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ON ONE MODULUS INEQUALITY FOR MAPPINGS WITH FINITE LENGTH DISTORTION

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The Väisälä inequality, which is well known in the theory of quasilinear mappings, is extended to the class of mappings with finite length distortion.

1. Introduction

Mappings with finite length distortion form one of the broadest classes of mappings including, in particular, mappings with bounded distortion in the sense of Reshetnyak, which are also known, due to works of Finnish mathematicians, as quasiregular mappings or quasiconformal mappings with branching (denoted by BD; see [1] and Theorem 4.7 in [2]). Martio and Väisälä also considered mappings with bounded length distortion (denoted by BLD) under which the lengths of all rectifiable curves are distorted finitely many times (see, e.g., [3]). Recently, the classes of mappings with finite length distortion (denoted by FLD) have been introduced under which almost all rectifiable curves are transformed into rectifiable curves with the condition of absolute continuity with respect to the length measure. This class was introduced by Ryazanov in 2002 and investigated by him together with Martio, Srebro, and Yakubov (see, e.g., [2]). This is one of the best-known classes of mappings closely related to the class of mappings with finite distortion (denoted by FD; see, e.g., [4] as well as Theorem 4.6 and Corollaries 4.9 and 4.16 in [2]).

A mapping $f\colon D\to\mathbb{R}^n$ is called a mapping with finite metric distortion if f possesses the Luzin (N)-property and distorts the distance between points by a finite factor almost everywhere. One of criteria is that f is differentiable almost everywhere and possesses the (N)- and (N^{-1})-properties (see Corollary 3.14 in [2]). A mapping $f\colon D\to\mathbb{R}^n$ is called a mapping with finite length distortion if f is a mapping with finite metric distortion possessing the (L)-property, i.e., first, the images of all curves γ in D are locally rectifiable and f possesses the (N)-property on γ with respect to the length measure, and, second, the (N)-property also takes place in the inverse direction for the liftings of curves. The relationship between the indicated classes can formally be denoted by the following imbeddings: $BLD\subset BD\subset FLD$.

Let us briefly describe the statement of the problem and the aim of investigations carried out in the present paper. For a mapping $f\colon D\to\mathbb{R}^n$ that has partial derivatives almost everywhere in D, we denote by $f'(x)$ the Jacobian matrix of the mapping f at a point x and by $J(x,f)$ the Jacobian of the mapping f at the point x, i.e., the determinant of $f'(x)$. In what follows, we use the notation

$$l(f'(x)) = \min_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|}.$$

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Recall that the *internal dilatation* of a mapping f at a point x is defined as follows:

$$K_I(x, f) = \frac{|J(x, f)|}{l(f'(x))^n} \quad \text{if} \quad J(x, f) \neq 0,$$

$$K_I(x, f) = 1 \quad \text{if} \quad f'(x) = 0,$$

$$K_I(x, f) = \infty \quad \text{otherwise.}$$

By analogy, the *external dilatation* of a mapping f at a point x is defined as follows:

$$K_O(x, f) = \frac{\|f'(x)\|^n}{|J(x, f)|} \quad \text{if} \quad J(x, f) \neq 0,$$

$$K_O(x, f) = 1 \quad \text{if} \quad f'(x) = 0,$$

$$K_O(x, f) = \infty \quad \text{otherwise.}$$

Note that mappings with finite metric distortion and, hence, mappings with finite length distortion are differentiable almost everywhere, and, moreover, $J(x, f) \neq 0$ almost everywhere (see Proposition 3.7 in [2]).

Recall that a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for a family Γ of curves γ in $\mathbb{R}^n$ if

$$\int_\gamma \rho(x)\, ds \geq 1 \quad \text{for all paths} \quad \gamma \in \Gamma.$$

In this case, we write $\rho \in \operatorname{adm} \Gamma$. The modulus of a family of curves Γ is defined as follows:

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int_D \rho^n(x) dm(x).$$

According to Theorem 6.10 in [2], the following lemma is true:

Lemma 1. *Let $f \colon D \to \mathbb{R}^n$ be a mapping with finite length distortion. Then*

$$M(f\Gamma) \leq \int_D K_I(x, f) \rho^n(x) dm(x) \tag{1}$$

for any family Γ of paths γ in D and every $\rho \in \operatorname{adm} \Gamma$.

Our aim is to generalize inequality (1) to mappings with finite length distortion with regard for the known Väisälä inequality, which was established earlier for quasiregular mappings (see, e.g., [5], Chap. II, Sec. 9). Namely, one can introduce a multiplier smaller than 1 into the right-hand side of inequality (1). It is clear that such inequalities are much more informative than (1) because they enable one to estimate the capacity of a condenser under a mapping more accurately (see the last section of the paper). Moreover, inequalities of the type (1) play an important role in the solution of problems of removal of singularities (see [5], Chap. III, Sec. 2). In the present work, we consider mappings with finite length distortion. Note that analogous estimates for the classes of mappings with finite distortion were obtained in [6]. A continuous mapping $f\colon D \to \mathbb{R}^n$ is called a *mapping with finite distortion* if $f \in W^{1,n}_{\mathrm{loc}}(D)$ and, almost everywhere,

$$\|f'(x)\|^n \leq K(x)\,J(x,f)$$

for a certain function $K(x)\colon D \to [1,\infty)$ (see, e.g., [4]). The conditions under which the main results of [6] were obtained are fairly restrictive because they demand the integrability of the inner dilatation of a mapping and the sufficiently strong summability of the derivative. It follows from the conditions indicated above that f belongs to $W^{1,n}_{\mathrm{loc}}$; moreover, the authors assume that the measure of the set B_f of the branch points of the mapping f is equal to zero. According to Remark 4.10 in [2], the open discrete mappings with finite distortion for which $K(x) \in W^{n-1}_{\mathrm{loc}}$ and the measure of the set of branch points is equal to zero always belong to the class of mappings with finite length distortion. The conditions under which the results of the present work are obtained do not contain any *a priori* assumptions concerning the dilatation $K_I(x,f)$, and, in particular, the local summability of K_I is not required. Moreover, we do not assume that $f \in W^{1,n}_{\mathrm{loc}}$. At the same time, as noted above, the results are formulated and proved for a broader class of mappings (see Remark 4.10 in [2]).

2. Definitions and Preliminary Remarks

Let us give several definitions. In what follows, D is a domain in $\mathbb{R}^n$, $n \geq 2$. A mapping $f\colon D \to \mathbb{R}^n$ is called *discrete* if the preimage $f^{-1}(y)$ of every point $y \in \mathbb{R}^n$ consists of isolated points, and it is called *open* if the image of any open set $U \subseteq D$ is an open set in $\mathbb{R}^n$. In what follows, the expression $f\colon D \to \mathbb{R}^n$ means that the mapping f is continuous in the domain of definition. We write $G \Subset D$ if $\overline{G}$ is a compact subset of the domain D. We say that a mapping f *preserves orientation* if the topological index $\mu(y,f,G)$ satisfies the inequality $\mu(y,f,G) > 0$ for an arbitrary domain $G \Subset D$ and arbitrary $y \in f(G)\setminus f(\partial G)$ (see, e.g., [1]). In what follows, we assume that the mapping f preserves orientation, unless otherwise stated. Let $f\colon D \to \mathbb{R}^n$ be an arbitrary mapping and let a domain $G \Subset D$ such that $\overline{G} \cap f^{-1}(f(x)) = \{x\}$ exist. Then the value $\mu(f(x),f,G)$, which is called the *local topological index,* is independent of the choice of the domain G, and we denote it by the symbol $i(x,f)$. Denote

$$B(x_0,r) = \left\{x \in \mathbb{R}^n : |x - x_0| < r\right\}.$$

For a mapping $f\colon D \to \mathbb{R}^n$, a set $E \subset D$, and $y \in \mathbb{R}^n$, we define a *multiplicity function* $N(y,f,E)$ as the number of preimages of the point y in the set E, i.e.,

$$N(y,f,E) = \operatorname{card}\left\{x \in E : f(x) = y\right\}, \qquad N(f,E) = \sup_{y \in \mathbb{R}^n} N(y,f,E).$$

A domain $G \Subset D$ is called *normal* if $fG \subset f(\partial G)$. For normal domains G, the value $\mu(y, f, G)$ is independent of y, and we denote it by $\mu(f, G)$. Let $f\colon D \to \mathbb{R}^n$ be an open discrete mapping. Then $\mu(f, G) = N(f, G)$ for any normal domain $G \Subset D$ (see Proposition 4.10 in [5], Chap. I).

Let $x \in E \subset \mathbb{R}^n$ and $\varphi\colon E \to \mathbb{R}^n$. We set

$$L(x, \varphi) = \limsup_{y \to x,\, y \in E} \frac{|\varphi(x) - \varphi(y)|}{|y - x|},$$

$$l(x, \varphi) = \liminf_{y \to x,\, y \in E} \frac{|\varphi(x) - \varphi(y)|}{|y - x|}.$$

A continuous mapping $f\colon D \to \mathbb{R}^n$ is called a mapping with *finite metric distortion* ($f \in FMD$) if f possesses the Luzin (N)-property and, for almost all $x \in D$,

$$0 < l(x, f) \le L(x, f) < \infty.$$

One says that a mapping $f\colon X \to Y$ between measure spaces (X, Σ, μ) and (X', Σ', μ') possesses the (N)-*property* if $\mu'(f(S)) = 0$ whenever $\mu(S) = 0$. Similarly, f possesses the (N^{-1})-*property* if $\mu(S) = 0$ whenever $\mu'(f(S)) = 0$. Let $\Delta \subseteq \mathbb{R}$ be an open interval of the number axis and let $\gamma\colon \Delta \to \mathbb{R}^n$ be a local rectifiable curve. Then there exists a unique nondecreasing length function $l_\gamma\colon \Delta \to \Delta_\gamma \subseteq \mathbb{R}$ with condition $l_\gamma(t_0) = 0$, $t_0 \in \Delta$, such that $l_\gamma(t)$ is equal to the length of the subcurve $\gamma|_{[t_0,t]}$ of the curve γ if $t > t_0$ and to $-l(\gamma|_{[t_0,t]})$ if $t < t_0$, $t \in \Delta$ Let $g\colon |\gamma| \to \mathbb{R}^n$ be a continuous mapping; here, $|\gamma| = \gamma(\Delta) \subseteq \mathbb{R}^n$. Assume that the curve $\tilde{\gamma} = g \circ \gamma$ is also locally rectifiable. Then there exists a unique nondecreasing function $L_{\gamma,g}\colon \Delta_\gamma \to \Delta_{\tilde{\gamma}}$ such that $L_{\gamma,g}(l_\gamma(t)) = l_{\tilde{\gamma}}(t)$ $\forall t \in \Delta$. We say that a mapping $f\colon D \to \mathbb{R}^n$ possesses the (L)-*property* if the following conditions are satisfied:

(L_1) for almost all curves $\gamma \in D$, the curve $\tilde{\gamma} = f \circ \gamma$ is locally rectifiable and the function $L_{\gamma,f}$ possesses the (N)-property;

(L_2) for almost all curves $\tilde{\gamma} \in f(D)$, every *(complete) lifting* γ of the curve $\tilde{\gamma}$ is locally rectifiable and the function $L_{\gamma,f}$ possesses the (N^{-1})-property.

A curve $\gamma \in D$ is called a *complete lifting of a curve* $\tilde{\gamma} \in \mathbb{R}^n$ *under a mapping* $f\colon D \to \mathbb{R}^n$ if $\tilde{\gamma} = f \circ \gamma$. We say that *almost all curves* of a domain D possess a certain property if all curves lying in D, except, possibly, a certain subfamily of them whose modulus is equal to zero, possess this property. We say that a mapping $f\colon D \to \mathbb{R}^n$, $n \ge 2$, is a *mapping with finite length distortion* ($f \in FLD$) if f belongs to FMD and possesses the (L)-property.

Recall that a mapping $\varphi\colon X \to Y$ between metric spaces X and Y is called a *Lipschitz* mapping if

$$\operatorname{dist}(\varphi(x_1), \varphi(x_2)) \le M \operatorname{dist}(x_1, x_2)$$

for a certain constant $M < \infty$ and all $x_1,\ x_2 \in X$. One says that a mapping $\varphi\colon X \to Y$ is a *bi-Lipschitz* mapping if, first, it is a Lipschitz mapping and, second,

$$M^* \operatorname{dist}(x_1, x_2) \ \le\ \operatorname{dist}(\varphi(x_1), \varphi(x_2))$$

for a certain constant $M^* > 0$ and all $x_1,\ x_2 \in X$.

The following result was obtained in [2] (see Lemma 3.20):

Lemma 2. *Let $f\colon D \to \mathbb{R}^n$ be a mapping with finite metric distortion. Then there exists a countable sequence of compact sets $C_k^* \subset D$ such that $|B| = 0$, where*

$$B = D \setminus \bigcup_{k=1}^{\infty} C_k^*$$

and $f|_{C_k^*}$ is a one-to-one bi-Lipschitz mapping for every $k = 1, 2, \ldots$. Moreover, f is differentiable for all $x \in C_k^*$ and $J(x, f) \neq 0$.

Recall that a mapping $f\colon D \to \mathbb{R}^n$ is called *absolutely continuous on lines* ($f \in ACL$) if, in any n-dimensional parallelepiped P such that its edges are parallel to coordinate axes and $\overline{P} \subset D$, all coordinate functions $f = (f_1, \ldots, f_n)$ are absolutely continuous on almost all straight lines parallel to coordinate axes. According to [5], a pair $E = (A, C)$, where A is an open set in $\mathbb{R}^n$ and C is a compact subset of A, is called a *condenser* in $\mathbb{R}^n$, $n \ge 2$. The *capacity* of a condenser E is defined as follows:

$$\operatorname{cap} E = \operatorname{cap}(A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^n \, dm(x), \tag{2}$$

where $W_0(E) = W_0(A, C)$ is the family of nonnegative continuous functions $u\colon A \to \mathbb{R}$ with compact support in A such that $u(x) \ge 1$ for $x \in C$ and $u \in ACL$, and, as usual,

$$|\nabla u| = \left(\sum_{i=1}^{n} (\partial_i u)^2 \right)^{1/2}.$$

The definitions presented below are taken from [5], Chap. II, Sec. 3. Let $f\colon D \to \mathbb{R}^n$, $n \ge 2$, be an open discrete mapping, let $\beta\colon [a, b) \to \mathbb{R}^n$ be a certain curve, and let $x \in f^{-1}(\beta(a))$. A curve $\alpha\colon [a, c) \to D$ is called the *maximum lifting* of the curve β under the mapping f with origin at the point x if the following conditions are satisfied:

(i) $\alpha(a) = x$;

(ii) $f \circ \alpha = \beta|_{[a,c)}$;

(iii) if $c < c' \le b$, then there is no curve $\alpha'\colon [a, c') \to D$ such that $\alpha = \alpha'|_{[a,c)}$ and $f \circ \alpha = \beta|_{[a,c')}$.

Let f be an open discrete mapping and let $x \in f^{-1}(\beta(a))$. In this case, the curve β has the maximum lifting under the mapping f with origin at the point x (see Corollary 3.3 in [5], Chap. II). We need the following statement (see Proposition 10.2 in [5], Chap. II):

Lemma 3. *Let* $E = (A, C)$ *be an arbitrary condenser in* $\mathbb{R}^n$ *and let* Γ_E *be the family of all curves of the form* $\gamma : [a, b) \to A$ *such that* $\gamma(a) \in C$ *and* $|\gamma| \cap (A \setminus F) \neq \varnothing$ *for an arbitrary compact set* $F \subset A$. *Then* $\operatorname{cap} E = M(\Gamma_E)$.

Let $x_1, \ldots, x_k$ be different points of the set $f^{-1}(\beta(a))$ and let

$$m = \sum_{i=1}^{k} i(x_i, f).$$

We say that a sequence of curves $\alpha_1, \ldots, \alpha_m$ is the maximum sequence of liftings of a curve β under a mapping f with origin at points $x_1, \ldots, x_k$ if the following conditions are satisfied:

(a) each curve α_j is the maximum lifting of the curve β under the mapping f;

(b) $\operatorname{card}\{j : a_j(a) = x_i\} = i(x_i, f)$, $1 \leq i \leq k$;

(c) $\operatorname{card}\{j : a_j(t) = x\} \leq i(x, f)$ for all $x \in D$ and all t.

Let f be an open discrete mapping and let $x_1, \ldots, x_k \in f^{-1}(\beta(a))$. Then the curve β has the maximum sequence of liftings under the mapping f at the points $x_1, \ldots, x_k$ (see Theorem 3.2 in [5], Chap. II).

According to Corollary 3.4 in [5], Chap. II, the following statement is true:

Lemma 4. *Let* $f\colon G \to \mathbb{R}^n$ *be an open discrete mapping, let* G *be the normal domain of the mapping* f, *let* $m = N(f, G)$, *and let* $\beta : [a, b) \to fG$ *be a certain curve. Then there exist curves* $\alpha_j\colon [a, b) \to G$, $1 \leq j \leq m$, *that possess the following properties:*

(i) $f \circ \alpha_j = \beta$;

(ii) $\operatorname{card}\{j : a_j(t) = x\} = i(x, f)$ *for all* $x \in G \cap f^{-1}\beta(t)$;

(iii) $|\alpha_1| \cup \ldots \cup |\alpha_m| = G \cap f^{-1}|\beta|$.

Let E be a set in $\mathbb{R}^n$ and let $\gamma\colon \Delta \to \mathbb{R}^n$ be a certain curve. Denote $\gamma \cap E = \gamma(\Delta) \cap E$. Assume that γ is locally rectifiable. Then

$$l(\gamma \cap E) = |E_\gamma|,$$

where

$$E_\gamma = l_\gamma(\gamma^{-1}(E)),$$

$|A|$ is the length of a set $A \subset \mathbb{R}$, and $l_\gamma\colon \Delta \to \Delta_\gamma$ is the function defined in Sec. 2 of the present paper. Note that

$$E_\gamma = \gamma_0^{-1}(E),$$

where $\gamma_0 : \Delta_\gamma \to \mathbb{R}^n$ is the natural parametrization of the curve γ and

$$l(\gamma \cap E) = \int\limits_{\Delta} \chi_E(\gamma(t))ds = \int\limits_{\Delta_\gamma} \chi_{E_\gamma}(s)ds.$$

3. Analog of the Väisälä Inequality

Let α and β be curves in $\mathbb{R}^n$. Then the expression $\alpha \subset \beta$ means that α is a subcurve of the curve β. We say that a family of curves Γ_1 is *minorized* by a family Γ_2 (in this case, we write $\Gamma_1 > \Gamma_2$) if, for every curve $\gamma \in \Gamma_1$, there exists a subcurve that belongs to the family Γ_2. Let Γ_1 and Γ_2 be arbitrary families of curves such that $\Gamma_1 > \Gamma_2$. Then $M(\Gamma_1) \le M(\Gamma_2)$ (see Theorem 6.4 in [7]).

Theorem 1. *Let $f\colon D \to \mathbb{R}^n$ be an open discrete mapping with finite length distortion, let Γ be a family of curves in D, let Γ' be a family of curves in $\mathbb{R}^n$, and let m be a natural number such that the following condition is satisfied: For every curve $\beta \in \Gamma'$, there exist curves $\alpha_1, \ldots, \alpha_m$ of the family Γ such that $f \circ \alpha_j \subset \beta$ for all j and the equality $\alpha_j(t) = x$ holds for all t and $x \in G$ for at most $i(x, f)$ indices j. Then*

$$M(\Gamma') \le \frac{1}{m}\int\limits_D K_I(x, f)\rho^n(x)dm(x)$$

for any family Γ of paths γ in D and any $\rho \in \operatorname{adm}\Gamma$.

Proof. Let B and C_k^* be the sets from Lemma 2 and let B_f be the set of the points of branching of the mapping f in D. Note that $|B_f| = 0$ (see Proposition 3.16 in [2]). Setting $B_0 = B \cup B_f$, $B_1 = C_1^* \setminus B_f$, $B_2 = C_2^* \setminus (B_1 \cup B_f), \ldots,$ and

$$B_k = C_k^* \setminus \left(\bigcup_{l=1}^{k-1} B_l \cup B_f \right),$$

we obtain the countable covering of the domain D by the Borel sets B_k, $k = 1, 2, \ldots$; furthermore, $|B_0| = 0$. Since the mapping f possesses the (N)-property, we have $|f(B_0)| = 0$. According to Lemma 2.13 in [2], we get $l(\bar{\gamma} \cap f(B_0)) = 0$ for almost all curves $\bar{\gamma}$ in the domain $f(D)$. Therefore, according to the (L)-property, we also have

$$l(\gamma \cap B_0) = 0 \tag{3}$$

for almost all curves $\bar{\gamma}$ in $f(D)$ and all γ such that $f \circ \gamma = \bar{\gamma}$. We now verify relation (3) for almost all $\tilde{\gamma} \in \Gamma'$ and all γ such that $f \circ \gamma \subset \tilde{\gamma}$.

Assume the contrary. Let Γ_1 be the family of all curves $\gamma' \in \Gamma'$ for which

$$l(\gamma \cap B_0) > 0 \tag{4}$$

and $f \circ \gamma \subset \gamma'$ for a certain curve γ. Assume that $M(\Gamma_1) > 0$. Let Γ_2 be the family of all subcurves γ'' of the family Γ_1 that have a complete lifting γ such that condition (4) is satisfied. Note that $\Gamma_2 < \Gamma_1$ and, hence, $M(\Gamma_2) \geq M(\Gamma_1) > 0$. The contradiction obtained disproves our assumption.

Let $\rho \in \operatorname{adm} \Gamma$. We set

$$\rho^*(x) = \begin{cases} \rho(x)/l(f'(x)), & x \in D \setminus B_0, \\ 0, & x \in B_0. \end{cases}$$

Consider the function

$$\tilde{\rho}(y) = \frac{1}{m} \chi_{f(D \setminus B_0)}(y) \sup_C \sum_{x \in C} \rho^*(x),$$

where C runs through all subsets $f^{-1}(y)$ of cardinality at most m in $D \setminus B_0$. Note that

$$\tilde{\rho}(y) = \frac{1}{m} \sup \sum_{i=1}^{s} \rho_{k_i}(y), \tag{5}$$

where the least upper bound is taken over all possible sets $\{k_{i_1}, \ldots, k_{i_s}\}$ such that $k_i \in \mathbb{N}$ and $k_i \neq k_j$ for $i \neq j$ and over all $s \leq m$, and

$$\rho_k(y) = \begin{cases} \rho^*(f_k^{-1}(y)), & y \in f(B_k), \\ 0, & y \notin f(B_k). \end{cases}$$

Here, $f_k = f|_{B_k}$, $k = 1, 2, \ldots$, is injective. It follows from (5) that $\tilde{\rho}(y)$ is a Borel function because $f(B_k)$ are Borel sets (see Sec. 2.3.2 in [8]). Let β be an arbitrary curve of the family Γ'. By assumption, there exist curves $\alpha_1, \ldots, \alpha_m$ of the family Γ such that $f \circ \alpha_j \subset \beta$ for all j and the equality $\alpha_j(t) = x$ holds for all t and $x \in G$ for at most $i(x, f)$ indices j. The last condition means that the curves α_j do not intersect on B_k because, at every point of B_k, the mapping f is a local homeomorphism, and, hence, $i(x, f) = 1$. According to the results of Sec. 3.2.5 in [8] for $m = 1$, by virtue of the additivity of the integral the following relation holds on each B_k for almost all curves $\beta \in \Gamma'$:

$$\int\limits_{\beta}\tilde{\rho}\,ds \;\geq\; \sum_{j=1}^{m}\int\limits_{f\circ\alpha_j}\tilde{\rho}\,ds \;=\; \sum_{j=1}^{m}\sum_{k=1}^{\infty}\int\limits_{(f\circ\alpha_j)\bigcap f(B_k)}\tilde{\rho}\,ds$$

$$\geq\; \frac{1}{m}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\int\limits_{\alpha_j\bigcap B_k}\rho(x)\,ds \;=\; \frac{1}{m}\sum_{j=1}^{m}\int\limits_{\alpha_j}\rho(x)\,ds \;\geq\; \frac{1}{m}m \;=\; 1.$$

Therefore, $\tilde{\rho}\in\operatorname{adm}\Gamma'\setminus\Gamma_0$, where $M(\Gamma_0)=0$, and, hence,

$$M(\Gamma') \;\leq\; \int\limits_{f(D)}\tilde{\rho}^{\,n}(y)\,dm(y). \tag{6}$$

According to the results of Sec. 2.3.5 in [8] for $m=n$, we have

$$\int\limits_{B_k}K_I(x,f)\rho^n(x)\,dm(x) \;=\; \int\limits_{f(D)}\rho_k^n(y)\,dm(y). \tag{7}$$

Also note that, according to the Hölder inequality for sums, we have

$$\left(\frac{1}{m}\sum_{i=1}^{s}\rho_{k_i}(y)\right)^n \;\leq\; \frac{1}{m}\sum_{i=1}^{s}\rho_{k_i}^n(y)$$

for any $1\leq s\leq m$ and any set $\{k_1,\ldots,k_s\}$ of length s such that $k_i\in\mathbb{N}$ and $k_i\neq k_j$ if $i\neq j$.

Therefore, by virtue of the Lebesgue theorem, relations (6)–(8) yield

$$\frac{1}{m}\int\limits_{D}K_I(x,f)\rho^n(x)\,dm(x) \;=\; \frac{1}{m}\int\limits_{f(D)}\sum_{k=1}^{\infty}\rho_k^n(y)\,dm(y)$$

$$\geq\; \frac{1}{m}\int\limits_{f(D)}\sup_{\substack{\{k_1,\ldots,k_s\},\,k_i\in\mathbb{N},\\ k_i\neq k_j,\,i\neq j}}\sum_{i=1}^{s}\rho_{k_i}^n(y)\,dm(y) \;\geq\; \int\limits_{f(D)}\tilde{\rho}^{\,n}(y)\,dm(y) \;\geq\; M(\Gamma').$$

The theorem is proved.

4. Applications

For a given family Γ of curves in $\mathbb{R}^n$, we denote

$$M_{K_I(\cdot,f)}(\Gamma) \;=\; \inf_{\rho\in\operatorname{adm}\Gamma}\int\limits_{\mathbb{R}^n}\rho^n(x)K_I(x,f)\,dm(x).$$

Let $E = (A, C)$ be an arbitrary condenser and let ω be a nonnegative measurable function. Then the *weighted* ω-*capacity* of the condenser E is defined as follows:

$$\operatorname{cap}_\omega E = \operatorname{cap}_\omega(A, C) = \inf \int\limits_A |\nabla u(x)|^n \omega(x)\, dm(x), \tag{9}$$

where the greatest lower bound is taken over all functions $u \in C_0^\infty(A)$ such that $u \geq 1$ on C. Note that if $\omega \equiv 1$, then $\operatorname{cap}_\omega E$ coincides with $\operatorname{cap} E$ in the sense of definition (2).

The statements presented below generalize known modulus and capacity inequalities for quasiregular mappings (see Secs. 9 and 10 in [5]).

Theorem 2. *Let* $f\colon D \to \mathbb{R}^n$ *be an open discrete mapping with finite length distortion, let* G *be the normal domain of* f, *let* Γ' *be a family of curves in* $G' = f(G)$, *and let* Γ *be the family of curves* α *in* G *such that* $f \circ \alpha \subset \Gamma'$. *Then*

$$M(\Gamma') \leq \frac{1}{N(f, G)} \int\limits_D K_I(x, f) \rho^n(x)\, dm(x)$$

for every $\rho \in \operatorname{adm} \Gamma$. *In particular,*

$$M(\Gamma') \leq \frac{1}{N(f, G)} M_{K_I(\cdot, f)}(\Gamma).$$

The proof follows directly from Theorem 1 and Lemma 4.

Let $f\colon D \to \mathbb{R}^n$ be an open discrete mapping and let $E = (A, C)$ be a condenser in D. The value

$$M(f, C) = \inf_{y \in f(C)} \sum_{x \in f^{-1}(y) \cap C} i(x, f)$$

is called the minimum multiplicity of the mapping f in C.

Theorem 3. *Let* $f\colon D \to \mathbb{R}^n$ *be an open discrete mapping with finite length distortion and let* $E = (A, C)$ *be a condenser in* D. *Then*

$$\operatorname{cap} fE \leq \frac{1}{M(f, C)} \operatorname{cap}_{K_I(\cdot, f)} E. \tag{10}$$

Proof. Let $E = (A, C)$ be a condenser in D. Then $fE = (fA, fC)$ is a condenser in $f(D)$. Let Γ_E and Γ_{fE} be families of curves in the sense of the notation used in Lemma 3. We set $m = M(f, C)$. Let $\beta\colon [a, b) \to f(A)$ be an arbitrary curve of the family Γ_{fE}. Then $C \cap f^{-1}(\beta(a))$ contains points $x_1, \ldots, x_k$ such that

$$m' = \sum_{l=1}^{k} i(x_l, f) \geq m.$$

According to Theorem 3.2 in [5], Chap. II, there exists the maximum sequence of liftings $\alpha_j : [a, c_j) \to D$ of the curve β, $1 \leq j \leq m'$, under the mapping f with origin at the points $x_1, \dots, x_k$. Then each curve α_j belongs to the family Γ_E. Hence, the families $\Gamma = \Gamma_E$ and $\Gamma' = \Gamma_{fE}$ satisfy the conditions of Theorem 1. Therefore, according to Lemma 3, we have

$$\operatorname{cap} fE \leq \frac{1}{M(f, C)} M_{K_I(\cdot, f)}(\Gamma_E).$$

Finally, relation (10) follows from the inequality

$$M_{K_I(\cdot, f)}(\Gamma_E) \leq \operatorname{cap}_{K_I(\cdot, f)} E$$

because the function $\rho(x) = |\nabla u(x)|$ is admissible for Γ_E for every value of u contained in the definition of $\operatorname{cap}_{K_I(\cdot, f)} E$.

REFERENCES

1. Yu. G. Reshetnikov, *Space Mappings with Bounded Distortion* [in Russian], Nauka, Novosibirsk (1982).
2. O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, "Mappings with finite length distortion," *J. Anal. Math.,* **93**, 215–236 (2004).
3. O. Martio and J. Väisälä, "Elliptic equations and maps of bounded metric distortion," *Math. Ann.,* **282**, 423–443 (1988).
4. T. Iwaniec and G. Martin, *Geometric Function Theory and Non-Linear Analysis,* Clarendon Press, Oxford (2001).
5. S. Rickman, *Quasiregular Mappings,* Springer, Berlin–New York (1993).
6. P. Koskela and J. Onninen, "Mappings of finite distortion: capacity and modulus inequalities," *J. Reine Angew. Math.,* **599**, 1–26 (2006).
7. J. Väisälä, *Lectures on n-Dimensional Quasiconformal Mappings,* Springer, Berlin–New York (1971).
8. H. Federer, *Geometric Measure Theory,* Springer, Berlin–Heidelberg–New York (1969).