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ON EQUIASYMPTOTIC STABILITY OF SOLUTIONS OF DOUBLY-PERIODIC IMPULSIVE SYSTEMS

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We consider a system of ordinary differential equations with pulse action at fixed times that admits the trivial solution. We establish sufficient conditions for the equiasymptotic stability of the trivial solution.

1. Introduction

In the mathematical description of the evolution of actual processes with short-time perturbations, it is often convenient to neglect the perturbation duration and assume that the perturbations are instantaneous. This idealization leads to the necessity of the investigation of dynamical systems with discontinuous trajectories, or, in other words, impulsive differential equations. At present, the theory of impulsive differential equations is an extensively developing field of mathematics, various aspects of which are described in [1–5]. In recent years, many applied works have been published in which impulsive differential equations have been used as mathematical models. This resulted in an increase in the number of works devoted to the investigation of various aspects of the theory of impulsive systems [6–13]. In the present paper, which is a continuation and development of [8], we study the stability of solutions of impulsive systems.

2. Main Definitions

Consider the following system of ordinary differential equations with pulse action:

$$\frac{dx}{dt} = f(t, x), \qquad t \neq \tau_i, \quad i = 1, 2, \ldots, \tag{1}$$

$$\Delta x|_{t=\tau_i} = J_i(x), \qquad i = 1, 2, \ldots, \tag{2}$$

where $t \in \mathbb{R}_+ := [0, \infty)$ is time, $i \in \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers, τ_i are constants, $x \in \mathbb{R}^n$, $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$, and $J_i\colon \mathbb{R}^n \to \mathbb{R}^n$. Equations (1), (2) describe the dynamics of the system, which consists of two parts: the continuous part (for $t \neq \tau_i$) described by ordinary differential equations and the discrete part (at times τ_i) in which the solutions of the system change jumpwise. Denote

$$B_H = \left\{x \in \mathbb{R}^n \colon \|x\| = \sqrt{x_1^2 + \ldots + x_n^2} \leq H\right\},$$

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$$G_i := \left\{(t, x) \in \mathbb{R}^{n+1} : \tau_{i-1} < t < \tau_i, x \in B_H\right\}, \qquad G := \bigcup_{i=1}^{\infty} G_i .$$

Below, we formulate hypotheses H_1–H_5 that may be true for system (1), (2).

H_1. The function $f = (f_1, \ldots, f_n) : G \to \mathbb{R}^n$ is uniformly continuous in $\mathbb{R}_+ \times B_H$, $f(t, 0) \equiv 0$, and there exists a constant $L > 0$ such that $\|f(t, x) - f(t, y)\| \le L\|x - y\|$ for $(t, x) \in G$, $(t, y) \in G$, $x \in B_H$, and $y \in B_H$.

H_2. The function $J_i : B_H \to \mathbb{R}^n$, $i \in \mathbb{N}$, is continuous and satisfies the Lipschitz condition with constant L in B_H, and $J_i(0) = 0$ for $i \in \mathbb{N}$.

H_3. There exists a constant $h \in (0, H)$ such that if $x \in B_h$, then $x + J_i(x) \in B_H$ for $i \in \mathbb{N}$.

H_4. The constants τ_i satisfy the conditions

$$0 = \tau_0 < \tau_1 < \tau_2 < \ldots, \qquad \lim_{i \to \infty} \tau_i = \infty.$$

H_5. The constants τ_i satisfy the following condition: For any $T > 0$ and $t > 0$, the segment $[t, t + T]$ contains at most p constants τ_i, where the number p depends only on T and does not depend on t.

Let $x(t, t_0, x_0)$, $t > t_0$, denote a solution of system (1), (2) that satisfies the condition $x(t_0, t_0, x_0) = x_0$ in the case where $t_0 \ne \tau_i$, $i \in \mathbb{N}$. If $t_0 = \tau_i$, then, for any natural i, the expression $x(t, t_0, x_0)$ means that $x\big(t, t_0 + 0, x_0 + J_i(x_0)\big)$ (for $t > t_0$). Denote the value of this solution at time t also by $x(t, t_0, x_0)$. We assume that this solution is continuously differentiable with respect to t on any set G_i and left-continuous at the points of discontinuity: $x(\tau_i, t_0, x_0) = x(\tau_i - 0, t_0, x_0)$.

If hypotheses H_1–H_3 are true, then system (1), (2) admits the trivial solution

$$x \equiv 0. \tag{3}$$

Let us introduce the notions of stability and attraction of the trivial solution of system (1), (2).

Definition 1. *The trivial solution of system (1), (2) is called stable if, for any* $\varepsilon > 0$ *and* $t_0 \in \mathbb{R}_+$, *there exists* $\delta = \delta(\varepsilon, t_0) > 0$ *such that if* $\|x_0\| \le \delta$, *then* $\|x(t, t_0, x_0)\| \le \varepsilon$ *for* $t > t_0$.

Definition 2. *Solution (3) of system (1), (2) is called*

attracting if, for any $t_0 \in \mathbb{R}_+$, *there exists* $\lambda = \lambda(t_0) > 0$ *and, for any* $\varepsilon > 0$ *and* $x_0 \in B_\lambda$, *there exists* $\sigma = \sigma(\varepsilon, t_0, x_0) > 0$ *such that* $\|x(t, t_0, x_0)\| \le \varepsilon$ *for all* $t \ge t_0 + \sigma$;

equiattracting if, for any $t_0 \in \mathbb{R}_+$, *there exists* $\lambda = \lambda(t_0) > 0$ *such that, for any* $\varepsilon > 0$, *one can find* $\sigma = \sigma(\varepsilon, t_0) > 0$ *such that, for any* $x_0 \in B_\lambda$ *and* $t \ge t_0 + \sigma$, *one has* $\|x(t, t_0, x_0)\| \le \varepsilon$;

uniformly attracting if there exists $\lambda > 0$ *such that, for any* $\varepsilon > 0$, *one can find* $\sigma = \sigma(\varepsilon) > 0$ *such that, for any* $t_0 \in \mathbb{R}_+$, $x_0 \in B_\lambda$, *and* $t \geq t_0 + \sigma$, *one has* $x(t, t_0, x_0) \in B_\varepsilon$.

In other words, solution (3) of system (1), (2) is called

attracting if, for any $t_0 \in \mathbb{R}_+$ and $x_0 \in B_\lambda$, the following limit relation is true:

$$\lim_{t \to \infty} \|x(t, t_0, x_0)\| = 0, \tag{4}$$

equiattracting if relation (4) holds uniformly in $x_0 \in B_\lambda$;

uniformly attracting if the limit relation (4) holds uniformly in $x_0 \in B_\lambda$ and $t_0 \in \mathbb{R}_+$.

Definition 3. *The trivial solution of system (1), (2) is called*

asymptotically stable if it is stable and attracting,

equiasymptotically stable if it is stable and equiattracting,

uniformly asymptotically stable if it is uniformly stable and uniformly attracting.

Definition 4. *We say that a function* $\omega: \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to a class* $\mathcal{K}$ $(\omega \in \mathcal{K})$ *if it is continuous, strictly increasing, and such that* $\omega(0) = 0$.

We also introduce the following definition:

Definition 5. *We say that a function* $V: \mathbb{R}_+ \times B_H \to \mathbb{R}$ *belongs to a class* $\mathcal{V}_0$ $(V \in \mathcal{V}_0)$ *if* V *is uniformly continuous on* $\mathbb{R}_+ \times B_H$.

We say that a function $V \in \mathcal{V}_0$ *belongs to a class* $\mathcal{V}_1$ $(V \in \mathcal{V}_1)$ *if* V *is continuously differentiable on* $\mathbb{R}_+ \times B_H$ *and its derivative is determined by the formula*

$$\frac{dV}{dx} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t, x).$$

By analogy with [8], we introduce the following definition:

Definition 6. *A function* $g: \mathbb{R}_+ \to \mathbb{R}^m$, $m \in \mathbb{N}$, *is called finally nonzero if, for any* $M > 0$, *there exists* $t > M$ *such that* $g(t) \neq 0$.

A number sequence $\{u_k\}_{k=1}^{\infty}$ *is called finally nonzero if, for any natural number* M, *there exists* $k > M$ *such that* $u_k \neq 0$.

3. Main Results

We consider the impulsive system (1), (2) under the assumption that hypotheses H_1–H_5 are true and there exist $\omega_1 > 0$, $\omega_2 > 0$, and $q \in \mathbb{N}$ such that

$$f(t+\omega_1, x) \equiv f(t, x), \qquad \tau_{k+q} = \omega_2 + \tau_k,$$
$$J_{k+q}(x) \equiv J_k(x) \quad \forall x \in B_H \quad \forall t \in \mathbb{R}_+ \quad \forall k \in \mathbb{N}. \tag{5}$$

If ω_1/ω_2 is a rational number, then system (1), (2) is periodic. The stability of solution (3) of the periodic system (1), (2) was studied in [8]. It was shown in [8] that asymptotically stable solutions of periodic systems are uniformly asymptotically stable, and sufficient conditions for their asymptotic stability were obtained. In what follows, we consider system (1), (2) under the assumption that conditions (5) are satisfied and the number ω_1/ω_2 is irrational. In the special case $J_i(x) \equiv 0$, $i \in \mathbb{N}$, system (1), (2) is periodic with period ω_1; in the case $f(t, x) \equiv 0$, it is periodic with period ω_2. In the general case where $f(t, x) \not\equiv 0$ and $J_i(x) \not\equiv 0$, system (1), (2) is called doubly periodic.

We now pass to the investigation of stability of the trivial solution of the doubly-periodic system (1), (2). First, we present the well-known Kronecker's result [14, p. 9].

Lemma 1. *Let ω_1 and ω_2 be arbitrary real numbers. For any $\varepsilon > 0$, there exists $L = L(\varepsilon) > 0$ such that every interval of length L contains at least one number T that satisfies the system of inequalities*

$$|T - s\omega_1| < \varepsilon, \qquad |T - m\omega_2| < \varepsilon,$$

where s and m are certain integers.

Consider a sequence of positive numbers $\{\varepsilon_i\}_{i=1}^{\infty}$ that tends monotonically to zero. According to Lemma 1, for every ε_i there exists a sequence of numbers $\{T_{i,k}\}$ such that $T_{i,k} < T_{i,k+1}$, $T_{i,k} \to \infty$ as $k \to \infty$, and the following inequalities are true:

$$\left|T_{i,k} - s_{i,k}\omega_1\right| < \varepsilon_i \qquad \text{and} \qquad \left|T_{i,k} - m_{i,k}\omega_1\right| < \varepsilon_i,$$

where $s_{i,k}$ and $m_{i,k}$ are integers. Without loss of generality, we can assume that $T_{i,k} < T_{i+1,k}$, $i \in \mathbb{N}$, $k \in \mathbb{N}$. We denote $T_{i,i} = T_i$, $s_{i,i} = s_i$, and $m_{i,i} = m_i$ and consider the "diagonal" sequence $\{T_i\}$. For this sequence, we obtain

$$|T_i - s_i\omega_1| < \varepsilon_i, \qquad |T_i - m_i\omega_2| < \varepsilon_i. \tag{6}$$

According to (6), we have

$$T_i = s_i\omega_1 + \delta_{i1} = m_i\omega_2 + \delta_{i2}, \qquad \text{where} \quad \left|\delta_{i1}\right| < \varepsilon_i, \ \left|\delta_{i2}\right| < \varepsilon_i. \tag{7}$$

Denote $x_k = x(t_0 + m_k\omega_2, t_0, x_0)$. Let $t^* > t_0$ be a fixed time. We show that

$$\lim_{k\to\infty}\left\|x(t^*, t_0, x_k) - x(t^* + m_k\omega_2, t_0, x_0)\right\| = 0. \tag{8}$$

Trajectories I and II of system (1), (2) that start from the point x_k at times t_0 and $t_0 + m_k\omega_2$ reach the points $x(t^*, t_0, x_k)$ and $x(t^* + m_k\omega_2, t_0 + m_k\omega_2, x_k) = x(t^* + m_k\omega_2, t_0, x_0)$, respectively, after the time $\Delta t = t^* - t_0$. Trajectory II of system (1), (2) that passes through the point x_k at $t = t_0 + m_k\omega_2$ can be interpreted as a trajectory of the system

$$\frac{dx}{dt} = f(t + m_k\omega_2, x), \quad t \neq \tau_i, \tag{9}$$

$$\Delta x = J_{i+m_kq}(x), \quad t = \tau_i \tag{10}$$

with the same initial point x_k and initial time t_0. Taking into account that $J_{i+m_kq}(x) \equiv J_i(x)$, $x \in B_H$, and using relation (7) and the fact that the function f is ω_1-periodic of with respect to t, i.e.,

$$f(t + m_k\omega_2, x) \equiv f(t + s_k\omega_1 + \delta_{k1} - \delta_{k2}, x) \equiv f(t + \delta_{k1} - \delta_{k2}, x), \quad x \in B_H, \quad t \in \mathbb{R},$$

we establish that the discrete system (10) coincides with system (2), and the right-hand sides of the continuous systems (9) and (1) possess the property

$$\lim_{k\to\infty}\|f(t + \delta_{k1} - \delta_{k2}, x) - f(t, x)\| = 0$$

uniformly in $t \in \mathbb{R}$ and $x \in B_H$ by virtue of the uniform continuity of the function $f(t, x)$. Since the number of points of pulse action τ_i on the segment $[t_0, t^*]$ is finite and a solution of the impulsive system depends continuously on the right-hand sides (see, e.g., Theorem 2.5 in [1]), the limit relation (8) is true.

Theorem 1. *Suppose that, for the doubly-periodic impulsive system of differential equations (1), (2), there exists a function* $V(t, x) = V_1(t, x) + V_2(t, x)$, $V_1 \in \mathcal{V}_1$, $V_2 \in \mathcal{V}_1$, *such that*

$$V_1(t + \omega_1, x) \equiv V_1(t, x), \qquad V_2(t + \omega_2, x) \equiv V_2(t, x), \qquad t \in \mathbb{R}_+, \quad x \in B_H, \tag{11}$$

$$V(t, x) \geq a(\|x\|), \quad a \in \mathcal{K}, \qquad V(t, 0) \equiv 0, \tag{12}$$

$$\frac{dV}{dt} \leq 0 \quad \text{for} \quad (t, x) \in G, \tag{13}$$

$$V(\tau_i + 0, x + J_i(x)) - V(\tau_i, x) \leq 0, \quad i \in \mathbb{N}, \quad x \in B_H. \tag{14}$$

If, along any finally nonzero solution of system (1), (2), at least one of the conditions

$$\frac{dV}{dt} \quad \textit{is a finally nonzero function}$$

and

$$\{V(\tau_i + 0, x + J_i(x)) - V(\tau_i, x)\} \quad \textit{is a finally nonzero sequence}$$

is satisfied, then solution (3) of system (1), (2) is equiasymptotically stable.

Proof. The stability of the trivial solution of system (1), (2) follows from the known Gurgula–Perestyuk theorem [15]. This means that, for arbitrary $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, one can find $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|x(t, t_0, x_0)\| \le \varepsilon$ for any $x_0 \in B_\delta$, . We show that, for any $x_0 \in B_\delta$, the following limit relation is true:

$$\lim_{t\to\infty} V(t, x(t, t_0, x_0)) = 0.$$

Assume that the converse statement is true, i.e., there exist $x_0 \in B_\delta$ and $\eta > 0$ such that $V(t, x(t, t_0, x_0)) > \eta > 0$ for $t \ge t_0$. By virtue of conditions (12)–(14), $V(t, x(t, t_0, x_0))$ is a nonnegative monotonically nonincreasing function of time. Therefore, the following limit exists:

$$\lim_{t\to\infty} V(t, x(t, t_0, x_0)) = V_0 \ge \eta > 0.$$

Consider a sequence of positive numbers $\{\varepsilon_i\}_{i=1}^{\infty}$ that tends monotonically to zero and the corresponding sequences of natural numbers $\{m_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=1}^{\infty}$ satisfying (6) and (7). Denote $x_k = x(t_0 + m_k\omega_2, t_0, x_0)$. Since solution (3) of system (1), (2) is stable, the sequence $\{x_k\}$ is bounded and has a limit point x^*. Without loss of generality, we can assume that the sequence $\{x_k\}$ itself converges to x^*. By virtue of the uniform continuity of the functions V_1 and V_2 and property (11), we get

$$V_2(t_0, x_k) = V_2(t_0 + m_k\omega_2, x_k),$$

$$V_1(t_0, x_k) = V_1(t_0 + s_i\omega_1, x_k) = V_1(t_0 + m_i + \delta_{i2} - \delta_{i1}, x_k),$$

$$\lim_{i\to\infty} V_1(t_0 + m_i\omega_2 + \delta_{i2} - \delta_{i1}, x_k) = V_1(t_0, x_k),$$

$$\begin{aligned} V(t_0, x^*) &= \lim_{k\to\infty} V(t_0, x_k) = \lim_{k\to\infty} [V_1(t_0, x_k) + V_2(t_0, x_k)] \\ &= \lim_{k\to\infty}\lim_{i\to\infty} V_1(t_0 + m_i\omega_2 + \delta_{i2} - \delta_{i1}, x_k) + \lim_{k\to\infty} V_2(t_0 + m_k\omega_2, x_k) \\ &= \lim_{k\to\infty} \left[V_1(t_0 + m_k\omega_2 + \delta_{k2} - \delta_{k1}, x_k) + V_2(t_0 + m_k\omega_2, x_k)\right] \end{aligned}$$

$$= \lim_{k\to\infty} [V_1(t_0 + m_k\omega_2, x_k) + V_2(t_0 + m_k\omega_2, x_k)]$$

$$= \lim_{k\to\infty} V(t_0 + m_k\omega_2, x_k) = \lim_{k\to\infty} V(t_0 + m_k\omega_2, x(t_0 + m_k\omega_2, t_0, x_0)) = V_0.$$

Now consider the trajectory $x(t, t_0, x^*)$ for $t_0 < t < \infty$. Assume that this trajectory is finally nonzero. In this case, this trajectory contains points at which either

$$\dot{V}(t, x(t, t_0, x^*)) < 0$$

or

$$V(\tau_i + 0, x + J_i(x)) - V(\tau_i, x) < 0,$$

i.e., we can indicate a time $t^* > t_0$ at which the following condition is satisfied:

$$V(t^*, x(t^*, t_0, x^*)) = V_1^* < V_0. \tag{15}$$

Since the sequence $\{x_k\}$ converges to the point x^*, by virtue of the continuous dependence of solutions on initial data [1, p. 21] we can write

$$x(t^*, t_0, x^*) = \lim_{k\to\infty} x(t^*, t_0, x_k)$$

whence

$$\lim_{k\to\infty} V(t^*, x(t^*, t_0, x_k)) = V_1^*. \tag{16}$$

Taking into account that, under the conditions of the theorem, the limit relation (8) is true, we obtain

$$\| x(t^*, t_0, x_k) - x(t^* + m_k\omega_2, t_0, x_0) \| \le \gamma_k,$$

where

$$\lim_{k\to\infty} \gamma_k = 0.$$

Since $V = V_1 + V_2$ and the function V_2 is periodic in t with period ω_2, we get

$$|V(t^*, x) - V(t^* + m_k\omega_2, x)| = |V_1(t^*, x) - V_1(t^* + m_k\omega_2, x)| < M(\varepsilon_k) \quad \forall x \in B_H, \tag{17}$$

where $M(\varepsilon_k) \to 0$ as $k \to \infty$. Using condition (16), we obtain

$$\left|V\big(t^*, x(t^* + m_k\omega_2, t_0, x_0)\big) - V_1^*\right| < \eta_k, \tag{18}$$

where

$$\lim_{k\to\infty} \eta_k = 0.$$

Using (17), we get

$$\left|V\big(t^*, x(t^* + m_k\omega_2, t_0, x_0)\big) - V\big(t^* + m_k\omega_2, x(t^* + m_k\omega_2, t_0, x_0)\big)\right| < M(\varepsilon_k). \tag{19}$$

It follows from (18) and (19) that

$$\left|V\big(t^* + m_k\omega_2, x(t^* + m_k\omega_2, t_0, x_0)\big) - V_1^*\right| < \eta_k + M(\varepsilon_k), \tag{20}$$

where $\eta_k + M(\varepsilon_k) \to 0$ as $k \to \infty$. At the same time, we have

$$\lim_{k\to\infty} V\big(t^* + m_k\omega_2, x(t^* + m_k\omega_2, t_0, x_0)\big) = V_0. \tag{21}$$

Relations (20) and (21) contradict the inequality $V_1^* < V_0$, i.e., assuming that the trajectory $x(t, t_0, x^*)$ is finally nonzero, we arrive at a contradiction. In the case where $x(t, t_0, x^*)$ is not finally nonzero, there exists $T_* > 0$ such that $x(t, t_0, x^*) \equiv 0$ for $t \geq T_*$. By virtue of properties of the function V, this implies that $V\big(t, x(t, t_0, x^*)\big) = 0$ for $t \geq T_*$. Hence, in this case, there also exists $t^* > t_0$ such that condition (15) is satisfied, and we arrive at a contradiction again. The contradiction obtained shows that any solution with $|x_0| \leq \delta$ possesses the property

$$\lim_{k\to\infty} V(t, x(t, t_0, x_0)) = 0. \tag{22}$$

Let us show that the limit relation (22) holds uniformly in $x_0 \in B_\delta$. It follows from (22) that, for any $\varepsilon > 0$, $t_0 \in \mathbb{R}_+$, and $x_0 \in B_\delta$, one can indicate $\sigma(\varepsilon, t_0, x_0) > 0$ for which

$$V\left(t_0 + \sigma, x(t_0 + \sigma, t_0, x_0)\right) \leq \frac{a(\varepsilon)}{2}.$$

By virtue of the continuity of the function $V(t, x)$ in x and the continuous dependence of solutions on initial conditions, the following inequality holds in a certain neighborhood $Q(x_0)$ of the point x_0:

$$V(t_0 + \sigma, x(t_0 + \sigma, t_0, x_0')) \leq a(\varepsilon) \quad \text{for} \quad x_0' \in Q(x_0). \tag{23}$$

By virtue of the fact that the function $V(t, x)$ increases monotonically along solutions, it follows from (23) that $V(t, x(t, t_0, x_0')) \leq a(\varepsilon)$ for $t \geq t_0 + \sigma(\varepsilon, t_0, x_0)$ and $x_0' \in Q(x)$. The compact domain B_δ is covered by a system of neighborhoods $\{Q(x_0)\}$. Therefore, according to the Heine–Borel lemma [16, p. 49], we can choose a finite subcovering $Q_1, \dots, Q_j$ with the corresponding numbers $\sigma_1, \dots, \sigma_j$. We set

$$\sigma(\varepsilon, t_0) = \max\{\sigma_1, \dots, \sigma_j\}.$$

Then

$$V(t, x(t, t_0, x_0)) \le a(\varepsilon) \quad \text{for any} \quad x_0 \in B_\delta, \quad t \ge t_0 + \sigma(\varepsilon, t_0). \tag{24}$$

Using estimates (12) and (24), we establish that

$$\|x(t, t_0, x_0)\| \le \varepsilon \quad \text{for} \quad x_0 \in B_\delta, \quad t \ge t_0 + \sigma(\varepsilon, t_0),$$

which proves that the trivial solution of system (1), (2) is equiasymptotically stable.

Theorem 2. *Suppose that, for the doubly-periodic impulsive system of differential equations (1), (2), there exists a function* $V(t, x) = V_1(t, x) + V_2(t, x)$, $V_1 \in \mathcal{V}_1$, $V_2 \in \mathcal{V}_1$, *satisfying conditions (11) and such that*

$$|V(t, x)| \le b(\|x\|), \quad b \in \mathcal{K}, \tag{25}$$

$$\frac{dV}{dt} \ge 0 \quad (t, x) \in G, \tag{26}$$

$$V(\tau_i + 0, x + J_i(x)) - V(\tau_i, x) \ge 0, \quad i \in \mathbb{N}. \tag{27}$$

Also assume that at least one of the following conditions is satisfied along any finally nonzero solution of the system of equations (1), (2):

$$\frac{dV}{dt} \quad \text{is a finally nonzero function}$$

and

$$\{V(\tau_i + 0,\ x + J_i(x)) - V(\tau_i, x)\} \quad \text{is a finally nonzero sequence.}$$

If, in an arbitrarily small neighborhood of the origin of coordinates, for any $t > 0$ *there exists a point* x *such that* $V(t, x) > 0$, *then solution (3) of system (1), (2) is unstable.*

Proof. Let $\varepsilon < H$ be a certain positive number. We choose an arbitrary $t_0 \in \mathbb{R}_+$ and an arbitrarily small $\delta > 0$. Let us show that there exist $x_0 \in B_\delta$ and $t > t_0$ such that $\|x(t, t_0, x_0)\| > \varepsilon$. For this purpose, we choose $x_0 \in B_\delta$ so that $V(t_0, x_0) = V_0 > 0$. Assume that the converse statement is true, i.e.,

$$\|x(t, t_0, x_0)\| \le \varepsilon \tag{28}$$

for all $t > t_0$. Using condition (25), we obtain

$$|V(t, x)| < V_0 \quad \text{for} \quad \|x\| < b^{-1}(V_0) = \eta, \quad t \in \mathbb{R}_+,$$

where b^{-1} is the function inverse to b. Taking assumptions (26)–(28) into account, we conclude that the semitrajectory $x(t) = x(t, t_0, x_0)$ satisfies the condition

$$\eta \le \|x(t, t_0, x_0)\| \le \varepsilon.$$

Consider a sequence of positive numbers $\{\varepsilon_i\}_{i=1}^{\infty}$ that tends monotonically to zero, the corresponding sequences of natural numbers $\{m_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=1}^{\infty}$ satisfying (6) and (7), and the sequence of points $\{x_j\}$, where $x_j = x(t_0 + m_j\omega_2, t_0, x_0)$, $j = 1, 2, \ldots$. In view of the fact that this sequence is located in a compact set, we can choose its subsequence that converges to a point x_* such that $\eta \le \|x_*\| \le \varepsilon$. Without loss of generality, we can assume that the sequence $\{x_j\}$ also converges to the point x_*.

The function $v(t) = V(t, x(t, t_0, x_0))$ is monotonically nondecreasing and bounded from above by $b(\varepsilon)$. Therefore, the following limit exists (see the proof of Theorem 1):

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} V(t, x(t, t_0, x_0)) = v_0 = V(t_0, x_*),$$

where

$$V(t, x(t, t_0, x_0)) \le v_0. \tag{29}$$

Now consider the semitrajectory $x(t, t_0, x_*)$ for $t > t_0$. By virtue of the conditions of the theorem, there exists either a time t_1 such that

$$\frac{dV\left(t_1, x(t_1, t_0, x_*)\right)}{dt} > 0$$

or a time of pulse action τ_k such that

$$V(\tau_k + 0, x(\tau_k, t_0, x_*) + J_k(x(\tau_k, t_0, x_*)) - V(\tau_k, x(\tau_k, t_0, x_*)) > 0.$$

This implies that there exists a time $t_* > t_0$ such that

$$V(t_*, x(t_*, t_0, x_*)) = v_1 > v_0.$$

Since the sequence $\{x_j\}$ converges to the point x_*, by virtue of the continuous dependence of solutions on initial data we get

$$\|x(t_*, t_0, x_*) - x(t_*, t_0, x_j)\| < \gamma$$

for all $j > N(\gamma)$ and any pregiven number $\gamma > 0$. Therefore,

$$\lim_{j\to\infty} V(t_*, x(t_*, t_0, x_j)) = v_1. \tag{30}$$

As in the proof of the previous theorem, we can show that

$$\lim_{j \to \infty} V\big(t_* + m_j \omega_2, x(t_* + m_j \omega_2, t_0, x_0)\big) = v_0 \tag{31}$$

and

$$\Big| V\big(t_* + m_j \omega_2, x(t_* + m_j \omega_2, t_0, x_0)\big) - v_1 \Big| < \eta_j + M(\varepsilon_j), \tag{32}$$

where $\eta_j + M(\varepsilon_j) \to 0$ as $j \to \infty$. Relations (30)–(32) lead to a contradiction because $v_1 > v_0$. The contradiction obtained proves that assumption (28) is not true, i.e., solution (3) of system (1), (2) is unstable.

4. Example

As an illustration, we consider the following impulsive system of differential equations:

$$\left.\begin{aligned} \frac{dx}{dt} &= -y\cos\alpha t + xy^2 \sin\alpha t - x^3 + 2x^2 y \\ \frac{dy}{dt} &= x\cos\alpha t - x^2 y \sin\alpha t - x^2 y \end{aligned}\right\} \quad \text{for} \quad t \neq \pi k, \quad k \in \mathbb{N},$$

$$\Delta x = -2x, \qquad \Delta y = -y \qquad \text{for} \quad t = \pi k, \quad k \in \mathbb{N},$$

where α is an irrational number. As a Lyapunov function, we take

$$V = \frac{1}{2}\left(x^2 + y^2\right).$$

Then

$$\frac{dV}{dt} = -x^4 + 2x^3 y - x^2 y^2 = -x^2 (x - y)^2 \leq 0,$$

$$\Delta V\big|_{t = \pi k} = \frac{1}{2}\left[(x + \Delta x)^2 + (y + \Delta y)^2 - x^2 - y^2\right] = -\frac{1}{2} y^2 \leq 0.$$

In the case where, for arbitrarily large t, the inequalities $x(t) \neq 0$ and $x(t) \neq y(t)$ hold along a trajectory of the system, the function $\frac{dV}{dt}$ is finally nonzero along this trajectory. In the cases where $x(t) = 0$ or $x(t) = y(t)$, we establish that ΔV is finally nonzero along any finally nonzero trajectory. Therefore, by virtue of Theorem 1, a nonzero solution of the system considered is equiasymptotically stable.

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