condensers E and fE, we consider the families of curves Γ_E and Γ_{fE}, respectively (see the notation of Lemma 4). Let Γ^* be the family of maximal f-liftings of the curves Γ_{fE} with origin in C that lie in A. We show that $\Gamma^*\subset\Gamma_E$.

Assume that the converse statement is true. Then there exists a curve $\beta\colon [a,b)\to\mathbb{R}^n$ of the family Γ_{fE} for which the corresponding maximal lifting $\alpha\colon [a,c)\to A$ lies, together with its closure $\overline{\alpha}$, in a certain compact set inside A. Therefore, $\overline{\alpha}$ is a compact set in A. Note that $c\ne b$ because otherwise $\overline{\beta}$ is a compact set in fA, which contradicts the condition $\beta\in\Gamma_{fE}$. Let G be the limit set of $\alpha(t)$ for $t\to c-0$. For $x\in G$, by virtue of the continuity of f, we have $f(\alpha(x_k))\to f(x)$ as $k\to\infty$, where $x_k\in[a,c)$ and $x_k\to c$ as $k\to\infty$. However, $f(\alpha(x_k)) = \beta(x_k)\to\beta(c)$ as $k\to\infty$. This implies that f is constant on G in A. By virtue of the Cantor condition, in the compact set $\overline{\alpha}$ [17, pp. 8, 9] we have

$$G = \bigcap_{k=1}^{\infty}\overline{\alpha([t_k,c))} = \limsup_{k\to\infty}\alpha([t_k,c)) = \liminf_{k\to\infty}\alpha([t_k,c)) \ne \varnothing$$

due to the monotonicity of the sequence of connected sets $\alpha([t_k,c))$; hence, G is connected with respect to I [see (9.12) in [18]]. Thus, by virtue of discreteness of f, G cannot consist of more than one point, and the curve $\alpha\colon [a,c)\to A$ can be extended to the closed curve $\alpha\colon [a,c]\to A$. Then $f(\alpha(c))=\beta(c)$, i.e., $\alpha(c)\in f^{-1}(\beta(c))$. On the other hand, by virtue of Corollary 3.3 in [12] (Chap. 2), we can construct a maximal lifting α' of the curve $\beta|_{[c,b)}$ with origin at the point $\alpha(c)$. Combining the liftings α and α', we obtain a new lifting α'' of the curve β defined on $[a,c')$, which contradicts the maximality of the lifting α.

Thus, $\Gamma^*\subset\Gamma_E$. Note that $\Gamma_{fE} > f\Gamma^*$ and, hence, $M(\Gamma_{fE})\le M(f\Gamma^*)$. Thus, according to Lemma 4 and the definition of Q-mapping, we have

$$\operatorname{cap} fE \leq \int_D Q(x)\rho^n(x)\,dm(x) \tag{7}$$

for an arbitrary function $\rho \in \operatorname{adm} \Gamma_E$.

Now let $A_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}$. According to the Luzin theorem [19, p. 69], there exists a Borel function $\psi_\varepsilon^*(t) = \psi_\varepsilon(t)$ for almost all t. Therefore,

$$\rho_\varepsilon(x) = \begin{cases} \psi_\varepsilon^*(|x - x_0|)/I(\varepsilon), & x \in A_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus A_\varepsilon \end{cases}$$

is a Borel function. Also let $\gamma \in \Gamma_E$. Then (see, e.g., Theorem 5.7 in [4])

$$\int_\gamma \rho_\varepsilon |dx| \geq \frac{1}{I(\varepsilon)} \int_\varepsilon^{\varepsilon_0} \psi_\varepsilon^*(t)\,dt = 1, \tag{8}$$

which implies that $\rho_\varepsilon(x) \in \operatorname{adm} \Gamma_E$. Combining (5), (7), and (8), we obtain

$$\operatorname{cap} fE \leq K I^{p-n}(\varepsilon). \tag{9}$$

Since $fA \subset B^n(r)$, by virtue of Lemma 3 applied to the condenser fE we get

$$\operatorname{cap} fE \geq \frac{\omega_{n-1}}{\left\{\log \dfrac{2\lambda_n^2}{h(fC)\,h(\overline{\mathbb{R}^n} \setminus B^n(r))}\right\}^{n-1}}. \tag{10}$$

Using the condition $h(\overline{\mathbb{R}^n} \setminus B^n(r)) \geq \delta$ and relations (9) and (10), we obtain

$$h(fC) \leq \frac{2\lambda_n^2}{\delta} \exp\left\{-\left(\frac{\omega_{n-1}}{K}\right)^{1/(n-1)} (I(\varepsilon))^{(p-n)/(n-1)}\right\}.$$

Denoting

$$\alpha_n = 2\lambda_n^2, \qquad \beta_n = \left(\frac{\omega_{n-1}}{K}\right)^{1/(n-1)}, \qquad \gamma_{n,p} = 1 - \frac{p-1}{n-1},$$

we obtain

$$h(fC) \leq \frac{\alpha_n}{\delta} \exp\left\{-\beta_n I^{\gamma_{n,p}}(\varepsilon)\right\}, \tag{11}$$

where $C = \overline{B(x_0, \varepsilon)}$. Now let $x \in D$ be such that $|x - x_0| = \varepsilon$, $0 < \varepsilon < \varepsilon_0$. Then $x \in \overline{B(x_0, \varepsilon)}$, $f(x) \in f\left(\overline{B(x_0, \varepsilon)}\right) = fC$, and relation (11) yields

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\delta} \exp\left\{-\beta_n I^{\gamma_{n,p}}(|x - x_0|)\right\} \qquad \forall\, \varepsilon \in (0, \varepsilon_0). \tag{12}$$

Since $\varepsilon \in (0, \varepsilon_0)$ is arbitrary, relation (12) holds in the entire ball $B(x_0, \varepsilon_0)$.

4. Estimates for the Distortion of a Distance under Q-Mappings

In what follows, $q_{x_0}(r)$ denotes the integral mean value of the function $Q(x)$ over the sphere $|x - x_0| = r$ and $\delta(x_0) = \operatorname{dist}(x_0, \partial D)$.

Theorem 1. *Let $f\colon D \to \mathbb{R}^n$, $n \ge 2$, be an open discrete Q-mapping for which $D' = f(D) \subset B^n(r)$ with $h(\overline{\mathbb{R}^n \setminus B^n(r)}) \ge \delta > 0$. Then, for every point $x \in B(x_0, \varepsilon(x_0))$, $\varepsilon(x_0) \le \operatorname{dist}(x_0, \partial D)$, and every $\beta \ge 1/(n-1)$, one has*

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\delta} \exp\left\{-\int\limits_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{r q_{x_0}^{\beta}(r)}\right\}, \tag{13}$$

where α_n is defined by (6).

Proof. Denote $\varepsilon_0 = \varepsilon(x_0)$. Consider the function

$$\psi(t) = \begin{cases} \dfrac{1}{t q_{x_0}^{\beta}(t)}, & t \in (0, \varepsilon_0), \\ 0, & t \in [\varepsilon_0, \infty). \end{cases}$$

We have

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x)\psi^n(|x - x_0|)\,dm(x) = \omega_{n-1}\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{r q_{x_0}^{\beta n-1}(r)} \le \omega_{n-1}\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dr}{r q_{x_0}^{\beta}(r)}.$$

The statement of the theorem follows from Lemma 5 for $p = 1$.

Remark 2. Of course, the mean value $q_{x_0}(r)$ of the function $Q(x)$ over certain spheres $|x - x_0| = r$ can be infinite. However, by virtue of the Fubini theorem (see, e.g., [19]), $q_{x_0}(r)$ is measurable with respect to the parameter r because $Q(x)$ is measurable with respect to x. Moreover,

$$\int\limits_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{r q_{x_0}^{\beta}(r)} < \infty$$

for $x \ne x_0$ because $q_{x_0}(r) \ge 1$. The integral may be equal to 0 if $q_{x_0}(r) = \infty$ almost everywhere. However, inequality (13) is obvious in this case because $\alpha_n \ge 32$, $\delta \le 1$, and $h(f(x), f(x_0))$ is less than or equal to 1.

Choosing $\beta = 1/(n-1)$ in Theorem 1, we obtain the following statement:

Corollary 1. *If*

$$q_{x_0}(r) \le \left[\log\frac{1}{r}\right]^{n-1} \tag{14}$$

for $r < \varepsilon(x_0) < \delta(x_0)$, *then*

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\delta}\frac{\log\frac{1}{\varepsilon(x_0)}}{\log\frac{1}{|x-x_0|}} \tag{15}$$

for all $x \in B(x_0, \varepsilon(x_0))$.

Remark 3. If, instead of relation (14), one has

$$q_{x_0}(r) \le c\left[\log\frac{1}{r}\right]^{n-1},$$

then

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\delta}\left[\frac{\log\frac{1}{\varepsilon(x_0)}}{\log\frac{1}{|x-x_0|}}\right]^{1/c^{1/(n-1)}}.$$

5. Finite Mean Oscillation

We say that a function $\varphi: D \to \mathbb{R}$, $\varphi \in L^1_{\mathrm{loc}}(D)$, has *bounded mean oscillation* in a domain D $(\varphi \in BMO(D))$ if

$$\|\varphi\|_* = \sup_{B\subset D}\frac{1}{|B|}\int_B |\varphi(x) - \varphi_B|\,dm(x) < \infty,$$

where the least upper bound is taken over all balls $B \subset D$ and

$$\varphi_B = |B|^{-1}\int_B \varphi(x)\,dm(x)$$

is the mean value of the function φ over the ball B. In what follows, for simplicity, we write

$$\fint_A f(x)\,dm(x) \;:=\; \frac{1}{|A|}\int_A f(x)\,dm(x),$$

where, as usual, $|A|$ is the Lebesgue measure of a set $A \subseteq \mathbb{R}^n$. It is known that $L^\infty(D) \subset BMO(D) \subset L^p_{\mathrm{loc}}(D)$ (see, e.g., [20]).

Following [20], we introduce the following definitions: We say that a function $\varphi : D \to \mathbb{R}$ has *finite mean oscillation* at a point $x_0 \in D$ $(\varphi \in FMO$ at $x_0)$ if

$$\limsup_{\varepsilon\to 0} \fint_{B(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon|\,dm(x) \;<\; \infty, \tag{16}$$

where

$$\overline{\varphi}_\varepsilon \;=\; \fint_{B(x_0,\varepsilon)} \varphi(x)\,dm(x).$$

Note that, under condition (16), it is possible that $\overline{\varphi}_\varepsilon \to \infty$ as $\varepsilon \to 0$.

We also say that $\varphi : D \to \mathbb{R}$ is a function of finite mean oscillation in the domain D $($we write $\varphi \in FMO(D)$ or, simply, $\varphi \in FMO)$ if φ has finite mean oscillation at every point $x \in D$. In particular, if, for certain numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$, one has

$$\limsup_{\varepsilon\to 0} \fint_{B(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon|\,dm(x) \;<\; \infty,$$

then the function φ has finite mean oscillation at the point x_0 [8]. For example, a function φ has finite mean oscillation at a point x_0 if the following relation holds at the point $x_0 \in D$:

$$\limsup_{\varepsilon\to 0} \fint_{B(x_0,\varepsilon)} |\varphi(x)|\,dm(x) \;<\; \infty.$$

The statement presented below plays the key role in studying mappings of finite mean oscillation. Earlier, it was widely used in the works of Ignat'ev and Ryazanov (see Corollary 2.3 in [8]) for the solution of the problem of homeomorphic (continuous) extension of homeomorphisms to the boundary.

Lemma 6. *Let* D_0 *be a domain in* $\mathbb{R}^n$, $n \geq 2$, *that contains the origin of coordinates and let* $\varphi : D_0 \to \mathbb{R}$ *be a nonnegative function that has finite mean oscillation at the point* 0. *Then*

$$\int_{\varepsilon<|x|<\varepsilon_0} \frac{\varphi(x)\,dm(x)}{\left(|x|\log\frac{1}{|x|}\right)^n} \;=\; O\left(\log\log\frac{1}{\varepsilon}\right)$$

as $\varepsilon \to 0$ *for a certain* $\varepsilon_0 \leq \operatorname{dist}(0, \partial D_0)$.

Theorem 2. *Let* $f\colon D \to \mathbb{R}^n$, $n \ge 2$, *be an open discrete* Q*-mapping such that* $D' = f(D) \subset B^n(r)$ *with* $h(\overline{\mathbb{R}^n} \setminus B^n(r)) \ge \delta > 0$. *If the function* $Q(x)$ *has finite mean oscillation at a point* $x_0 \in D$, *then*

$$h(f(x), f(x_0)) \le \frac{\alpha_n}{\delta} \left\{ \frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x - x_0|}} \right\}^{\beta_0}$$

for $x \in B(x_0, \varepsilon_0)$ *and a certain* $\varepsilon_0 < \operatorname{dist}(x_0, \partial D)$; *here,* α_n *depends only on* n, *and* $\beta_0 > 0$ *depends only on* n *and* Q.

Proof. Let

$$\varepsilon_0 < \min\{1, \operatorname{dist}(x_0, \partial D)\}.$$

Assume that the function $Q(x)$ has finite mean oscillation in the domain D. According to Lemma 6, for the function

$$\psi(t) = \frac{1}{t \log \frac{1}{t}}$$

we have

$$\int\limits_{\varepsilon < |x - x_0| < \varepsilon_0} Q(x) \psi^n(|x - x_0|)\, dm(x) = \int\limits_{\varepsilon < |y| < \varepsilon_0} Q(y + x_0) \psi^n(|y|)\, dm(y)$$

$$= \int\limits_{\varepsilon < |y| < \varepsilon_0} \frac{Q(y + x_0)}{\left(|y| \log \frac{1}{|y|}\right)^n}\, dm(y) = O\left(\log \log \frac{1}{\varepsilon}\right). \tag{17}$$

Here, we have used the fact that the function $Q_1(y) := Q(y + x_0)$ has finite mean oscillation at the point 0. Also note that

$$I(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt = \log\left(c \log \frac{1}{\varepsilon}\right), \tag{18}$$

where

$$c = \frac{1}{\log \frac{1}{\varepsilon_0}}.$$

Using relations (17) and (18), we establish that the function ψ thus chosen satisfies relation (5) with $p = 1$. The remaining part of the statement of the theorem follows from Lemma 5.

6. On Normal Families of Q-Mappings

Let $\mathfrak{F}_{Q,\delta}(D)$ denote the class of all open discrete Q-mappings $f\colon D \to \mathbb{R}^n$, $n \geq 2$, such that $D' = f(D) \subset B^n(r)$ with $h(\overline{\mathbb{R}^n \setminus B^n(r)}) \geq \delta > 0$. The following statements are true:

Theorem 3. *If $Q \in FMO$, then the class $\mathfrak{F}_{Q,\delta}(D)$ forms a normal family of mappings in $\mathbb{R}^n$.*

Corollary 2. *The family $\mathfrak{F}_{Q,\delta}(D)$ is normal in $\mathbb{R}^n$ if, for every $x_0 \in D$, one has*

$$\limsup_{\varepsilon \to 0} \fint_{B(x_0,\varepsilon)} Q(x)\,dm(x) < \infty.$$

Corollary 3. *The class $\mathfrak{F}_{Q,\delta}(D)$ forms a normal family of mappings in $\mathbb{R}^n$ if every point $x_0 \in D$ is a Lebesgue point of the function $Q(x)$.*

Theorem 4. *The family $\mathfrak{F}_{Q,\delta}(D)$ is normal in $\mathbb{R}^n$ if the condition of the divergence of the integral*

$$\int_0^{\delta(x_0)} \frac{dr}{r q_{x_0}^{\beta}(r)} = \infty$$

is satisfied for a certain $\beta \geq 1/(n-1)$ and, in particular, for $\beta = 1$ at every point $x_0 \in D$.

By virtue of Remark 3, the following statement is true:

Corollary 4. *The class $\mathfrak{F}_{Q,\delta}(D)$ is normal in $\mathbb{R}^n$ if $Q(x)$ has logarithmic singularities of at most $(n-1)$th order at every point $x \in D$.*

7. On Mappings with Finite Distortion of Length

Following [5], we say that $f\colon D \to \mathbb{R}^n$, $n \geq 2$, is a mapping with *finite metric distortion* ($f \in FMD$) if f possesses the Luzin (N)-property and

$$0 < l(x,f) \leq L(x,f) < \infty \qquad \text{almost everywhere,}$$

where

$$L(x,f) = \limsup_{y \to x,\; y \in D} \frac{|f(x)-f(y)|}{|y-x|}, \qquad l(x,f) = \liminf_{y \to x,\; y \in D} \frac{|f(x)-f(y)|}{|y-x|}.$$

Recall that a mapping $f\colon X \to Y$ between the spaces with measure (X, Σ, μ) and (X', Σ', μ') possesses the (N)-*property* if $\mu'(f(S)) = 0$ whenever $\mu(S) = 0$. By analogy, f possesses the (N^{-1})-*property* if $\mu(S) = 0$ whenever $\mu'(f(S)) = 0$.

Let $\Delta \subseteq \mathbb{R}$ be an open interval of the number axis and let $\gamma\colon \Delta \to \mathbb{R}^n$ be a locally rectifiable curve. Then there exists a unique nondecreasing function of length $l_\gamma\colon \Delta \to \Delta_\gamma \subseteq \mathbb{R}$ such that $l_\gamma(t_0) = 0$, $t_0 \in \Delta$, and the value $l_\gamma(t)$ is equal to the length of the subcurve $\gamma|_{[t_0,t]}$ of the curve γ if $t > t_0$ and to $-l(\gamma|_{[t_0,t]})$ if $t < t_0$, $t \in \Delta$. Let $g\colon |\gamma| \to \mathbb{R}^n$ be a continuous mapping; here, $|\gamma| = \gamma(\Delta) \subseteq \mathbb{R}^n$. Assume that the curve $\tilde{\gamma} = g \circ \gamma$ is also locally rectifiable. Then there exists a unique nondecreasing function $L_{\gamma,g}\colon \Delta_\gamma \to \Delta_{\tilde{\gamma}}$ such that $L_{\gamma,g}(l_\gamma(t)) = l_{\tilde{\gamma}}(t)$ $\forall\, t \in \Delta$. We say that a mapping $f\colon D \to \mathbb{R}^n$ possesses the (L)-*property* if the following conditions are satisfied:

(L_1) for almost all curves $\gamma \in D$, the curve $\tilde{\gamma} = f \circ \gamma$ is locally rectifiable and the function $L_{\gamma,f}$ possesses the (N)-property;

(L_2) for almost all curves $\tilde{\gamma} \in f(D)$, every *lifting* γ of the curve $\tilde{\gamma}$ is locally rectifiable and the function $L_{\gamma,f}$ possesses the (N^{-1})-property.

Here, a curve $\gamma \in D$ is called the *lifting of a curve* $\tilde{\gamma} \in \mathbb{R}^n$ *under a mapping* $f\colon D \to \mathbb{R}^n$ if $\tilde{\gamma} = f \circ \gamma$. We say that *almost all curves* of the domain D possess a certain property if all curves lying in D except, possibly, a certain family whose modulus is equal to zero possess this property.

Following [5], we say that a mapping $f\colon D \to \mathbb{R}^n$, $n \geq 2$, is a *mapping with finite distortion of length* ($f \in FLD$) if f belongs to FMD and possesses the (L)-property.

Note that the theory developed above can be applied to families of mappings $f\colon D \to \mathbb{R}^n$ with finite distortion of length because mappings of the class FLD are Q-mappings with $Q(x) = K_I(x, f)$, where $K_I(x, f)$ is the inner dilatation of the mapping f at the point $x \in D$ (see Theorem 6.10 in [5]).

For a mapping $f\colon D \to \mathbb{R}^n$ that has partial derivatives almost everywhere in D, we assume that $f'(x)$ is the Jacobian matrix of the mapping f' at a point x, and $J(x, f)$ is the Jacobian of the mapping f at the point x, i.e., the determinant of $f'(x)$. Furthermore,

$$l(f'(x)) = \min_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|}.$$

Recall that the *inner dilatation* of a mapping f at a point x is defined as follows:

$$K_I(x, f) = \frac{|J(x, f)|}{l(f'(x))^n} \quad \text{if} \quad J(x, f) \neq 0,$$

$$K_I(x, f) = 1 \quad \text{if} \quad f'(x) = 0,$$

and $K_I(x, f) = \infty$ at the other points.

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$. By $\mathcal{L}_{Q,\Delta}(D)$ we denote the family of all open discrete mappings of finite distortion of length $f\colon D \to \mathbb{R}^n$ such that $D' = f(D) \subset B^n(r)$ with $h(\overline{\mathbb{R}^n} \setminus B^n(r)) \geq \Delta > 0$ and $K_I(x, f) \leq Q(x)$ almost everywhere.

Theorem 5. *Suppose that* $\Delta > 0$ *and* $Q\colon D \to [1, \infty]$ *is a measurable function such that*

$$\int_0^{\delta(x_0)} \frac{dr}{r q_{x_0}^{1/(n-1)}(r)} = \infty \qquad \forall\, x_0 \in D.$$

Then the family of mappings $\mathcal{L}_{Q,\Delta}(D)$ *is normal in* $\mathbb{R}^n$.

Corollary 5. *The family of mappings* $\mathcal{L}_{Q,\Delta}(D)$ *is normal in* $\mathbb{R}^n$ *if*

$$q_{x_0}(r) = O\left(\left(\log\frac{1}{r}\right)^{n-1}\right)$$

for every $x_0 \in D$ *as* $r \to 0$.

Corollary 6. *The family of mappings* $\mathcal{L}_{Q,\Delta}(D)$ *is normal in* $\mathbb{R}^n$ *if the function* $Q(x)$ *has logarithmic singularities of at most* $(n-1)$*th order at every point* $x_0 \in D$.

Corollary 7. *If* $Q \in FMO$, *then the family of mappings* $\mathcal{L}_{Q,\Delta}(D)$ *is normal in* $\mathbb{R}^n$.

Corollary 8. *The family of mappings* $\mathcal{L}_{Q,\Delta}(D)$ *is normal in* $\mathbb{R}^n$ *if*

$$\limsup_{\varepsilon \to 0} \fint_{B(x_0,\varepsilon)} Q(x)\,dm(x) < \infty \qquad \forall\, x_0 \in D.$$

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