ON THE EXISTENCE OF LYAPUNOV FUNCTIONS IN PROBLEMS OF STABILITY OF INTEGRAL SETS

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A nonautonomous system of ordinary differential equations is considered. If this system admits a uniformly asymptotically stable integral set, then a neighborhood of this set contains a function similar to the Lyapunov function.

Consider a system of ordinary differential equations

$$\frac{dx}{dt} = X(t, x), \tag{1}$$

where $x,\ X \in R^n$ and $t \in I = [0; \infty)$. Assume that the conditions for the existence and uniqueness of solutions to system (1) hold for

$$(t, x) \in \Gamma_{H_1} = I \times B_{H_1}, \quad B_{H_1} = \{x \in R^n : \|x\| < H_1\}. \tag{2}$$

In what follows, we assume that $\|x\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$ and H_1 is a positive number. Let us introduce several definitions similar to those used in the books [1–6].

***Definition* 1.** *A set M in the space (t, x) is called integral if the inclusion $(t, x(t)) \in M,\ t \geq t_0,$ holds for any point $(t_0, x_0) \in M$; here, $x(t) = x(t, t_0, x_0)$ is a solution of system* (1) *with the initial data $x(t_0) = x_0$.*

Let $M \subset I \times R^n$. Denote by M_s an intersection of this set with a hyperplane $t = s$ and by $\rho(x, M_s)$ the distance between the point x and the set M_s. The set M is called periodic with period ω if

$$M_s = M_{s+\omega} \tag{3}$$

for any $s \in I$. We say that an integral set M is *independent of time* if (3) holds identically for all s and ω.

***Definition* 2.** *An integral set M of system* (1) *is called uniformly stable if, for all $t_0 \in I$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that the inequality $\rho(x_0, M_{t_0}) < \delta$ implies the inequality $\rho(x(t), M_t) < \varepsilon$ for $t \geq t_0$.*

Denote $S(M_t, r) = \{x \in R^n : \rho(x, M_t) < r\}$.

***Definition* 3.** *An integral set M is called uniformly attracting if, for some $\eta > 0$ and any $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon) > 0$ such that the inequality $\rho(x(t, t_0, x_0), M_t) < \varepsilon$ holds for all $t_0 \in I$, $x_0 \in S(M_{t_0}, \eta)$ and $t \geq t_0 + \sigma$.*

An integral set M is called uniformly asymptotically stable if it is uniformly stable and uniformly attracting.

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We say that the region $\{(t,x)\in R^{n+1}:\ t\in I,\ x\in S(M_t,\eta)\}$ *belongs to the domain of attraction of the integral set* M.

Following Lyapunov, we consider real functions $v(t,x)$ of the variables t and x defined and continuously differentiable in the domain $U_H(M)=\{(t,x)\in R^{n+1}: t\in I, x\in S(M_t,H)\}$ such that $U_H(M)\in\Gamma_{H_1}$. Unless otherwise stated, we suppose that

$$v(t,x)=0 \quad \text{for} \quad t\in I \quad \text{and} \quad x\in M_t. \tag{4}$$

***Definition* 4.** *A function* $v(t,x)$ *is called definitely positive with respect to the integral set* M *of system* (1) *if conditions* (4) *are satisfied and* $v(t,x)\ge a(\rho(x,M_t))$, $a\in K$, *where* K *is the Hahn function class* [5]. *Similarly, the function* $v(t,x)$ *is called definitely negative with respect to the set* M *if* $v(t,x)\le -a(\rho(x,M_t))$, $a\in K$.

***Definition* 5.** *A function* $v(t,x)$ *admits an infinitesimal upper bound with respect to* M *in the domain* $U_H(M)$ *if there exists a function* $b\in K$ *such that* $|v(t,x)|\le b(\rho(x,M_t))$.

Let us examine the problem of existence for a function v satisfying the conditions of a theorem similar to the Lyapunov theorem on the uniform asymptotic stability of equilibrium state. To establish the existence of a function of this sort, we use the method of Krasovskii [7]. Let U_H be a domain lying in Γ_{H_1} together with its closure $\overline{U}_H$. Below, we always assume that M is such that, for all positive h and H $(h<H)$, we have the inclusion $\overline{U}_h\subset U_H$. In addition, we assume that the right-hand sides of Eq. (1) satisfy the Lipschitz conditions in the variables x in any closed domain $\overline{U}_\lambda\subset\Gamma_{H_1}$, i.e.,

$$|X_i(t,x_1)-X_i(t,x_2)|\le \|x_1-x_2\| \quad (i=1,\ldots,n;\ L_\lambda=\text{const}). \tag{5}$$

Let T be a positive number. Assume that the entire trajectory determined by the solution $x(t,t_0,x_0)$ of system (1) lies in $\overline{U}_h$ for $0<t_0<t<t_0+\theta$, where $0<\theta\le T$.

Lemma 1. *Assume that* $\gamma>0$, $T>0$, *and* $0<h<H$ *are arbitrary given numbers. Then there exists a function* $V(t,x)$ *continuous with all its partial derivatives with respect to* $x_1,\ldots,x_n$, *and* t *in the region* Γ_{H_1} *and satisfying the conditions*

$$V(t,x)=0 \quad \textit{for} \quad \|x-x(t,t_0,x_0)\|\ge\gamma \quad \textit{and} \quad t\in R,$$

$$\frac{dV}{dt}=\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}X_i+\frac{\partial V}{\partial t}>d \quad \textit{for} \quad \|x-x(t,t_0,x_0)\|<\alpha \quad \textit{and} \quad t\in[t_0-\tau;\ t_0+\theta],$$

$$\frac{dV}{dt}\ge 0 \quad \textit{for} \quad \|x\|<\infty \quad \textit{and} \quad t\in(-\infty;\ t_0+\theta+\tau)$$

$$\textit{and for} \quad \|x-x(t_0+\theta,\ t_0,x_0)\|\ge\gamma \quad \textit{and} \quad t\in[t_0+\theta+\tau;\ t_0+\theta+2\tau],$$

$$V>d \quad \textit{for} \quad \|x-x(t,t_0,x_0)\|<\alpha \quad \textit{and} \quad t\in[t_0-\tau;\ t_0+\theta],$$

$V = 0 \qquad\qquad for \qquad t \in (-\infty;\ t_0 - 2\tau) \cup (t_0 + \theta + 2\tau;\ +\infty),$

where τ, α, *and* d *are positive constants whose bounds are determined by the choice of the numbers* γ, T, h, *and* H *but do not depend on the choice of the point* $x_0 \in S(M_{t_0}, h)$.

Consider an arc of the trajectory corresponding to the solution $x(t, t_0, x_0)$ of system (1) and lying in the region M_h ($\overline{M}_h \subset M_H$) for times $t \in [t_0 - \theta, t_0]$, where $0 < \theta \le T$. Then the following statement is true:

Lemma 2. *Assume that* $\gamma > 0$, $T > 0$, *and* $0 < h < H$ *are arbitrary given numbers. Then there exists a function* $V(t, x)$ *continuous with all its partial derivatives with respect to* $x_1, \dots, x_n$, *and* t *in the region* Γ_{H_1} *and satisfying the conditions*

$$V(t,x) = 0 \quad for \quad \|x - x(t, t_0, x_0)\| \ge \gamma \quad and \quad t \in R,$$

$$\frac{dV}{dt} > d \quad for \quad \|x - x(t, t_0, x_0)\| < \alpha \quad and \quad t \in [t_0 - \theta;\ t_0 + \tau],$$

$$\frac{dV}{dt} \ge 0 \quad for \quad \|x\| < \infty \quad and \quad t \in (t_0 - \theta - \tau;\ \infty)$$

$$and\ for \quad \|x - x(t, t_0, x_0)\| \ge \gamma \quad and \quad t \in [t_0 - \theta - 2\tau;\ t_0 - \theta - \tau],$$

$$V < -d \quad for \quad \|x - x(t, t_0, x_0)\| < \alpha \quad and \quad t \in [t_0 - \theta;\ t_0 + \tau],$$

$$V = 0 \quad for \quad t \in (-\infty;\ t_0 - \theta - 2\tau) \cup (t_0 + 2\tau;\ \infty),$$

where τ, α, *and* d *are positive constants whose bounds are determined by the choice of the numbers* γ, T, h, *and* H *but do not depend on the choice of the point* $x_0 \in S(M_{t_0}, h)$.

Remark **1.** In Lemmas 1 and 2, the functions $V(t, x)$ possess partial derivatives $\frac{\partial V}{\partial t}$ and $\frac{\partial V}{\partial x_i}$ of the first order, which are uniformly bounded by some constant N_0, i.e., $\left|\frac{\partial V}{\partial t}\right| < N_0$ and $\left|\frac{\partial V}{\partial x_i}\right| < N_0$, where the bound N_0 is determined only by the choice of the numbers γ, T, h, and H.

Remark **2.** Assume that the functions $X_i(t, x)$ are uniformly continuous in time t in each domain U_δ. Then, for every natural number k, one can indicate a constant $N_k > 0$ such that there exists a function V satisfying the conditions of Lemma 1 (or Lemma 2) and the inequalities

$$\left|\frac{\partial^k V}{\partial x_1^{k_1} \dots \partial x_n^{k_n} \partial t^{k_{n+1}}}\right| < N_k, \qquad k = 1, 2, \dots .$$

Assume that $0 < h < H$, $\lambda_0 = H$, $\lambda_k = 2^{-k} H$, $k = 1, 2, \dots$, and $R(k) = U_h \backslash U_{\lambda_k}$. Introduce the following definition:

Definition **6.** *We say that a property* (A) *holds in the domain* U_H *with respect to the integral set* M *if,* *for any* $0 < h < H$ *and any natural number* k, *one can find a number* $T_k > 0$ *such that there is no segment of the trajectory* $x(t) = x(t, t_0, x_0)$, *such that* $(t, x(t)) \in U_h$ *for* $t \in [t_0 - T_k;\ t_0 + T_k]$, $t_0 \geq T_k$, *and* $(t_0, x_0) \in R(k)$.

The property (A) with respect to M is a necessary condition for the existence of a function v admitting an infinitesimal upper bound with respect to M and such that dv/dt is of fixed sign with respect to M. We now prove this.

Theorem 1. *Assume that* U_h *is a domain lying with its closure in the domain* U_H. *If it is possible to construct (in each* U_h*) a function* $v(t, x)$ *admitting an infinitesimal upper bound with respect to* M *in* U_h *and having, by Eq. (1), a derivative of fixed sign with respect to* M, *then the property* (A) *is satisfied in the domain* U_H *with respect to* M.

Proof. Let $\overline{U}_h \subset U_H$. Assume that a function $v(t, x)$ having the properties indicated in the assertion of Theorem 1 is constructed in the domain U_h. Choose s such that $\lambda_s < h$ $(\overline{U}_{\lambda_s} \subset U_h)$, $(t_0, x_0) \in R(s)$, and $v(t_0, x_0) \geq 0$. First, we show that within the period $\Delta t = \theta_s$, where θ_s is a fixed positive number, all the points on the trajectory $x(t)$ satisfy the condition

$$\rho(x(t), M_t) > \lambda_s \exp(-nL_H\theta_s) = \mu_s, \tag{6}$$

where L_H is the Lipschitz constant. Let $y(t) = y(t, t_0, y_0) \in M_t$ be an arbitrary trajectory of system (1) belonging to the integral set. Further, we have

$$\frac{d}{dt}(\|x(t) - y(t)\|)^2 = 2\sum_{i=1}^{n}(x_i - y_i)\left(\frac{dx_i}{dt} - \frac{dy_i}{dt}\right) = 2\sum_{i=1}^{n}(x_i - y_i)(X_i(t, x) - X_i(t, y))$$

$$\leq 2\sum_{i=1}^{n}|x_i - y_i||X_i(t, x) - X_i(t, y)| \leq 2L_H\|x - y\|\sum_{i=1}^{n}|x_i - y_i| \leq 2nL_H\|x - y\|^2,$$

$$\frac{d}{dt}(\|x - y\|)^2 \geq -2L_H n\|x - y\|^2. \tag{7}$$

By integrating inequalities (7), we obtain

$$\|x_0 - y_0\| e^{-2L_H n(t - t_0)} \leq \|x(t) - y(t)\| \leq \|x_0 - y_0\| e^{2nL_H(t - t_0)}, \tag{8}$$

whence, by virtue of the condition $\|x_0 - y_0\| \geq \lambda_s$, we get inequality (6).

Choose an arbitrary positive r satisfying the conditions $r < h$ and $\overline{U}_r \subset U_h$. Assume, for definiteness, that the function dv/dt is definitely positive with respect to M in U_h:

$$|v(t, x)| \leq b(\rho(x, M_t)) \quad \text{and} \quad \frac{dv}{dt} \geq c(\rho(x, M_t)), \quad b, c \in K. \tag{9}$$

Taking conditions (6) and (9) into account and assuming that an arc of the trajectory given by the solution $x(t, t_0, x_0)$ of system (1) lies in U_r for $t \in [t_0;\ t_0 + \theta_s]$, we obtain

$$v(t_0+\theta_s,\ x(t_0+\theta_s,\ t_0,\ x_0)) \geq \theta_s c(\mu_s). \tag{10}$$

Since the function v possesses an infinitesimal upper bound with respect to M, there exists a natural number $N(s)$ such that $b(\lambda_{N(s)}) < c(\mu_s)\theta_s$. The role of $N(s)$ can be played by an arbitrary natural number satisfying the inequality $N(s) > \log_2 H - \log_2 b^{-1}(c(\mu_s)\theta_s)$.

Condition (10) and the definite positivity of dv/dt enable us to conclude that

$$(t, x(t)) \in U_h \backslash U_{\lambda_{N(s)}}$$

for any $t > 0$; this and the second inequality in (9) imply that $dv(t,x(t))/dt \geq c(\lambda_{N(s)})$. Hence, the inequality $v(t,x(t)) \geq c(\mu_s)\theta_s + c(\lambda_{N(s)})(t-\theta_s)$ holds for $t > \theta_s$. Denote $M_1 = b(r)$. It follows from the last inequality that the arc of the considered trajectory $(t, x(t))$ corresponding to $t \in [t_0;\ t_0+T_s]$ may not lie entirely in the set U_r; here,

$$T_s = \frac{b(r)-c(\mu_s)\theta_s}{c(\lambda_{N(s)})} + \theta_s.$$

If $v(t_0, x_0) < 0$, we act as above and investigate the variation of the function $v(t_0, x(t, t_0, x_0))$ as time t decreases. We can repeat all the estimates given above with t replaced by $-t$; as a result, we conclude that the region U_r may not contain the entire arc of the considered trajectory $(t, x(t, t_0, x_0))$ for $t \in [t_0 - T_s;\ t_0]$, where T_s has the same sense as before. This completes the proof of the theorem, because U_r is an arbitrary *a priori* fixed domain in U_h.

Thus, we have shown that property (A) with respect to M is a necessary condition for the existence of the function v, which admits an infinitesimal upper bound and has the derivative dv/dt of fixed sign with respect to M.

Let us show that this property is not only necessary but also a sufficient condition for the existence of the function v with the indicated properties.

Theorem 2. *Assume that condition* (A) *with respect to* M *holds in the region* U_H $(\overline{U}_H \subset \Gamma_{H_1})$. *Then, for any positive number* h, $h < H$, *the following statements are valid:*

(i) *There exists a function* $v(t, x)$ *having, in view of Eq.* (1), *the derivative* dv/dt *of fixed sign with respect to* M *in the domain* U_h *and admitting in this domain an infinitesimal upper bound with respect to* M; *furthermore, the function* $v(t, x)$ *has the partial derivatives* $\partial v/\partial t$, $\partial v/\partial x_i$, $i = 1, 2, \ldots, n$, *continuous and uniformly bounded in the region* U_h.

(ii) *If the functions* X_i *are uniformly continuous in time in the region* U_H *for* $t > 0$, *then the function* $v(t,x)$ *possesses the continuous partial derivatives of an arbitrary order with respect to all arguments; moreover, all these derivatives are uniformly bounded in* U_h *(each derivative is bounded by its own constant).*

(iii) *If* X_i *are periodic functions of time* t *with the same period* ω, M *is the periodic integral set with period* ω, *and condition* (A) *with respect to* M *is satisfied, then the function* $v(t, x)$ *exists in the region* U_h *and is periodic in* t *with period* ω; *if the functions* X_i *and the set* M *do not depend on time explicitly and condition* (A) *with respect to* M *is satisfied, then the function* $v(x)$ *exists and does not depend on* t *explicitly.*

Proof. Denote by γ a natural number satisfying the inequality $\gamma < \frac{1}{3}(H-h)$. Let k be the first natural number such that $\lambda_k < h$. As before, we denote $R(s) = U_h \backslash U_{\lambda_s}$, $s \geq k$. First, let us show that, for any number $s \geq k$, one can find a number $N(s)$ such that the arc of the trajectory corresponding to the solution $x(t, t_0, x_0)$ and lying inside the domain U_H for $t \in [t_0; t^*]$ (or for $t \in [t^*; t_0]$), where $|t^* - t_0| \leq T_s$, has no points inside the domain $U_{N(s)}$ provided that $(t_0, x_0) \in R(s)$. Here and below, when saying "an arc of the trajectory $x(t)$ lies in the domain U_H," we mean that $(t, x(t)) \in U_H$. It follows from inequalities (8) that

$$\| x(t) - y(t) \| \geq \| x_0 - y_0 \| \exp(-nL_H T_s).$$

The condition $(t_0, x_0) \in R(s)$ gives $\| x_0 - y_0 \| \geq 2^{-s} H$. Since $y(t)$ is an arbitrary trajectory in M, this proves the assertion formulated above provided that the role of $N(s)$ is played by the least natural number satisfying the inequality $2^{-N(s)} < 2^{-s} \exp(-nL_H T_s)$.

Let γ_s be a number satisfying the inequalities

$$\gamma_s \leq 2^{-N(s)-2} H, \quad \gamma_s \leq \gamma.$$

Let us construct the function $v(t, x)$. Consider a point $(t_0, x_0) \in R(s)$ for $s \geq k$. Since condition (A) with respect to M is satisfied, one can indicate a number θ such that $(t_0 + \theta, x(t_0 + \theta, t_0, x_0)) \in U_{h+\gamma}$, where $|\theta| \leq T_s$ (or $t_0 < T_s$). Assume, for definiteness, that $\theta > 0$. Then, according to Lemma 1, there exists a continuous function $V = v(t, x, t_0, x_0)$ defined in the domain $R \times B_{H_1}$ with the continuous uniformly bounded derivatives $\partial v / \partial t$ and $\partial v / \partial x_i$, $i = 1, 2, \ldots, n$. Moreover, by setting $\gamma = \gamma_s$ in Lemma 1, we guarantee, in view of the the choice of γ_s, the validity of the following conditions:

$$
\begin{aligned}
&v(t, x, t_0, x_0) = 0 && \text{in} \quad U_{\lambda_{N(s)+1}} \quad \text{and} \quad \Gamma_{H_1} \backslash U_{h+2\gamma}, \\
&\frac{dv(t, x(t), t_0, x_0)}{dt} \geq 0 && \text{in} \quad U_h, \\
&\frac{dv(t, x(t), t_0, x_0)}{dt} \geq d_s && \text{in} \quad \| x - x(t, t_0, x_0) \| < \alpha_s \quad \text{for} \quad t \in (t_0 - \tau_s;\, t_0 + \tau_s),
\end{aligned}
\tag{11}
$$

where positive numbers α_s, d_s, and τ_s depend only on s but not on the choice of the point $(t_0, x_0) \in R(s)$; the symbol $\frac{dv(t, x(t), t_0, x_0)}{dt}$ denotes the total derivative of the function $v(t, x(t), t_0, x_0)$; $x(t)$ is a solution of the differential equations (1). Taking the fact that the derivatives dx_i / dt are uniformly bounded on the set $U_{h+\gamma}$ and inequalities (11) into account, we can guarantee, by reducing the numbers τ_s and α_s, the validity of the inequality

$$\frac{dv(t, x(t), t_0, x_0)}{dt} > d_s \quad \text{for} \quad \| x - x_0 \| < \alpha_s \quad \text{and} \quad t \in (t_0 - \tau_s;\, t_0 + \tau_s). \tag{12}$$

Below, we assume that condition (12) is satisfied. If condition (A) with respect to M is satisfied for the point $(t_0, x_0) \in R(s)$ and $t < t_0$, then it is necessary to investigate the function $v(t, x, t_0, x_0) = V$ in Lemma 2 by using the same procedure (in the case of $t_0 < T_s$, one should also use Lemma 2 with $\theta = -t_0$).

The intersection of the set $R(s)$ with the hyperplane $t = t_0$ is compact for any $t_0 \in I$. Hence, one can indicate a finite set of points $x_0^{(l)}$, $l = 1, 2, \ldots, N_s$ from this set such that the system of neighborhoods $\| x - x_0^{(l)} \| < \alpha_s$,

$l = 1, 2, \ldots, N_s$, covers this intersection. Denote $t_0^{(k)} = \frac{1}{2} k \tau_k$, $k = 0, 1, \ldots$. For every pair $(t_0^{(k)}, x_0^{(l)})$, we construct a function $v(t, x, t_0^{(k)}, x_0^{(l)})$ according to the rule given above.

Consider a function

$$v_s(t, x) = \sum_{k=0}^{\infty} \sum_{l=1}^{N_s} v(t, x, t_0^{(k)}, x_0^{(l)}). \tag{13}$$

This function has the following properties: $v_s(t, x) = 0$ in the domain $U_{N(s)+1}$; the first partial derivatives of the function $v_s(t, x)$ are continuous and bounded with respect to all variables in the domain Γ_{H_1}. Indeed, a neighborhood of any point $(t, x) \in \Gamma_{H_1}$ contains, for any $t > 0$, at most finitely many nonzero terms on the right-hand side of equality (13), and the number of points is bounded by a constant independent of the choice of the point. In the domain $R(s)$, the derivative dv_s/dt of the function $v_s(t, x)$ is definitely positive with respect to M because every point $(t, x) \in R(s)$ belongs, at least, to one neighborhood of the form $\| x - x_0^{(l)} \| < \alpha_s$, $t \in [t_0^{(k)}; t_0^{(k+1)}]$, where inequality (12) holds for the corresponding term in (13).

Assume that the inequalities

$$\left| \frac{\partial v_s}{\partial x_i} \right| < P_s, \quad \left| \frac{\partial v_s}{\partial t} \right| < P_s, \quad | v_s | < P_s \tag{14}$$

hold. The function

$$v(t, x) = \sum_{s=k}^{\infty} \frac{1}{2^s P_s} v_s(t, x) \tag{15}$$

satisfies all the conditions of the theorem. Indeed, the series on the right-hand side of equality (15) and the series

$$\frac{\partial v}{\partial x_i} = \sum_{s=k}^{\infty} \frac{1}{2^s P_s} \frac{\partial v_s}{\partial x_i} \quad \text{and} \quad \frac{\partial v}{\partial t} = \sum_{s=k}^{\infty} \frac{1}{2^s P_s} \frac{\partial v_s}{\partial t}$$

are absolutely and uniformly convergent in the region Γ_{H_1} in view of inequalities (14). This proves that the function v exists and is continuously differentiable. The existence of the infinitesimal upper bound of the function $v(t, x)$ and the positive definiteness of the function dv/dt (both with respect to M) follow immediately from the properties of the functions v_s established above. The first assertion of the theorem is proved.

Let us prove the second assertion of theorem. If the functions $X_i(t, x)$ are uniformly continuous in t in the region Γ_{H_1}, then, by virtue of Remark 2 to Lemmas 1 and 2, the partial derivatives of the functions $V = v(t, x, t_0, x_0)$ of any order exist and are uniformly bounded with respect to all variables. However, in this case, the functions v_s also possess the continuous partial derivatives of any order with respect to all variables, and the following inequalities hold:

$$\left| \frac{\partial^k v_s}{\partial x_1^{k_1} \ldots \partial x_n^{k_n} \partial t^{k_{n+1}}} \right| < P_s^{(k)}. \tag{16}$$

Consider function (15) with the numbers P_s defined in the following way:

$$P_s = \max P_\mu^{(k)} \quad \text{for} \quad \mu = 1, 2, \ldots, s; \quad k = 1, 2, \ldots, s. \tag{17}$$

The function v given by (15) is defined in the domain Γ_{H_1} and has continuous bounded partial derivatives of any order with respect to all variables. Indeed, by virtue of inequalities (16) and the choice of numbers P_s in (17), the series on the right-hand side of (15) and all the series composed of the partial derivatives of an arbitrary order are uniformly and absolutely convergent in the region Γ_{H_1}. This proves the second assertion of the theorem.

We now prove the third assertion of the theorem. Let M be a periodic integral set with period ω and let $X_i(t, x)$ be ω-periodic functions. In this case, summation in expression (13) for v_s can be carried out over all values of k between $-\infty$ and $+\infty$, and the number τ_s in (12) may be regarded as a divisor of the number ω. Then the functions $v_s(t, x)$ are periodic in time t with period ω and preserve the other properties established earlier, i.e., the function $v(t, x)$ determined by expression (15) is also ω-periodic in time.

Assume that the right-hand sides of Eq. (1) do not depend on time explicitly. In this case, when constructing the function v_s, we first choose the numbers $\tau_s > 0$ to be divisors of a number $\omega > 0$. The functions v_s obtained as a result are denoted by $v_s^{(1)}(t, x)$. Consider a sequence of functions $l^{-1} v_s^{(l)}(t, x)$, $l = 1, 2, \ldots$, where v_s are constructed by using the numbers $\tau_s l^{-1}$ instead of τ_s. Clearly, the functions $v_s^{(l)}(t, x)$ are periodic in time with period ωl^{-1}. In addition, the functions $v_s^{(l)} / l$ are uniformly bounded in l in the domain Γ_{H_1}. The functions

$$\frac{1}{l} \frac{\partial^k v_s^{(l)}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n} \partial t^{k_{n+1}}}$$

are also uniformly bounded in l (each function is bounded by its own constant). Indeed, if the number of nonzero terms $v(t, x, t_0, x_0)$ in the formula for $v_s^{(1)}$ can be bounded by a certain constant N in the neighborhood of every point $(t, x) \in \Gamma_{H_1}$, then the number of nonzero terms in the formula for $v_s^{(l)}(t, x)$ can be bounded by the constant Nl in the neighborhood of the same point. Since the expression $\frac{v_s^{(l)}}{l}$ contains, in this case, the factor $1/l$, the assertion concerning the uniform boundedness of $\frac{v_s^{(l)}}{l}$ and all partial derivatives of $\frac{v_s^{(l)}}{l}$ is, in fact, proved. Every function $\frac{v_s^{(l)}}{l}$ possesses in Γ_{H_1} the derivative $\frac{1}{l}\frac{dv_s^{(l)}}{dt}$, which is definitely positive with respect to M and satisfies the inequality

$$\frac{1}{l} \frac{dv_s^{(l)}}{dt} > d_s > 0, \qquad d_s = \text{const}, \tag{18}$$

uniformly in l. Indeed, at any point $(t, x) \in R(t)$, the number of terms $v(t, x, t_0, x_0)$ in relation (13) for $v_s^{(l)}$, which satisfy inequality (12), can be bounded from below by l. This proves inequality (18). The functions $\frac{v_s^{(l)}}{l}$, as well as all the functions

$$\frac{1}{l} \frac{\partial^k v_s^{(l)}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n} \partial t^{k_{n+1}}},$$

form a family of uniformly bounded equicontinuous functions. Therefore, one can construct a subsequence $v^{(l_\nu)}(t, x)$ such that the functions

$$v_s^{(l_\nu)} \quad \text{and} \quad \frac{\partial^k v_s^{(l_\nu)}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n} \partial t^{k_{n+1}}}$$

converge uniformly in Γ_{H_1} to certain functions

$$v_s \quad \text{and} \quad \frac{\partial^k v_s}{\partial x_1^{k_1} \ldots \partial x_n^{k_n} \partial t^{k_{n+1}}}.$$

Clearly, the limiting function v_s does not depend on time t explicitly; moreover, this function satisfies the inequality $dv_s/dt \geq d_s$ for $(t, x) \in R(s)$. Function (15) with P_s chosen according to (17) is, clearly, a stationary function satisfying all conditions of the theorem. Theorem 2 is proved.

Theorem 3. *If the integral set M of the system of differential equations* (1) *is uniformly asymptotically stable with respect to M and the domain U_h lies in the domain of attraction, then the domain U_h contains a function $v(t, x)$ having, by virtue of Eq. (1), the derivative dv/dt definitely negative with respect to M. The function v is definitely positive with respect to M, admits an infinitesimal upper bound with respect to M, and possesses uniformly bounded continuous partial derivatives of the first order with respect to all variables in this domain.*

If the functions $X_i(t, x)$ are uniformly continuous in time $t \in I$ in the domain U_{h_1}, $h_1 > h$, then the function $v(t, x)$ possesses the partial derivatives of an arbitrary order with respect to all variables, and these derivatives are uniformly bounded in the domain U_h (each is bounded by its own constant).

If M and $X_i(t, x)$ are periodic with period ω or do not depend on time, then one can construct the function v as periodic with period ω or independent of time, respectively.

Proof. First, we show that there exists a positive number $\lambda > h$ such that M is uniformly asymptotically stable and U_λ belongs to the domain of attraction. Assume that $0 < \eta_0 < \frac{1}{2} h$ and $t_0 \in I$. By Definition 3, there exists a number $H > 0$ such that $\overline{U}_H \subset \Gamma_{H_1}$ and $x(t, t_0, x_0) \in S(M_t, H)$ for $t \geq t_0$, provided that $x_0 \in S(M_{t_0}, h)$. Denote by γ a positive number satisfying the relation $2\gamma < \rho(\partial\Gamma_{H_1}, \partial U_H)$. According to Definition 3, we have $x(t_0 + T(\eta_0), t_0, x_0) \in S(M_{t_0+T(\eta_0)}, \eta_0)$ for all $x_0 \in S(M_{t_0}, h)$. By the choice of γ, we get the inclusion $\overline{U}_{H+\gamma} \subset \Gamma_{H_1}$. Denote $\lambda = h + \gamma$ and $2\delta = \min(\gamma, \eta_0) \exp(-nL_\lambda T(\eta_0))$, where L_λ is the Lipschitz constant in conditions (5) in the domain $U_{h+\gamma}$. If $(t_0, x_0') \in U_{h+\delta}$ and $(t_0, x_0'') \in U_{h+\delta}$, then the inequalities

$$\| x(t, t_0, x_0') - x(t, t_0, x_0'') \| < \| x_0' - x_0'' \| \exp(nL_\gamma |t - t_0|) \tag{19}$$

hold along the trajectories $x(t, t_0, x_0')$ and $x(t, t_0, x_0'')$ for the values of t at which the arcs of these trajectories remain entirely in the domain $U_{H+\gamma}$. Consider the trajectories defined by the solutions $x(t, t_0, x_0')$ and $x(t, t_0, x_0'')$ for $t \in [t_0; t_0 + T(\eta_0)]$; here, x_0'' is chosen arbitrarily, namely, $x_0'' \in S(M_{t_0}, h + \delta)$, and x_0' lies on the segment connecting the points x_0'' and y_0, where y_0 is the point in M_{t_0} nearest to x_0''. In addition, x_0' satisfies the conditions $x_0' \in S(M_{t_0}, h)$ and $\| x_0'' - x_0' \| < 2\delta$. Further, we take into account that $(t, x(t, t_0, x_0')) \in U_H$, since $x_0' \in S(M_{t_0}, h)$. By using inequality (19), we obtain

$$\| x(t, t_0, x_0') - x(t, t_0, x_0'') \| < 2\delta \exp(nL_\gamma |t - t_0|) \leq \gamma, \tag{20}$$

whence it is possible to conclude that $x(t, t_0, x_0'') \in S(M_t, H + \gamma)$. Moreover, relation (20) implies that $x(t_0 + T(\eta_0), t_0, x_0) \in S(M_{t_0+T(\eta_0)}, 2\eta_0)$, i.e., $x(t_0 + T(\eta_0), t_0, x_0) \in S(M_{t_0+T(\eta_0)}, h)$ by virtue of the choice of the number

η_0. For $t = t_0 + T(\eta_0)$, the trajectory defined by the solution $x(t, t_0, x_0'')$ belongs to U_h. Therefore, by Definition 3, we have

$$x(t_0 + T(\eta_0) + t, t_0, x_0) \in S(M_{t_0+T(\eta_0)+t}, \eta)$$

$$\text{for all} \quad t > T(\eta), \quad x_0 \in S(M_{t_0}, \lambda), \quad \text{and} \quad \lambda = h + \delta.$$

Hence, the domain U_λ satisfies all conditions of Definition 3 necessary for the uniform asymptotic stability of M if the constants $T_\lambda(\eta)$ for the initial conditions $(t_0, x_0) \in U_\lambda$ are determined from the relation $T_\lambda(\eta) = T(\eta_0) + T(\eta)$, where T are the constants from Definition 3 for the initial conditions (t_0, x_0) in the domain U_h.

Let us show that property (A) with respect to M holds in the domain U_λ. Indeed, let

$$t_0 \geq T_\lambda(\eta) \quad \text{and} \quad x_0 \in S(M_{t_0}, \lambda) \backslash S(M_{t_0}, \eta). \tag{21}$$

Then the point $x(t_0 - T_\lambda(\eta), t_0, x_0)$ cannot lie in $S(M_{t_0 - T_\lambda(\eta)}, \lambda)$ since, otherwise,

$$x(t_0, t_0 - T_\lambda(\eta), x(t_0 - T_\lambda(\eta), t_0, x_0)) = x_0,$$

but this contradicts assumption (21).

Thus, condition (A) with respect to M is satisfied in the domain U_λ for the arcs of negative semitrajectories. By virtue of Theorem 2, we can now conclude that the domain U_h contains a function $v(t, x)$ admitting an infinitesimal upper bound with respect to M and having in this domain the bounded continuous partial derivatives $\partial v / \partial t$ and $\partial v / \partial x_i$, $i = 1, 2, \dots, n$; moreover, its derivative dv/dt is definitely negative with respect to M. In this case, the role of the function v is played by the function in Theorem 2 with inverse sign. In the case where the right-hand sides of Eq. (1) and the integral set M are periodic with period ω (or do not depend on time explicitly), the function v is also periodic with period ω (does not depend on time). The definite positivity of the function v with respect to M in the domain U_h can be established immediately, if we note that all the terms $v(t, x, t_0^{(k)}, x_0^{(l)})$ in relations (13) and (15) for the functions from Lemma 2 are nonnegative and, in addition, at least one term $v(t, x, t_0^{(k)}, x_0^{(l)})$ in relation (13) satisfies, by virtue of condition (12) and the choice of $t_0^{(k)}$, the inequality

$$v(t, x, t_0^{(k)}, x_0^{(l)}) < -\Delta_s < 0, \quad \Delta_s = \text{const},$$

at every point $(t, x) \in R(s)$.

If the functions $X_i(t, x)$ are uniformly continuous in $t \in I$ in the domain U_{h_1}, $h_1 > h$, then Theorem 2 implies that the constructed function $v(t, x)$ possesses partial derivatives of an arbitrary order with respect to all variables, and these derivatives are uniformly bounded in the domain U_h (each derivative is bounded by its own constant). The theorem is proved.

Consider the autonomous system of differential equations

$$\frac{dx}{dt} = f(x), \quad x, f \in R^n, \tag{22}$$

where the function f satisfies the Lipschitz condition. In this case, the following corollary of Theorem 3 is valid:

Corollary. *If system* (22) *admits a bounded invariant asymptotically stable set* G, *then a neighborhood of* G *contains a function* $v(x)$, *which is definitely positive with respect to* G; *by virtue of* (22), *its derivative* dv/dx *is definitely negative with respect to* G.

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