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Asymptotic Stability and Instability of the Solutions of Systems with Impulse Action

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Abstract—In this paper, we study the stability of the zero solution of a system of ordinary differential equations subject to impulse action. Using the method of Lyapunov functions, we obtain tests for asymptotic stability or instability of the system. Illustrative examples are given.

Key words: *differential equation with impulse action, dynamical system with discontinuous trajectories, asymptotic stability, Lyapunov function.*

1. INTRODUCTION

In the mathematical description of the evolution of processes with short-term perturbations, it is convenient, in the many cases, to neglect the duration of such perturbations and assume that these perturbations are instantaneous. Such an idealization leads to the study of dynamical systems with discontinuous trajectories or, in other words, to the study of differential equations with impulse action. This problem was first studied in [1], [2]. The early work on differential equations with impulse action was summarized in the monograph [3], in which the foundations of this theory were described. In recent years, an increasing number of mathematical papers dealing with various aspects of the theory of impulsive systems [4]–[16] have appeared, which is due to the needs of modern technology. The present paper is devoted to the study of the stability of the solutions of systems with impulse action. It is a continuation and further development of [17], [18].

2. DEFINITIONS AND PRELIMINARY RESULTS

Consider the following system of ordinary differential equations with impulse action:

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \qquad && t \neq \tau_k, \\ \Delta x &= I_k(x), && t = \tau_k, \end{aligned} \tag{2.1}$$

where $t \in \mathbb{R}_+ := [0,\infty)$ is the time, $k \in \mathbb{N}$, ($\mathbb{N}$ is the set of natural numbers), the τ_k are constants, $x \in \mathbb{R}^n$, $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$, and $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$. Equations (2.1) describe the dynamics of the system consisting of two parts: the continuous part (for $t \neq \tau_k$) described by the ordinary differential equations and the discrete part (at times τ_k) when the solutions of the system undergo stepwise jumps. We denote

$$B_H := \Big\{ x \in \mathbb{R}^n \colon \|x\| = \sqrt{x_1^2 + \cdots + x_n^2} < H \Big\},$$

$$G_k := \{(t,x) \in \mathbb{R}^{n+1} \colon \tau_{k-1} < t < \tau_k,\ \ x \in B_H\}, \qquad G := \bigcup_{k=1}^{\infty} G_k.$$

Let us state Assumptions (A1)–(A5) for system (2.1).

Assumption (A1). The function $f = (f_1, \dots, f_n) \colon G \to \mathbb{R}^n$ is continuous in each of the sets G_k, $k \in \mathbb{N}$, $f(t,0) \equiv 0$, and there exists a constant $L > 0$ such that

$$\|f(t,x) - f(t,y)\| \le L\|x - y\| \qquad \text{for} \quad (t,x) \in G, \quad (t,y) \in G, \quad x \in B_H, \quad y \in B_H.$$

Assumption (A2). The functions $I_k \colon B_H \to \mathbb{R}^n$, $k \in \mathbb{N}$, are continuous in B_H and $I_k(0) = 0$ for $k \in \mathbb{N}$.

Assumption (A3). There exists a constant $h \in (0,H)$ such that if $x \in B_h$, then $x + I_k(x) \in B_H$ for $k \in \mathbb{N}$.

Assumption (A4). The constants τ_k satisfy the conditions

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots, \qquad \lim_{k \to \infty} \tau_k = \infty.$$

Assumption (A5). The constants τ_k satisfy the condition: for all $T > 0$ and $t > 0$, the closed interval $[t, t+T]$ contains at most p constants τ_k and the number p depends only on T and is independent of t.

By $x(t,t_0,x_0)$, $t > t_0$, we denote the solution of system (2.1) such that $x(t_0,t_0,x_0) = x_0$ in the case $t_0 \ne \tau_k$, $k \in \mathbb{N}$. But if $t_0 = \tau_k$ for some natural number k, then by the expression $x(t,t_0,x_0)$ we shall mean $x(t, t_0+0, x_0 + I_k(x_0))$ (for $t > t_0$).

Under Assumptions (A1)–(A3) system (2.1) admits the trivial solution

$$x \equiv 0. \tag{2.2}$$

Definition 2.1. The trivial solution of system (2.1) is said to be *stable* if, for all $\varepsilon > 0$, $t_0 \in \mathbb{R}_+$, we can indicate a $\delta = \delta(\varepsilon, t_0) > 0$ such that if $\|x_0\| < \delta$, then $\|x(t,t_0,x_0)\| < \varepsilon$ for $t > t_0$.

Definition 2.2. The solution (2.2) of system (2.1) is said to be *attracting* if, for any $t_0 \in \mathbb{R}_+$, there exists an $\eta = \eta(t_0) > 0$ and, for all $\varepsilon > 0$ and $x_0 \in B_\eta$, there exists a $\sigma = \sigma(\varepsilon, t_0, x_0) > 0$ such that $\|x(t,t_0,x_0)\| < \varepsilon$ for all $t \ge t_0 + \sigma$.

Definition 2.3. The trivial solution of system (2.1) is said to be *asymptotically stable* if it is stable and attracting.

Definition 2.4. We say that a function $\omega \colon \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the *class* $\mathscr{K}$ ($\omega \in \mathscr{K}$) if it is continuous, strictly increasing, and $\omega(0) = 0$.

Following [4], we introduce some definitions.

Definition 2.5. We say that a function $V \colon \mathbb{R}_+ \times B_H \to \mathbb{R}$ belongs to the *class* $\mathscr{V}_0$ ($V \in \mathscr{V}_0$) if V is continuous on any of the sets G_k, $V(t,0) \equiv 0$ for $t \in \mathbb{R}_+$, the finite limits

$$V(t_0 - 0, x_0) = \lim_{\substack{(t,x) \to (t_0,x_0) \\ (t,x) \in G_k}} V(t,x), \qquad V(t_0 + 0, x_0) = \lim_{\substack{(t,x) \to (t_0,x_0) \\ (t,x) \in G_{k+1}}} V(t,x)$$

exist, and the relation $V(t_0 - 0, x_0) = V(t_0, x_0)$ holds.

Suppose that $f(t,x)$ is a C^{j-1}-function: $f\colon G \to \mathbb{R}^n$, and V is a C^j-function: $V\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$. Let us define $V_s\colon \mathbb{R}_+ \times B_H \to \mathbb{R}^n$:

$$V_s(t,x) := \frac{\partial V_{s-1}}{\partial t} + \sum_{i=1}^{n} \frac{\partial V_{s-1}}{\partial x_i} f_i(t,x), \qquad s = 1,2,\dots,j,$$

for $t \neq \tau_k$ and $V_s(\tau_k, x) = V_s(\tau_k - 0, x)$, where $V_0(t,x) := V(t,x)$. In particular,

$$V_1(t,x) = \frac{dV}{dx}\bigg|_{(2.1)} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x).$$

Definition 2.6. We say that a function $V\colon \mathbb{R}_+ \times B_H \to \mathbb{R}$ belongs to the *class* $\mathscr{V}_m$ ($V \in \mathscr{V}_m$) if V is m times continuously differentiable on any of the sets G_k and the finite limits

$$V_r(t_0 - 0, x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\ (t,x)\in G_k}} V_r(t,x), \quad V_r(t_0 + 0, x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\ (t,x)\in G_{k+1}}} V_r(t,x), \qquad r = 0,1,\dots,m.$$

exist. Here $V_0 = V$.

It was shown in [10], [18] that if system (2.1) is such that Assumptions (A1)–(A4) hold and there exist functions $V \in \mathscr{V}_1$, $a, c \in \mathscr{K}$ for which

$$\begin{gathered} V(t,x) \ge a(\|x\|) \qquad \text{for} \quad (t,x) \in \mathbb{R}_+ \times B_H, \\ \frac{dV}{dt}\bigg|_{(2.1)} \le -c(\|x\|) \qquad \text{for} \quad (t,x) \in G, \\ V(\tau_k + 0, I_k(x) + x) - V(\tau_k, x) \le 0 \qquad \text{for} \quad k \in \mathbb{N}, \end{gathered} \tag{2.3}$$

then the solution (2.2) of system (2.1) is asymptotically stable.

The goal of this paper is to derive less stringent conditions for asymptotic stability in the situation where the function V satisfies the condition $dV/dt \le 0$ instead of (2.3).

In what follows, we shall need the following assertion.

Lemma 2.1. *Suppose that $h(t)$ is a scalar function having points of discontinuity of the first kind at*

$$t = b_1, \quad t = b_2, \quad \dots, \qquad \text{where} \quad 0 < b_1 < b_2 < \cdots, \quad \lim_{i\to\infty} b_i = \infty,$$

$h(b_i - 0) = h(b_i)$, the function $h(t)$ is $j+1$ times continuously differentiable in each of the intervals (b_i, b_{i+1}), its derivatives $h'(t), h''(t), \dots, h^{(j+1)}$ are bounded for $t \in \bigcup_{i=1}^{\infty}(b_i, b_{i+1}]$, the constants b_i satisfy Assumption (A5), *and the following limit relations hold*:

$$\lim_{i\to\infty}[h(b_i + 0) - h(b_i - 0)] = 0, \qquad \lim_{i\to\infty}\big[h^{(r)}(b_i + 0) - h^{(r)}(b_i - 0)\big] = 0, \quad r = 1,\dots,j.$$

If

$$\lim_{t\to\infty} h(t) = 0, \tag{2.4}$$

then

$$\lim_{t\to\infty} h^{(r)}(t) = 0, \qquad r = 1,2,\dots,j.$$

Remark 2.1. By derivatives of any order at the points b_i we shall mean left derivatives.

Proof the lemmas. Let us first show that

$$\lim_{t\to\infty} h'(t) = 0. \tag{2.5}$$

Assume the converse: suppose that there exists a $\xi > 0$ and instants of time $T_m \in \mathbb{R}_+$, $m \in \mathbb{N}$ such that $T_m \to \infty$ as $m \to \infty$ and $|h'(T_m)| \geq 2\xi$. This implies that either $h'(T_m) \geq 2\xi$, or $h'(T_m) \leq -2\xi$. Since $h''(t)$ is bounded and the condition

$$\lim_{i\to\infty}[h'(b_i + 0) - h'(b_i - 0)] = 0$$

holds, it follows that there exists a $\zeta > 0$ and an $M_1 \in \mathbb{N}$ such that

$$h'(t) \geq \frac{3}{2}\xi \tag{2.6}$$

or

$$h'(t) \leq -\frac{3}{2}\xi \tag{2.7}$$

for

$$t \in [T_m - \zeta, T_m + \zeta], \qquad m \geq M_1.$$

Hence, using the conditions of the lemma, we find that there exists an $M_2 \in \mathbb{N}$, $M_2 \geq M_1$, such that, for $m \geq M_2$, the following inequality is satisfied:

$$h(T_m + \zeta) \geq h(T_m) + \xi\zeta \tag{2.8}$$

in case (2.6) or the inequality

$$h(T_m + \zeta) \leq h(T_m) - \xi\zeta \tag{2.9}$$

in case (2.7). It follows from the limit relation (2.4) that there exists an $M \geq T_{M_2} > 0$ such that $|h(t)| < \xi\zeta/2$ for $t \geq M$. On the other hand, inequalities (2.8), (2.9) imply the following inequalities:

$$|h(T_m + \zeta)| \geq \frac{1}{2}\xi\zeta \qquad \text{for} \quad m \in \mathbb{N}, \quad T_m \geq M.$$

But these inequalities contradict condition (2.4). The resulting contradiction proves the limit relation (2.5).

Similarly, we can show that

$$\lim_{t\to\infty} h''(t) = 0, \quad \dots, \quad \lim_{t\to\infty} h^{(j)}(t) = 0.$$

The lemma is proved. $\square$

3. MAIN RESULTS

Theorem 3.1. *Suppose that system* (2.1) *is such that* $f \in C^j$ *for* $(t,x) \in G$, *Assumptions* (A1)–(A4) *hold, and there exist* $V \in \mathscr{V}_{j+1}$, $w_1 \in \mathscr{K}$, $w_2 \in \mathscr{K}$ *such that*

(A) $V(t,x) \geq w_1(\|x\|)$ *for* $(t,x) \in \mathbb{R}_+ \times B_H$;
(B) $V_1(t,x) = dV/dt \leq 0$ *for* $(t,x) \in G$;
(C) *the* $V_s(t,x)$, $s = 1,\dots,j+1$, *are bounded for* $(t,x) \in G$;
(D) $\sum_{s=1}^{j} V_s^2(t,x) \geq w_2(\|x\|)$;
(E) $V(\tau_k + 0, x + I_k(x)) - V(\tau_k, x) \leq 0$ *for* $x \in B_H$, $k \in \mathbb{N}$;
(F) *the constants* τ_k *satisfy Assumption* (A5) *and the following limit relations hold*:

$$\lim_{k\to\infty}[V_r(\tau_k + 0, x + I_k(x)) - V_r(\tau_k, x)] = 0, \qquad r = 0, 1, \dots, j,$$

uniformly in $x \in B_H$.

Then the solution (2.2) *of system* (2.1) *is asymptotically stable.*

Proof. In view of [18], it follows from conditions (A), (B), (E) that the solution (2.2) of system (2.1) is stable. Take arbitrary $\varepsilon \in (0, H)$, $t_0 \in \mathbb{R}_+$ and $x_0 \in B_\delta$, where $\delta > 0$ satisfies $\|x(t, t_0, x_0)\| < \varepsilon$ for $t > t_0$. Denote $v(t) = V(t, x(t, t_0, x_0))$[1]. The function $v(t)$ is positive and not increasing; therefore, the limit $\lim_{t\to\infty} v(t)$ exists and is nonnegative:

$$\lim_{t\to\infty} v(t) = \eta \geq 0. \tag{3.1}$$

Let us show that

$$\lim_{t\to\infty} v_1(t) = 0, \tag{3.2}$$

where $v_1(t) = V_1(t, x(t, t_0, x_0))$. Assume the converse, i.e., assume that the limit relation (3.2) does not hold. Then there exists a $\beta > 0$ and a sequence $\{T_i\}_{i=1}^{\infty}$ $(T_i \in \mathbb{R}_+,\ i \in \mathbb{N},\ T_{i+1} > T_i$, and $T_i \to \infty$ as $i \to \infty)$ such that $v_1(T_i) \leq -2\beta < 0$. By conditions (A), (B), (C), (F) of the theorem, there exists a $\gamma > 0$ and an $M_1 \in \mathbb{N}$ such that $v_1(t) \leq -\beta$ for $t \in [T_i, T_i + \gamma]$, $i \geq M_1$. It follows from conditions (B), (E) that

$$v(T_i + \gamma) \leq v(T_i) - \beta\gamma \tag{3.3}$$

for $i \geq M_1$. Since

$$\lim_{i\to\infty} v(T_i) = \eta,$$

in view of (3.3), we have

$$\lim_{i\to\infty} v(T_i + \gamma) \leq \eta - \beta\gamma.$$

But since $T_i + \gamma \to \infty$ as $i \to \infty$, the resulting inequality contradicts the limit relation (3.1). This contradiction proves relation (3.2).

Using the limit relation (3.2) and Lemma 2.1, we find that

$$\lim_{t\to\infty} \sum_{s=1}^{j} v_s^2(t) = 0, \tag{3.4}$$

where $v_s(t) = V_s(t, x(t, t_0, x_0))$. Let us now show that the zero solution of system (2.1) is attracting. To do this, we must show that

$$\lim_{t\to\infty} \|x(t, t_0, x_0)\| = 0. \tag{3.5}$$

Assume the converse: suppose that there exists an $\alpha > 0$ and a sequence $\{t_i\}$ $(i \in \mathbb{N},\ t_i \to \infty$ as $i \to \infty)$ such that $\|x(t_i, t_0, x_0)\| \geq \alpha$. It follows from condition (D) that

$$\sum_{s=1}^{j} v_s^2(t_i) \geq w_2(\|x(t_i, t_0, x_0)\|) \geq w_2(\alpha) > 0.$$

The resulting inequality contradicts (3.4); hence the limit relation (3.5) holds. The theorem is proved. ☐

Remark 3.1. If, in this theorem, we set $j = 1$, $I_k(x) \equiv 0$, $k \in \mathbb{N}$, $x \in B_H$, then we obtain the well-known Marachkov theorem [19].

Denote

$$Q_h := \{(t, x) \in \mathbb{R}_+ \times B_h \colon V(t, x) > 0\}.$$

Theorem 3.2. *Suppose that system* (2.1) *satisfies* $f \in C^j$ *for* $(t, x) \in G$, *Assumptions* (A1)–(A4) *hold, and there exist functions* $V \in \mathscr{V}_{j+1}$ *and* $w \in \mathscr{K}$ *such that*

(a) *for all* $t \in \mathbb{R}_+$, $\varepsilon \in (0, H)$, *the set* Q_ε *is not empty*;

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[1] By $v(\tau_k)$ we mean, as before, $v(\tau_k - 0)$.

(b) $V(t,x), V_1(t,x), \dots, V_{j+1}(t,x)$ *are bounded in* Q_H;
(c) $dV/dt = V_1(t,x) \geq 0$ *in* Q_H;
(d) $\sum_{s=1}^{j} V_s^2(t,x) \geq w(V(t,x))$;
(f) $\Delta V|_{t=\tau_k} = V(\tau_k + 0, x + I_k(x)) - V(\tau_k, x) \geq 0$ *for* $x \in B_H$, $k \in \mathbb{N}$;
(f) *condition* (F) *of Theorem* 3.1 *holds.*

Then the solution (2.2) *of system* (2.1) *is unstable.*

Proof. Choose arbitrary $\varepsilon \in (0, H)$, $t_0 \in \mathbb{R}_+$, and suppose that δ is an arbitrarily small positive number. Let us show that there exists an $x_0 \in B_\delta$ and a $t_1 > t_0$ such that $\|x(t_1, t_0, x_0)\| \geq \varepsilon$. We choose $x_0 \in B_\delta$ so that $V(t_0, x_0) > 0$. Such a choice is possible by condition (a).

Assume the converse: for any $t > t_0$, the following inequality holds:

$$\|x(t, t_0, x_0)\| < \varepsilon, \tag{3.6}$$

whence it follows that $(t,x) \in Q_\varepsilon$ and the function $V(t, x(t, t_0, x_0))$ is bounded. From conditions (c), (f), we obtain

$$v(t) = V(t, x(t, t_0, x_0)) \geq V(t_0, x_0) = \beta > 0.$$

The function $v(t)$ is bounded and nondecreasing; therefore, the limit

$$\lim_{t\to\infty} v(t) = \eta \geq \beta > 0$$

exists. As in the proof of Theorem 3.1, we can show that

$$\lim_{t\to\infty} v_s(t) = 0, \quad s = 1, \dots, j,$$

where $v_s = V_s(t, x(t, t_0, x_0))$; hence we see that the limit relation (3.4) holds. It follows from assumption (d) that

$$\sum_{s=1}^{j} v_s^2(t) = \sum_{s=1}^{j} V_s^2(t, x(t, t_0, x_0)) \geq w(V(t, x(t, t_0, x_0)) \geq w(\beta) > 0 \tag{3.7}$$

for $t \geq t_0$. However, relations (3.4) and (3.7) contradict each other. This contradiction shows that inequality (3.6) is false, i.e., the solution (2.2) of system (2.1) is unstable. $\square$

Example 3.1. Consider the following system of differential equations with impulse action:

$$\begin{gathered} \frac{dx}{dt} = -y - a(t)x, \qquad \frac{dy}{dt} = x, \qquad t \neq \tau_k, k = 1, 2, \dots, \\ \Delta x|_{t=\tau_k} = -\frac{x}{k}, \qquad \Delta y|_{t=\tau_k} = 0, \end{gathered} \tag{3.8}$$

where

$$a(t) = 2 - \cos 2\pi t - \cos(2\pi \log_2(t+1)), \qquad \tau_k = 3k, \quad k \in \mathbb{N}.$$

System (3.8) admits the trivial solution

$$x = 0, \qquad y = 0. \tag{3.9}$$

To study the stability of this solution, we use the Lyapunov function

$$V = \frac{1}{2}(x^2 + y^2),$$

successively obtaining

$$\begin{aligned}
V_1 &= \frac{dV}{dt} = -a(t)x^2, \qquad V_2 = \frac{dV_1}{dt} = (2a^2 - a')x^2 + 2axy,\\
V_3 &= \frac{dV_2}{dt} = (6aa' - a'' - 4a^3 + 2a)x^2 + (4a' - 6a^2)xy - 2ay^2,\\
V_4 &= \frac{dV_3}{dt} = (6a'^2 + 8aa'' - a''' - 24a^2a' + 6a' + 8a^4 - 10a^2)x^2\\
&\qquad + (-28aa' + 6a'' + 14a^3 - 8a)xy + (6a^2 - 6a')y^2,\\
V_5 &= \frac{dV_4}{dt} = (20a'a'' + 10aa''' - a'''' - 60aa'^2 - 40a^2a'' + 12a''\\
&\qquad + 80a^3a' - 60aa' - 16a^5 + 34a^3 - 8a)x^2\\
&\qquad + (-40a'^2 - 50aa'' + 8a''' + 118a^2a' - 32a' - 30a^4 + 40a^2)xy\\
&\qquad + (40aa' - 12a'' - 14a^3 + 8a)y^2.
\end{aligned}$$

The derivatives of the function $a(t)$ are of the form

$$a^{(s)}(t) = \alpha_s(t) + \beta_s \frac{\sin(2\pi \log_2(t+1))}{(t+1)^s} + \gamma_s \frac{\cos(2\pi \log_2(t+1))}{(t+1)^s},$$

where β_s, γ_s, $s \in \mathbb{N}$, are constants, with $\beta_1 = 2\pi/\ln 2$, $\gamma_1 = 0$,

$$\alpha_1 = 2\pi \sin 2\pi t, \quad \alpha_2 = (2\pi)^2 \cos 2\pi t, \quad \alpha_3 = -(2\pi)^3 \sin 2\pi t, \quad \alpha_4 = -(2\pi)^4 \cos 2\pi t.$$

The function $a(t)$ is nonnegative, with $a(t) = 0$ for $t = t_n = 2^n - 1$, $(n = 0, 1, 2, \dots)$.

Consider the function $W = V_1^2 + V_2^2 + V_3^2 + V_4^2 + V_5^2$. Let us show that $W(t, x, y)$ is a positive definite function with respect to x and y. Let us first show that $W(t_n, x, y)$ is a positive definite function. Taking the relations $a(t_n) = 0$, $a'(t_n) = 0$ into account, we successively obtain:

$$\begin{aligned}
&V_1(t_n, x, y) = 0, \qquad V_2(t_n, x, y) = 0, \qquad V_3(t_n, x, y) = -a''(t_n)x^2,\\
&V_4(t_n, x, y) = -a'''(t_n)x^2 + 6a''(t_n)xy,\\
&V_5(t_n, x, y) = (-a''''(t_n) + 12a''(t_n))x^2 + 8a'''(t_n)xy - 12a''(t_n)y^2.
\end{aligned}$$

Note that $\gamma_2 = (2\pi)^2/\ln^2 2 > 0$; hence we find

$$a''(t_n) > (2\pi)^2. \tag{3.10}$$

Using (3.10), we obtain

$$W(t_n, x, y) \ge V_3^2(t_n, x, y) + V_5^2(t_n, x, y) \ge (2\pi)^4 x^4 + V_5^2(t_n, x, y).$$

Using the relation $V_5(t_n, 0, y) = -12a''(t_n)y^2$, and inequality (3.10), we see that $W(t_n, x, y) > 2w(x^2 + y^2)$, where $w \in \mathscr{K}$. The functions $a(t)$ and $a^{(s)}(t)$, $s = 1, 2, 3, 4$, are uniformly continuous for $t \in \mathbb{R}_+$; therefore, there exists an $\varepsilon > 0$ such that

$$W(t, x, y) \ge w(x^2 + y^2) \tag{3.11}$$

for $t_n - \varepsilon \le t \le t_n + \varepsilon$. Let us show that, for

$$t \notin [t_n - \varepsilon, t_n + \varepsilon], \tag{3.12}$$

the function W also satisfies property (3.11) for any $n \in \mathbb{N}$. Note that, under condition (3.12), there exists an $\omega > 0$ such that $a(t) > \omega$. Let us estimate W for such a t:

$$\begin{aligned}
W(t, x, y)\big|_{t \notin [t_n - \varepsilon, t_n + \varepsilon]} &\ge (V_1^2 + V_3^2)\big|_{t \notin [t_n - \varepsilon, t_n + \varepsilon]} \ge \psi(t, x, y)\\
&= \omega^2 x^4 + [(6aa' - a'' - 4a^3 + 2a)x^2 + (4a' - 6a^2)xy - 2ay^2]^2.
\end{aligned}$$

Let us find $\psi(t,x,y)$ for $x=0$:

$$\psi(t,0,y)=4a^2(t)y^4>4\omega^2y^4.$$

Since $a(t)$, $a'(t)$, $a''(t)$ are bounded functions for $t\in\mathbb{R}_+$, we find that, for $t\notin[t_n-\varepsilon,t_n+\varepsilon]$, the function W is positive definite with respect to x, y. Therefore, there exists a $w\in\mathscr{K}$ such that inequality (3.11) holds for $t\in\mathbb{R}_+$, $(x,y)\in\mathbb{R}^2$. Let us now find

$$\Delta V|_{t=\tau_k}=\frac{1}{2}\big[(x+\Delta x|_{t=\tau_k})^2+(y+\Delta y|_{t=\tau_k})^2\big]-\frac{1}{2}(x^2+y^2)=-\frac{x^2}{k}\left(1-\frac{1}{2k}\right)\leq 0.$$

Note that V_1, V_2, V_3, V_4, V_5, $V_6:=dV_5/dt$ can be expressed as

$$V_m=F_m(t)x^2+P_m(t)xy+Q_m(t)y^2,\qquad m=1,2,3,4,5,6,$$

where F_m, P_m, Q_m are bounded continuous functions of t. Therefore, these functions are bounded for $x^2+y^2<H^2$, and

$$\begin{aligned}\Delta V_m|_{t=\tau_k}&=F_m(\tau_k)\big[(x+\Delta x|_{t=\tau_k})^2-x^2\big]+P_m(\tau_k)\big[(x+\Delta x|_{t=\tau_k})(y+\Delta y|_{t=\tau_k})-xy\big]\\&\quad+Q_m(\tau_k)\big[(y+\Delta y|_{t=\tau_k})^2-y^2\big]=-F_m(\tau_k)\frac{x^2}{k}\left(1-\frac{1}{2k}\right)-P_m(\tau_k)\frac{xy}{k},\end{aligned}$$

whence we obtain $\lim_{k\to\infty}\Delta V_m|_{t=\tau_k}=0$, $m=1,\dots,5$.

Thus, conditions (A) –(E) and (F) of Theorem 3.1 are satisfied, and we can conclude that the solution (3.9) of system (3.8) is asymptotically stable.

Example 3.2. Consider the following system of ordinary differential equations with impulse action

$$\begin{gathered}\frac{dx}{dt}=-y+a(t)x,\qquad\frac{dy}{dt}=x,\qquad t\neq\tau_k,\quad k=1,2,\dots,\\ \Delta x|_{t=\tau_k}=\frac{x}{k},\qquad\Delta y|_{t=\tau_k}=0,\end{gathered}\tag{3.13}$$

where $a(t)$ is the same as in the previous example. System (3.13) admits the zero solution (3.9). Let us show that this solution is unstable. Consider the function

$$V=\frac{1}{2}(x^2+y^2).$$

We obtain

$$\frac{dV}{dt}=a(t)x^2\geq 0,\qquad\Delta V\big|_{t=\tau_k}=\frac{x^2}{k}+\frac{x^2}{2k^2}\geq 0.$$

Just as in the example given above, we can show that there exists a $w\in\mathscr{K}$ such that

$$W=V_1^2+V_2^2+V_3^2+V_4^2+V_5^2\geq w(x^2+y^2).$$

Thus, all the conditions of Theorem 3.2 are satisfied; therefore, the zero solution of system (3.13) is unstable.

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