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Isolated Singularities of Solutions of Quasilinear Anisotropic Elliptic Equations

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Abstract

We consider a wide class of degenerate quasilinear second order elliptic equations with model representative $\sum_{i=1}^{n} \left(|u_{x_i}|^{p_i-2}u_{x_i}\right)_{x_i} = 0$, $1 < p_1 \le p_2 \le \ldots \le p_n$, whose solutions have singularity at a point. There are established sharp pointwise conditions for removable isolated singularity of solutions of such equations. For solutions with non-removable point singularity (source type or fundamental solution), precise upper and lower estimates near the singularity point are obtained.

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1 Introduction

The paper is devoted to the study of behavior of an arbitrary generalized solution to quasilinear anisotropic elliptic equation of general form

$$Lu := \sum_{i=1}^{n} \frac{d}{dx_i} a_i\left(x, u, \frac{\partial u}{\partial x}\right) - a_0\left(x, u, \frac{\partial u}{\partial x}\right) = 0, \quad \forall x \in \Omega\backslash\{x_0\}, \tag{1.1}$$

near the isolated singularity point $x_0 \in \Omega \subset \mathbb{R}^n$, $n \geq 2$. Here Ω is a bounded domain. We assume that functions $a_i(x,s,\xi)$, $i = 0, 1, \ldots, n$, are defined in $\Omega \times \mathbb{R}^1 \times \mathbb{R}^n$, satisfy the Caratheodory conditions, and there exist constants K_1, K_2 such that for every $x \in \Omega$, $s \in R^1$, $\xi \in R^k$ the following inequalities hold:

$$\begin{gathered}\sum_{i=1}^{n} a_i(x,s,\xi)\xi_i \geq K_1 \sum_{i=1}^{n} |\xi_i|^{p_i} - g_1(x)|s|^p - f_1(x), \quad K_1 > 0, \\ |a_i(x,s,\xi)| \leq K_2 \left(\sum_{j=1}^{n} |\xi_j|^{p_j}\right)^{1-\frac{1}{p_i}} + g_2(x)|s|^{p\left(1-\frac{1}{p_i}\right)} + f_2(x),\ i = \overline{1,n},\ K_2 < \infty, \\ a_0(x,s,\xi) \leq \sum_{i=1}^{n} h_i(x)|\xi_i|^{p_i\left(1-\frac{1}{p}\right)} + g_3(x)|s|^{p-1} + f_3(x).\end{gathered} \tag{1.2}$$

Here nonnegative functions $h_j(x), g_i(x), f_i(x)$, $i = 1,2,3$, $j = \overline{1,n}$, are contained in certain definite Lebesgue classes over the domain Ω (see § 2),

$$1 < p_1 \leq p_2 \leq p_3 \leq \ldots \leq p_n < \infty, \quad \frac{1}{p} := \frac{1}{n}\sum_{i=1}^{n} \frac{1}{p_i}, \quad p \leq n. \tag{1.3}$$

As the simplest model example of equation from (1.1) we can keep in the mind

$$L_0 u := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = 0 \ \text{ in } \ \Omega\backslash\{x_0\}. \tag{1.4}$$

In the case

$$1 < p_1 = p_2 = \cdots = p_n = p \leq n \tag{1.5}$$

equation (1.4) has the source type (fundamental) solution of the form

$$u(x) = |x - x_0|^{-\frac{n-p}{p-1}}, \quad p < n; \quad u(x) = \ln|x - x_0|, \quad p = n; \tag{1.6}$$

which exhibits the solution with "minimal" singularity at $x = x_0$. J. Serrin [6] firstly proved that even for general isotropic equation of the structure (1.1) with conditions (1.2), (1.5) singular at one point x_0, a solution $u(x)$ with the following asymptotic

$$u(x) = O\left(|x - x_0|^{-\frac{n-p}{p-1}+\delta}\right), \quad \delta > 0, \quad 1 < p < n, \tag{1.7}$$

$$u(x) = O\left(|\ln|x - x_0||^{1-\delta}\right), \quad \delta > 0, \quad p = n, \tag{1.8}$$

does not exist.

Further analysis of sufficient conditions for removability of singular solutions has been made by many authors for different classes of nonlinear elliptic and parabolic equations (see, for example, the book [8] and references therein). Firstly the precise point-wise sufficient condition for removability of isolated singularity of solution to general isotropic equation (1.1) (case (1.5)) was found in [5] in the form

$$\max_{r \le |x - x_0| \le R_0} |u(x)| = o\left(r^{-\frac{n-p}{p-1}}\right), \quad 1 < p < n, \tag{1.9}$$

$$\max_{r \le |x - x_0| \le R_0} |u(x)| = o\left(\left|\ln \frac{1}{r}\right|\right), \quad p = n. \tag{1.10}$$

As to anisotropic equation like (1.4), we do not know whether there exists explicit source type solution in the form similar to (1.6). Existence of nonnegative fundamental solution to equation (1.4) was proved in [9] under the following additional restriction:

$$1 < p_1 \le p_2 \le \ldots \le p_n < (n-1)(n-p)^{-1}p \ \text{ if } \ p < n. \tag{1.11}$$

Moreover, it was proved ([9]) that such a fundamental solution belongs to the anisotropic Sobolev space

$$u(x) \in W^1_{(\overline{q})}(\Omega) := \left\{ v \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{q_i}(\Omega), \quad i = \overline{1, n} \right\}, \tag{1.12}$$

where real q_i must satisfy the following conditions

$$1 < q_i < n(p-1)p^{-1}(n-1)^{-1}p_i, \quad i = \overline{1, n}. \tag{1.13}$$

It is easy to check that function

$$u_\delta(x) := \left(\sum_{i=1}^{n} |x_i - x_{o,i}|^{b_i}\right)^{-\frac{n-p}{p-1}-\delta}, \quad b_i := \frac{p_i(p-1)}{p(p_i - 1) + n(p - p_i)}, \quad 1 < p < n, \tag{1.14}$$

satisfies condition (1.12) with some $q_i = q_i(\delta)$ from (1.13) for an arbitrary $\delta > 0$. Here $q_i(\delta) \to n(p-1)p^{-1}(n-1)^{-1}p_i$, $i = \overline{1, n}$, as $\delta \to 0$. This fact yields the following:

Conjecture 1.1. The function $u_0(x)$ define the asymptotic of an arbitrary removable singularity point of solution to equation (1.1)

$$|u(x)| = o(u_0(x)), \quad 1 < p < n.$$

Conjecture 1.2. The function $u_0(x)$ ($u_\delta(x)$ with $\delta = 0$) defines the asymptotic of source type solution as for model anisotropic equation (1.4) and for general equation (1.1) of the structure (1.2) as $x \to x_0$.

In this paper we will exploit the method of local point-wise estimates elaborated by I.V. Skrypnik [7] to get these results.

The paper is organized as follows. In Section 2 assumptions and main results are formulated. The auxiliary integral estimates for solutions with isolated singularity are established in Section 3. We prove the fundamental point-wise estimate for solution in Section 4. Theorem on removable singulary is established in Section 5. In Section 6 the upper and below point-wise estimates showing the sharpness of removability conditions from Theorem 2.1 for fundamental solution of equation (1.4) are stated.

2 Formulation of assumptions and main results

Firstly we describe assumptions for the functions $g_i(x), f_i(x), h_i(x)$ from conditions (1.2), which define the structure of equation (1.1). Namely let us introduce functions

$$H_1(x) := \sum_{i=1}^{n} h_i^p(x) + g_1(x) + g_3(x) + f_1(x) + f_3(x), \quad H_2(x) := \sum_{i=1}^{n} (g_2(x) + f_2(x))^{\frac{p_i}{p_i - 1}}$$

and suppose that for some $\delta,\ 0 < \delta < \min((1, p_1 - 1),$

$$H_1(x) + H_2(x) \in L_{\frac{n}{p-\delta}}(\Omega), \quad 1 < p \le n. \tag{2.1}$$

Definition 2.1. A function $u(x)$ is siad to be a generalized solution of equation (1.1) in $\Omega \backslash \{x_0\}$, if $u(x) \in W^1_{(\overline{p})}(\Omega') := W^1_{p_1,p_2,\ldots,p_n}(\Omega')$ for an arbitrary subdomain $\Omega' \subset \Omega$ such that $x_0 \notin \overline{\Omega}'$ and the following integral identity holds

$$\sum_{i=1}^{n} \int\limits_{\Omega} a_i\left(x, u, \frac{\partial u}{\partial x}\right) \frac{\partial}{\partial x_i}(\psi\varphi)dx + \int\limits_{\Omega} a_0\left(x, u, \frac{\partial u}{\partial x}\right) \psi\varphi\ dx = 0. \tag{2.2}$$

Here $\varphi(x) \in \overset{\circ}{W}{}^1_{(p)}(\Omega),\ \psi(x) \in C^1(\overline{\Omega})$ are an arbitrary functions. $\psi(x)$ is equal to zero in a neighborhood of x_0.

Now we introduce the family of subdomains

$$\Omega(r) := \left\{ x \in \Omega : l(x) := \left(\sum_{i=1}^{n} |x_i - x_{o,i}|^{\frac{b_i}{b_1}} \right)^{b_1} < r \right\},\ b_i \text{ from (1.14)},\ 1 < p < n,$$

$$\Omega(r) := \left\{ x \in \Omega : l(x) := \left(\sum_{i=1}^{n} |x_i - x_{o,i}|^{\frac{p_i}{p_1}} \right)^{\frac{p_1}{n}} < r \right\} \ \forall r > 0, \quad p = n. \tag{2.3}$$

Let R_0 be an arbitrary number satisfying the inequality

$$0 < R_0 < \{\text{dist } (\{x_0\}, \partial\Omega)\}. \tag{2.4}$$

Define now the function $M(r)$ connected with a solution $u(x)$ of equation (1.1)

$$M(r) := \text{esssup}\ \{|u(x)| : x \in \Omega(R_0)\backslash\Omega(r)\}, \quad 0 < r < R_0. \tag{2.5}$$

Regularity result from [1] yields that $M(r) < \infty$ for an arbitrary $r > 0$. Now we shall formulate our main results.

Proposition 2.1. *Let $u(x)$ be an arbitrary solution of equation (1.1) in $\Omega\backslash\{x_0\}$ in the sense of Definition 2.1. Let (1.3), (1.11) hold. Suppose that function $M(r)$ satisfies the following conditions:*

$$\lim_{r\to 0} M(r) r^{\frac{n-p}{p-1}} = 0 \quad \textit{if} \quad 1 < p < n, \quad \textit{or} \quad \lim_{r\to 0} M(r) \left(\ln\frac{1}{r}\right)^{-1} = 0 \quad \textit{if} \quad p = n. \tag{2.6}$$

Then there exist positive constants C_0, γ, which depend on known parameters only, such that

$$M(r) \leq C_0 r^{-\frac{n-p}{p-1}+\gamma} \quad \forall\ r : 0 < r < R_0, \quad 1 < p < n; \tag{2.7}$$

$$M(r) \leq C_0 \left[\ln\frac{1}{r}\right]^{1-\gamma} \quad \forall\ r : 0 < r < R_0, \quad p = n. \tag{2.8}$$

Theorem 2.1. *Let $u(x)$ be an arbitrary solution of equation (1.1) in $\Omega\backslash\{x_0\}$. Let all conditions of Proposition 2.1 hold. Let additional structural conditions (1.2) be satisfied with*

$$g_1(x) = f_1(x) = 0. \tag{2.9}$$

Then the singularity of solution $u(x)$ at the point $\{x_0\}$ is removable and the integral identity (2.2) holds for $\psi(x) = 1$.

Now we consider source type (fundamental) solution $U(x)$ of equation (1.4), that is weak solution of the boundary value problem

$$L_0 U = -\delta(x - x_0) \ \text{ in } \ \Omega, \quad U(x) = 0 \ \text{ on } \ \partial\Omega. \tag{2.10}$$

If condition (1.11) is satisfied, then such nonnegative solution exists ([9]) and (1.12) holds. The validity of inequalities $p_i - 1 < \dfrac{n(p-1)p_i}{p(n-1)}$, $i = \overline{1,n}$ follows from (1.11). Therefore (1.12) implies that

$$\frac{\partial U}{\partial x_i} \in L^{p_i - 1}(\Omega), \quad i = \overline{1,n}.$$

Then $U(x)$ satisfies the following integral identity

$$\sum_{i=1}^{n} \int_{\Omega} \left|\frac{\partial U(x)}{\partial x_i}\right|^{p_i - 2} \frac{\partial U(x)}{\partial x_i} \frac{\partial \varphi(x)}{\partial x_i} dx = \varphi(0), \tag{2.11}$$

for an arbitrary function $\varphi(x) \in C_0^1(\Omega)$.

Introduce now the following function, connected with fundamental solution $U(x)$

$$m(\rho) := \text{ess inf } \{U(x) : x \in \partial\Omega(\rho)\}, \quad \forall \rho > 0, \quad 1 < p \leq n. \tag{2.12}$$

The following statement proves Conjecture 1.2 with respect to the model anisotropic equation.

Theorem 2.2. *Let $U(x)$ be a fundamental solution of equation (1.4). Then there exist positive constants R_0, d_0, d_1, depending on parameters $n, p_1, \dots, p_n$ only, such that*

$$d_0 \rho^{-\frac{n-p}{p-1}} \leq M(\rho), \quad m(\rho) \leq d_1 \rho^{-\frac{n-p}{p-1}} \quad \forall \rho < R_0, \quad \text{if} \quad 1 < p < n, \tag{2.13}$$

$$d_0 \ln \frac{1}{\rho} \leq M(\rho), \quad m(\rho) \leq d_1 \ln \frac{1}{\rho} \quad \forall \rho < R_0, \quad \text{if} \quad p = n. \tag{2.14}$$

3 Auxiliary integral estimates of the solution

We will realize our analysis of singularity satisfying condition (2.6) in some steps. Firstly we deduce some integral estimates of solution under consideration. Without loss of generality it can be assumed that the function $M(r)$ satisfies

$$\lim_{r \to 0} M(r) = \infty. \tag{3.1}$$

For the constant R_0 suppose that

$$M(R_0) \geq 1. \tag{3.2}$$

Let us set

$$u_R(x) = (u(x) - M(R), 0)_+,$$
$$E(R) = \{x \in \Omega(R) \backslash \{x_0\} : u(x) > M(R)\}, \tag{3.3}$$

for an arbitrary $R \in (0, R_0)$.

Later on, by c, c_i, C, C_j we will denote different positive constants which depend on known parameters of the problem under consideration only. Without loss of generality it can be assumed that $x_0 = 0$. Now we introduce the nonnegative cutoff function $\psi(t) \in C^\infty(\mathbb{R}_1)$ satisfying conditions

$$\psi(t) = 0 \quad \text{for} \quad t \leq 1, \quad \psi(t) = 1 \quad \text{for} \quad t \geq 2, \quad \left| \frac{d\psi(t)}{dt} \right| \leq 2 \quad \text{for} \quad t \in [1, 2]. \tag{3.4}$$

Denote by $\psi_r(x)$ our main cutoff function

$$\psi_r(x) := \psi \left(r^{-1} \left(\sum_{i=1}^n |x_i|^{\frac{b_i}{b_1}} \right)^{b_1} \right) \quad \forall r > 0, \quad 1 < p < n; \tag{3.5}$$

$$\psi_r(x) := 2 - 2\,(\ln r)^{-1} \ln\left(\sum_{i=1}^{n} |x_i|^{\frac{p_i}{p_1}}\right)^{\frac{p_1}{n}} \quad \text{in} \quad \Omega(\sqrt{r})\backslash\Omega(r),$$

$$\psi_r(x) = 0 \quad \text{in} \quad \Omega(r), \quad \psi_r(x) = 1 \quad \text{outside} \quad \Omega(\sqrt{r}), \quad \Omega(r) \quad \text{from} \quad (2.3), \quad p = n. \tag{3.6}$$

It is easy to check:

$$\begin{aligned} &\text{mes}\,\Omega(r) \le cr^n; \\ &\int\limits_{\Omega(R_0)} \left|\frac{\partial \psi_r}{\partial x_i}\right|^{p_i} dx \le cr^{\frac{(n-p)(p_i-1)}{p-1}}, \ \forall r > 0, \quad 0 < p < n; \\ &\int\limits_{\Omega(R_0)} \left|\frac{\partial \psi_r}{\partial x_i}\right|^{p_i} dx \le c\left(\ln r^{-1}\right)^{-(p_i-1)}, \ \forall r > 0, \quad p = n. \end{aligned} \tag{3.7}$$

Define also function $\omega(r)$:

$$\omega(r) := \left(M(r) r^{\frac{n-p}{p-1}}\right)^{p_1-1}, \ \forall r > 0, \quad 1 < p < n; \tag{3.8}$$

$$\omega(r) := \left(M(r)\left(\ln r^{-1}\right)^{-1}\right)^{p_1-1}, \ \forall r > 0, \quad p = n. \tag{3.9}$$

Lemma 3.1. *Assume that conditions of Proposition 2.1 are satisfied. Then there exist positive constants* $C_1 \ne C_1(r,R)$, $R_0 \ne R_0(r,R)$ *such that*

$$\sum_{i=1}^{n} \int\limits_{E(R)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \psi_r^{p_n}(x)dx \le C_1\{M(r)\omega(r) + M^{p_n}(R)\} \qquad \forall r, R : 0 < r < R < R_0, \quad 1 < p \le n, \tag{3.10}$$

where $\omega(r)$ *is from (3.8) if* $1 < p < n$ *and from (3.9) if* $p = n$.

Proof. Let us substitute into the integral identity (2.2) the test functions:

$$\varphi(x) = u_R(x)\psi_r(x)^{p_n-1}, \quad \psi(x) = \psi_r(x),$$

where $\psi_r(x)$ is defined by (3.5) if $1 < p < n$ and by (3.6) if $p = n$. After simple computations, using structural conditions (1.2) and the Young inequality, we deduce

$$\sum_{i=1}^{n}\int\limits_{E(R)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^{p_n}(x)dx \le c_1\int\limits_{E(R)} H_1(x)u_R^p(x)\psi_r^{p_n}(x)dx+$$
$$+c_1M^{p_n}(R)\int\limits_{E(R)\setminus\Omega(r)}\Big(g_1(x)+g_3(x)\Big)dx+$$
$$+c_1\sum_{i=1}^{n}\int\limits_{K(r)\cap E(R)}\left[u_R^{p_i}(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i}+g_2^{\frac{p_i}{p_i-1}}(x)u^p(x)\right]dx+ \tag{3.11}$$
$$+c_1\int\limits_{E(R)\setminus\Omega(r)}\left[f_1(x)+\sum_{i=1}^{n}[f_2(x)]^{\frac{p_i}{p_i-1}}+f_3(x)\right]dx,$$

where $H_1(x)$ is from (2.1), $K(r):=\Omega(2r)\setminus\Omega(r)$ if $1<p<n$, $K(r):=\Omega(\sqrt{r})\setminus\Omega(r)$ if $p=n$.

First we shall consider the case $1<p<n$. We estimate the first term in the right-hand side of (3.11) using conditions (2.1), the Hölder inequality, and inequality (2.1) from lemma 7.1 with $\alpha_1=\dots=\alpha_n=0$. As result we obtain

$$\int\limits_{E(R)\setminus\Omega(r)} H_1(x)u_R^p(x)\psi_r^{p_n}(x)dx \le \int\limits_{E(R)\setminus\Omega(2r)} H_1(x)u_R^p(x)dx+$$
$$+\int\limits_{K(r)} H_1(x)u_R^p(x)dx \le c_2\left\{R^\delta\sum_{i=1}^{n}\int\limits_{E(R)\setminus\Omega(2r)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+M^p(r)r^{n-p+\delta}\right\}. \tag{3.12}$$

By using conditions (2.1), property (3.7), and the Hölder inequality we have

$$\sum_{i=1}^{n}\int\limits_{K(r)\cap E(R)}\left[u_R^{p_i}(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i}+[g_2(x)]^{\frac{p_i}{p_i-1}}u^p(x)\right]dx \le$$
$$\le c_3\sum_{i=1}^{n}M^{p_i}(r)r^{\frac{n-p}{p-1}(p_i-1)}+c_3M^p(r)r^{n-p+\delta}. \tag{3.13}$$

Using assumption (2.1) we estimate the second and fourth integral terms in the right-hand side of (3.11) by constant. Let us suppose that R_0 satisfies the following smallness condition also

$$c_1c_2R_0^\delta \le 2^{-1}. \tag{3.14}$$

It is obvious that

$$p_1-1=\min\left\{p-1,\ \min_{1\le i\le n}\{p_i-1\},\ \min_{1\le i\le n}\left\{\frac{p(p_i-1)}{p_i}\right\}\right\}. \tag{3.15}$$

It follows from inequalities (3.11)-(3.13), assumption (3.14), property (3.15), and assumptions (2.6), (3.1) that (3.10) holds.

For the case $p = n$ this statement can be proved in the same way. In this case instead of inequalities (3.12), (3.13) we get:

$$\int\limits_{E(R)\setminus\Omega(r)} H_1(x)u_R^p(x)\psi_r^{p_n}(x)dx \le c_4\left\{R^{\frac{\delta}{2}}\sum_{i=1}^{n}\int\limits_{E(R)\setminus\Omega(2r)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx + M^p(r)r^{\frac{\delta}{2}}\right\}; \tag{3.16}$$

$$\begin{aligned}&\sum_{i=1}^{n}\int\limits_{K(r)\cap E(R)}\left[u_R^{p_i}(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i} + [g_2(x)]^{\frac{p_i}{p_i-1}}u^p(x)\right]dx \le\\ &\le c_5\sum_{i=1}^{n}M^{p_i}(r)\left(\ln r^{-1}\right)^{-(p_i-1)} + c_5M^p(r)r^{\frac{\delta}{2}}.\end{aligned} \tag{3.17}$$

Then using the same arguments completes the proof of Lemma 3.1.

For an arbitrary $R < R_0$ we define the number $\rho_0 = \rho_0(R)$ by equality

$$M(\rho_0(R)) = \max\left\{2M(R),\ M\left(\frac{R}{2}\right)\right\}, \tag{3.18}$$

and the set

$$E(\rho, R) := \{x \in \Omega : 0 < u_R(x) \le M(\rho) - M(R)\}\ \ \forall\rho \le \rho_0(R). \tag{3.19}$$

Lemma 3.2. *Assume that conditions of Proposition 2.1 are satisfied. Then there exists positive constant C_2 such that the following estimate holds*

$$\begin{aligned}&\sum_{i=1}^{n}\int\limits_{E(\rho,R)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^{p_n}(x)dx \le C_2(M(\rho)\omega(r) + M^{p_n}(R)) + C_2M(\rho)\times\\ &\times\int\limits_{E(\rho)}\left\{\sum_{j=1}^{n}h_j(x)\left|\frac{\partial u}{\partial x_j}\right|^{p_j\left(1-\frac{1}{p}\right)} + g_3(x)u^{p-1}(x)\right\}\psi_r^{p_n}(x)dx, \quad 1 < p \le n,\end{aligned} \tag{3.20}$$

for every $0 < r < \rho < \rho_0(R)$.

Proof. Let us substitute into the integral identity (2.2) the test functions:

$$\varphi(x) = \min\left[u_R(x), M(\rho) - M(R)\right]\psi_r^{p_n-1}(x), \quad \psi(x) = \psi_r(x), \quad 1 < p \le n. \tag{3.21}$$

Using assumptions (1.2) and Young's inequality we obtain

$$\sum_{i=1}^{n}\int\limits_{E(\rho,R)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^{p_n}(x)dx\le c_1\int\limits_{E(\rho,R)}H_1(x)u_R^p(x)\psi_r^{p_n}(x)dx+c_1M(\rho)\times$$
$$\times\sum_{i=1}^{n}\int\limits_{K(r)\cap E(R)}\left[\left(\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\right)^{1-\frac{1}{p_i}}+g_2(x)u^{p\left(1-\frac{1}{p_i}\right)}+f_2(x)\right]\left|\frac{\partial\psi_r}{\partial x_i}\right|\psi_r^{p_n-1}dx+$$
$$+c_1M(\rho)\int\limits_{E(\rho)}\left\{\sum_{j=1}^{n}h_j(x)\left|\frac{\partial u}{\partial x_j}\right|^{p_j\left(1-\frac{1}{p}\right)}+g_3(x)u^{p-1}(x)+f_3(x)\right\}\psi_r^{p_n}(x)dx+$$
$$+c_1M^p(R)\int\limits_{E(\rho,R)}\left(f_3(x)+g_3(x)\right)dx. \tag{3.22}$$

Consider the case $p<n$. The first term in the right-hand side of (3.22) we estimate analogously to (3.12)

$$\int\limits_{E(\rho,R)}H_1(x)u_R^p(x)\psi_r^{p_n}(x)dx\le$$
$$\le c_2\left\{R^\delta\sum_{i=1}^{n}\int\limits_{E(\rho,R)\setminus\Omega(2r)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+M^p(r)r^{n-p+\delta}\right\}. \tag{3.23}$$

By using the Hölder inequality, Lemma 3.1, definition (3.8), and inequality (3.7), we estimate the second integral term in the right-hand side of (3.22) as follows

$$\sum_{i=1}^{n}\int\limits_{K(r)\cap E(R)}\left(\sum_{j=1}^{n}\int\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\right)^{1-\frac{1}{p_i}}\left|\frac{\partial\psi_r}{\partial x_i}\right|\psi_r^{p_n-1}(x)dx\le$$
$$\le\sum_{i=1}^{n}\left(\int\limits_{K(r)\cap E(R)}\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\psi_r^{p_n}(x)dx\right)^{\frac{p_i-1}{p_i}}\left(\int\limits_{K(r)}\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}\le \tag{3.24}$$
$$\le c_3\sum_{i=1}^{n}(M(r)\omega(r))^{\frac{p_i-1}{p_i}}r^{\frac{n-p}{p-1}\cdot\frac{p_i-1}{p_i}}\le c_4\omega(r).$$

By the same arguments and due to assumption (2.1), we deduce

$$\sum_{i=1}^{n}\int\limits_{K(r)\cap E(R)}g_2(x)u^{p\left(1-\frac{1}{p_i}\right)}(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|\psi_r^{p_n}(x)dx\le c_5\omega(r). \tag{3.25}$$

Let us suppose that constant R_0 satisfies the following additional smallness condition

$$c_1 c_2 R_0^{\delta} \le 2^{-1}. \tag{3.26}$$

Taking into account (3.23)–(3.25), we derive from (3.22) estimate (3.20).

The proof of inequality (3.20) in the case $p = n$ is realized by the same arguments, substituting into the integral identity (2.2) test function (3.21) with $\psi_r(x)$ from (3.6). Similarly to the proof of inequalities (3.16), (3.17) it is not hard to show analogues of estimates (3.23)–(3.25) for the case $p = n$. This concludes the proof.

Lemma 3.3. *Let all assumptions of Proposition 2.1 be satisfied. Then there exist positive constant C_3 such that the following estimates hold:*

$$\sum_{i=1}^{n} \int\limits_{E(\rho)} u_R^{-q}(x) \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \psi_r^{p_n}(x) dx \le C_3 \left(\omega(r) + r^{p \frac{n-p}{(p-1)^2} \cdot \frac{p_1 - 1}{p_1}} + \rho^{\frac{\delta}{2}} \right), \quad 1 < p < n; \tag{3.27}$$

$$\sum_{i=1}^{n} \int\limits_{E(\rho)} u_R^{-q}(x) \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \psi_r^{p_n}(x) dx \le C_3 \left(\omega(r) + \left[\ln \frac{1}{r} \right]^{-p \frac{p_1 - 1}{p_1}} + \rho^{\frac{\delta}{2n}} \right), \quad p = n; \tag{3.28}$$

for all $0 < r < \rho < \rho_0(R) < R_0$. *Here* $q = 1 + 2^{-1}(n-p)^{-1}\delta$ *if* $1 < p < n$, $q > 1$ *if* $p = n$.

Proof. Let us substitute into (2.2) the test functions

$$\varphi(x) := \Big\{ [M(\rho) - M(R)]^{1-q} - [\max(u_R(x), M(\rho) - M(R))]^{1-q} \Big\} \psi_r^{p_n - 1}(x),$$
$$\psi(x) := \psi_r(x), \tag{3.29}$$

where $\psi_r(x)$ is defined by (3.5) if $1 < p < n$, and by (3.6) if $p = n$.

After simple computations, using structural conditions (1.2) and the Young inequality, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \int\limits_{E(\rho)} u_R^{-q}(x) \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \psi_r^{p_n}(x) dx \le c_1 \int\limits_{E(\rho)} u_R^{-q}(x) \Big[g_1(x) u^p(x) + \\ + f_1(x) \Big] \psi_r^{p_n}(x) dx + c_1 M^{1-q}(\rho) \sum_{i=1}^{n} \int\limits_{K(r) \cap E(\rho)} \left\{ \left(\sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right|^{p_j} \right)^{1 - \frac{1}{p_i}} + \right. \\ \left. + g_2(x) u^{p\left(1 - \frac{1}{p_i}\right)}(x) + f_2(x) \right\} \left| \frac{\partial \psi_r}{\partial x_i} \right| \psi_r^{p_n - 1}(x) dx + \\ + c_1 M^{1-q}(\rho) \int\limits_{E(\rho)} \left\{ \sum_{j=1}^{n} h_j^p(x) u_R^{q(p-1)}(x) + g_3(x) u^{p-1}(x) + f_3(x) \right\} \psi_r^{p_n}(x) dx. \end{aligned} \tag{3.30}$$

Let us set

$$A_1 := \sum_{j=1}^{n} \int\limits_{E(\rho)} h_j^p(x) u_R^{q(p-1)}(x) \psi_r^{p_n}(x) dx.$$

It is easy to see that $E(\rho) \subset \Omega(\rho)\ \forall\ \rho < \rho_0(R_0)$. Therefore due to condition (2.1), Hölder's inequality, and assumption (2.6), we have

$$A_1 < c_2 \left(\int\limits_{E(\rho)\setminus\Omega(r)} u_R^{\frac{q(p-1)n}{n-p+\delta}} dx \right)^{\frac{n-p+\delta}{n}} \le c_3 \left(\int\limits_{\Omega(\rho)} \left(\sum_{i=1}^{n} |x_i|^{b_i} \right)^{-\frac{qn}{n-p+\delta}(n-p)} dx \right)^{\frac{n-p+\delta}{n}},$$

where b_i are from (1.14), $1 < p < n$;

$$A_1 < c_4 \left(\int\limits_{E(\rho)\setminus\Omega(r)} u_R^{\frac{q(p-1)n}{\delta}}(x) dx \right)^{\frac{\delta}{n}} \le c_5 \left(\int\limits_{\Omega(\rho)} \left(\ln \left(\sum_{i=1}^{n} |x_i|^{\frac{p_i}{p_1}} \right)^{-\frac{p_1}{n}} \right)^{\frac{qn(p-1)}{\delta}} dx \right)^{\frac{\delta}{n}},$$

$p = n$;

$$\tag{3.31}$$

here $0 < r < \rho < \rho_0(R) < R_0$.

Introduce a new independent variables

$$\begin{aligned} x_i &:= y_i^{\frac{\theta}{b_i}} \operatorname{sign} y_i, \quad i = \overline{1,n}, \quad \theta = 2 \max_{1\le i\le n}(1, b_i),\ 1 < p < n; \\ x_i &:= y_i^{\theta \frac{n}{p_i}} \operatorname{sign} y_i, \quad i = \overline{1,n}, \quad \theta = 2 \max_{1\le i\le n}(1, \frac{p_i}{n}),\ p = n. \end{aligned} \tag{3.32}$$

Therefore after simple computations, by using (3.32), we estimate the right-hand side of (3.31) as follows:

$$\begin{aligned} A_1 &\le c_6 \left(\int\limits_{|y|^\theta<\rho} \left(\sum_{i=1}^{n} |y_i|^\theta \right)^{-\frac{qn(n-p)}{n-p+\delta}} \prod_{i=1}^{n} |y_i|^{\left(\frac{\theta}{b_i}-1\right)} dy \right)^{\frac{n-p+\delta}{n}} \le \\ &\le c_7 \left(\int\limits_0^{\rho^{\frac{1}{\theta}}} |y|^{-\theta\frac{qn(n-p)}{n-p+\delta} + \sum\limits_{i=1}^{n} \frac{\theta}{b_i} - 1} d|y| \right)^{\frac{n-p+\delta}{n}} \le c_8 \rho^{\frac{\delta}{2}},\ 1 < p < n; \\ A_1 &\le c_9 \rho^{\frac{\delta}{n}} (\ln \rho^{-1})^{q(p-1)} < c_{10} \rho^{\frac{\delta}{2n}},\ p = n; \end{aligned} \tag{3.33}$$

where $0 < r < \rho < \rho_0(R) < R_0$. In the same way we deduce

$$\int\limits_{E(\rho)} g_3(x) u^{p-1}(x) \psi_r^{p_n}(x) dx \le c_{11} \rho^{\frac{\delta}{2n}},\ 1 < p \le n. \tag{3.34}$$

Consider the first term in the right-hand side of (3.30)

$$\int\limits_{E(\rho)} g_1(x)u_R^{-q}(x)u^p(x)\psi_r^{p_n}(x)dx \le c_{12} \int\limits_{E(\rho)} g_1(x)u_R^{p-q}(x)\psi_r^{p_n}(x)dx+$$
$$+c_{12}M^{p-q}(\rho)\int\limits_{E(\rho)} g_1(x)dx \le c_{13}\left(M^{p-q}(\rho)\rho^{n-p+\delta}+\rho^{\delta\left(\frac{1}{2(p-1)}+1\right)}\right),\ 1<p<n;$$
$$\int\limits_{E(\rho)} g_1(x)u_R^{-q}(x)u^p(x)\psi_r^{p_n}(x)dx \le c_{14}\left(M^{p-q}(\rho)\rho^{\delta}+\left(\ln\rho^{-1}\right)^{p-q}\rho^{\frac{\delta}{n}}\right),\ p=n. \tag{3.35}$$

Using condition (2.1), we deduce easily too:

$$\int\limits_{E(\rho)} [f_1(x)+f_3(x)]dx \le c_{15}\rho^{n-p+\frac{\delta}{2n}},\ 1<p\le n; \tag{3.36}$$

$$\sum_{i=1}^{n}\int\limits_{E(\rho)} f_2(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|dx \le c_{16}\sum_{i=1}^{n} r^{\frac{n-p}{(p-1)^2}\cdot\frac{p_i-1}{p_i}p} \le c_{17}r^{\frac{n-p}{(p-1)^2}\cdot\frac{p_1-1}{p_1}p},\ 1<p<n;$$
$$\sum_{i=1}^{n}\int\limits_{E(\rho)} f_2(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|dx \le c_{18}\left(\ln r^{-1}\right)^{-p\frac{p_1-1}{p_1}},\ p=n. \tag{3.37}$$

Therefore after straightforward computation, by using (3.33)–(3.37) and inequalities (3.24), (3.25) for $1<p<n$ (and its analogues for $p=n$), we obtain (3.27), (3.28) from (3.30). Lemma 3.3 is proved.

Now we introduce the main integral a priory estimate of solution.

Proposition 3.1. *Let all assumptions of Proposition 2.1 be satisfied. Then there exists positive constant C_4 such that*

$$\sum_{i=1}^{n}\int\limits_{E(\rho,R)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^{p_n}(x)dx \le$$
$$\le C_4\left\{M(\rho)\left[\omega(r)+r^{p\frac{n-p}{(p-1)^2}\cdot\frac{p_1-1}{p_1}}+\rho^{\frac{\delta}{2}}\right]+1\right\},\quad \textit{if}\ \ 1<p<n; \tag{3.38}$$

$$\sum_{i=1}^{n}\int\limits_{E(\rho,R)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^{p_n}(x)dx \le$$
$$\le C_4\left\{M(\rho)\left[\omega(r)+\left(\ln\frac{1}{r}\right)^{-p\frac{p_1-1}{p_1}}+\rho^{\frac{\delta}{2n}}\right]+1\right\},\quad \textit{if}\ \ p=n. \tag{3.39}$$

Proof. Let us estimate the second term in the right hand-side of (3.20) from Lemma 3.2 by inequality (3.27). Using Hölder's inequality and estimate (3.33), we obtain

$$M(\rho)\int\limits_{E(\rho)} h_j(x)\left|\frac{\partial u}{\partial x_j}\right|^{p_j\left(1-\frac{1}{p}\right)} u_R^{-q\left(1-\frac{1}{p}\right)}(x)u_R^{q\left(1-\frac{1}{p}\right)}(x)\psi_r^{p_n}(x)dx \le$$

$$\le M(\rho)\left(\int\limits_{E(\rho)}\left|\frac{\partial u}{\partial x_j}\right|^{p_j} u_R^{-q}(x)\psi_r^{p_n}(x)dx\right)^{\frac{p-1}{p}}\left(\int\limits_{E(\rho)} h_j^p(x)u_R^{q(p-1)}(x)\psi_r^{p_n}(x)dx\right)^{\frac{1}{p}} \le$$

$$\le cM(\rho)\left(\omega(\rho)+r^{\frac{p(n-p)(p_1-1)}{(p-1)^2 p_i}}+\rho^{\frac{\delta}{2}}\right)^{\frac{p-1}{p}}\rho^{\frac{\delta}{2p}}.$$

By this estimate and inequality (3.34), it follows from (3.20) that (3.38) holds. In the same way we derive (3.39).

Corollary 3.1. *Let all assumptions of Proposition 2.1 be satisfied. Then there exist positive constant C_3 such that*

$$\sum_{i=1}^{n}\int\limits_{E(\rho,R)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx \le C_4\left(M(\rho)\rho^{\frac{\delta}{2n}}+1\right), \quad i=\overline{1,n}, \quad 1<p\le n. \qquad (3.40)$$

It is not hard to prove that this Corollary is an immediate consequence of Proposition 3.1. Indeed, in virtue of assumption (2.6) we have that $\omega(r)\to 0$ as $r\to 0$. Then, passing to the limit in (3.38), (3.39) as $r\to 0$, we obtain (3.40).

4 Point-wise estimates of the solution

Let us define the new family of cutoff functions. Fix a positive numbers a_i, $i=1,2,3,4$, $0<a_1<a_2<a_3<a_4$, and define function $\chi_{a_1,a_2}^{a_3,a_4}(s)\in C^1(\mathbb{R}^1)$ such that

$$0\le\chi_{a_1,a_2}^{a_3,a_4}(s)\le 1,\ \forall\ s\in\mathbb{R}^1,$$

$$\chi_{a_1,a_2}^{a_3,a_4}(s)=1 \text{ in } (a_2,a_3),\ \chi_{a_1,a_2}^{a_3,a_4}(s)=0 \text{ in } \mathbb{R}^1\backslash(a_1,a_4),$$

$$\left|\frac{d\chi_{a_1,a_2}^{a_3,a_4}(s)}{ds}\right| < c\max\left\{(a_2-a_1)^{-1},(a_4-a_3)^{-1}\right\}.$$

Let us set

$$\chi_l(x):=\chi_{a_1,a_2}^{a_3,a_4}(l(x)),$$

where function $l(x)$ is from (2.3), $1<p\le n$. It is easy to check that

$$\chi_l(x)=1 \ \text{ in } \ \Omega(a_3)\backslash\Omega(a_2),$$

$$\chi_l(x)=0 \ \text{ in } \ \mathbb{R}^n\backslash\{\Omega(a_4)\backslash\Omega(a_1)\},$$

$$\left|\frac{\partial\chi_l(x)}{\partial x_i}\right| \le c\max\left\{\frac{a_1^{1-\frac{1}{b_i}}+a_2^{1-\frac{1}{b_i}}}{a_2-a_1}, \frac{a_3^{1-\frac{1}{b_i}}+a_4^{1-\frac{1}{b_i}}}{a_4-a_3}\right\}, \tag{4.1}$$

where constant c does not depend on a_i, $i=1,2,3,4$, $1<p<n$.

Let $u(x)$ be a solution from Theorem 2.1 and $u_R(x)$ is defined by (3.3) for an arbitrary $R, 0<R<R_0$. Then we consider the family of sets

$$A_k(s,t) := \{x\in\Omega : u_R(x)>k\}\cap(\Omega(t)\backslash\Omega(s)), \quad 0<s<t. \tag{4.2}$$

Fix also a constant $\sigma\in(0,2^{-1})$.

Lemma 4.1. *Let all assumptions of Proposition 2.1 be satisfied. Then there exists positive constant $C_5 \neq C_5(k,r,\sigma)$ such that the following estimates hold:*

$$\sum_{i=1}^{n}\int\limits_{A_k\left(\frac{r(1+\sigma)}{2},\frac{3r(1-\sigma)}{2}\right)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx \le C_5\sigma^{-p_n}r^{-\frac{p(n-p)}{p-1}-\delta}\left(\text{mes }A_k\left(\frac{r}{2},\frac{3r}{2}\right)\right)^{1-\frac{p}{n}+\frac{\delta}{n}},$$

$1<p<n$,

$$\sum_{i=1}^{n}\int\limits_{A_k\left(\frac{r(1+\sigma)}{2},\frac{3r(1-\sigma)}{2}\right)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx \le C_5\sigma^{-p_n}r^{-\delta}(\ln r^{-1})^p\left(\text{mes }A_k\left(\frac{r}{2},\frac{3r}{2}\right)\right)^{\frac{\delta}{n}}, \quad p=n. \tag{4.3}$$

Proof. Let us substitute into the integral identity (2.2) the test functions:

$$\varphi(x) = (u_R(x)-k)_+\chi_r^{p_n-1}(x), \quad \psi(x)=\chi_r(x), \quad 1<p<n,$$

where

$$\chi_r(x) := \chi_{a_1,a_2}^{a_3,a_4}(l(x)) \ \text{ with } \ l(x) \ \text{ from } \ (2.3),\ k>0,\ r>0,$$

and

$$a_1=\frac{r}{2}, \quad a_2=\frac{r}{2}(1+\sigma), \quad a_3=\frac{3r}{2}(1-\sigma), \quad a_4=\frac{3r}{2}. \tag{4.4}$$

Denoting $A_{k,r}=A_k\left(\frac{r}{2},\frac{3r}{2}\right)$, $A_{k,r,\sigma}=A_k\left(\frac{r(1+\sigma)}{2},\frac{3r(1-\sigma)}{2}\right)$, using structural conditions (1.2) and Young's inequality, we deduce after standard computations

$$\begin{aligned}\sum_{i=1}^{n}\int\limits_{A_{k,r}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\chi_r^{p_n}(x)dx &\le c_1\int\limits_{A_{k,r}}H_1(x)u^p(x)\chi_r^{p_n}(x)dx+\\ &+c_1\int\limits_{A_{k,r}}(f_1(x)+f_3(x))dx+c_1\sum_{i=1}^{n}\int\limits_{A_{k,r}\backslash A_{k,r,\sigma}}u^{p_i}(x)\left|\frac{\partial\chi_r(x)}{\partial x_i}\right|^{p_i}dx+\\ &+c_1\sum_{i=1}^{n}\int\limits_{A_{k,r}\backslash A_{k,r,\sigma}}\left(g_2(x)u^{p\left(1-\frac{1}{p_i}\right)}(x)+f_2(x)\right)(u_R(x)-k)\left|\frac{\partial\chi_r(x)}{\partial x_i}\right|dx,\end{aligned} \tag{4.5}$$

where $H_1(x)$ is from (2.1). Using conditions (2.1) and property (4.1), we estimate integrals in the right-hand side of (4.5) as follows

$$\int\limits_{A_{k,r}} H_1(x)u^p(x)\chi_r^{p_n}(x)dx \le c_2 M^p\left(\frac{r}{2}\right)(\text{mes } A_{k,r})^{1-\frac{p}{n}+\frac{\delta}{n}}, \tag{4.6}$$

$$\int\limits_{A_{k,r}} (f_1(x)+f_2(x))dx \le c_3 (\text{mes } A_{k,r})^{1-\frac{p}{n}+\frac{\delta}{n}}, \tag{4.7}$$

$$\int\limits_{A_{k,r}\backslash A_{k,r,\sigma}} u^{p_i}(x)\left|\frac{\partial\chi_r}{\partial x_i}\right|^{p_i} dx \le c_4\sigma^{-p_i}r^{-\frac{p_i}{b_i}}M^{p_i}\left(\frac{r}{2}\right)\text{mes }(A_{k,r}\backslash A_{k,r,\sigma}), \tag{4.8}$$

$$\begin{aligned}\int\limits_{A_{k,r}\backslash A_{k,r,\sigma}} \left(g_2(x)u^{p\left(1-\frac{1}{p_i}\right)}+f_2\right)(u_R(x)-k)\left|\frac{\partial\chi_r}{\partial x_i}\right| dx \le c_5\sigma^{-1}r^{-\frac{1}{b_i}}\times \\ \times\left(M\left(\frac{r}{2}\right)+M\left(\frac{r}{2}\right)^{p\left(1-\frac{1}{p_i}\right)+1}\right)[\text{mes}(A_{k,r}\backslash A_{k,r,\sigma})]^{1-\frac{p}{n}+\frac{p}{p_i n}+\frac{\delta(p_i-1)}{p_i n}}.\end{aligned} \tag{4.9}$$

Due to inequalities (3.7), it follows from $A_{k,r}\subset\Omega(2r)$ that mes $A_{k,r}\le cr^n$. Using this fact, condition (2.6), inequality (4.5), and estimates (4.6)–(4.9), we obtain (4.3). This completes the proof of Lemma 4.1.

Let us fix constants $k_0>0$, $\sigma_0\in\left(0,2^{-1}\right)$, $\rho>0$, and introduce the sequences:

$$\begin{aligned} k_h &= k_0(2-2^{-h}), \quad h=0,1,2,\ldots \\ t_h &= \frac{3}{2}\rho(1-\sigma_0(1-2^{-h})), \quad h=0,1,2,\ldots \\ s_h &= \frac{\rho}{2}(1+\sigma_0(1-2^{-h})), \quad h=0,1,2,\ldots \end{aligned} \tag{4.10}$$

According to definition (4.2) consider the sequence of sets

$$A_h := A_{k_h}(s_h,t_h), \quad h=0,1,\ldots. \tag{4.11}$$

Now we shall prove the following generalization of Theorem 5.1 ([2]) to the anisotropic case.

Lemma 4.2. *Let all conditions of Proposition 2.1 are fulfilled. Then there exists positive constant C_6 such that the following a priory estimates hold:*

$$\left(\operatorname*{vraimax}_{\Omega\left(\frac{3\rho}{2}(1-\sigma_0)\right)\backslash\Omega\left(\frac{\rho}{2}(1+\sigma_0)\right)} u(x)\right)^{p\left(1+\frac{n}{\delta}\right)} \le C_6\rho^{-\left(1+\frac{p(n-p)}{(p-1)\delta}\right)n}\int\limits_{\Omega\left(\frac{3}{2}\rho\right)\backslash\Omega\left(\frac{\rho}{2}\right)} u_R^p(x)dx, \quad 1<p<n, \tag{4.12}$$

$$\left(\operatorname*{vraimax}_{\Omega\left(\frac{3\rho}{2}(1-\sigma_0)\right)\backslash\Omega\left(\frac{\rho}{2}(1+\sigma_0)\right)} u(x)\right)^{p\left(1+\frac{n}{\delta}\right)} \le C_6\rho^{-2n}(\ln\rho^{-1})^{\frac{2n^2}{\delta}}\int\limits_{\Omega\left(\frac{3}{2}\rho\right)\backslash\Omega\left(\frac{\rho}{2}\right)} u_R^n(x)dx, \quad p=n. \tag{4.13}$$

Proof. Consider the case $1 < p < n$. We introduce the additional sequences of numbers:

$$\overline{t}_h := \frac{1}{2}(t_h + t_{h+1}) = \frac{3}{2}\rho\left(1 - \sigma_0\left(1 - \frac{3}{4}2^{-h}\right)\right), \quad h = 0, 1, \ldots$$

$$\overline{s}_h := \frac{1}{2}(s_h + s_{h+1}) = \frac{\rho}{2}\left(1 + \sigma_0\left(1 - \frac{3}{4}2^{-h}\right)\right), \quad h = 0, 1, \ldots$$

Define the family of cutoff functions

$$\zeta_{h+1}(x) := \chi^{a_3(h), a_4(h)}_{a_1(h), a_2(h)}(l(x)), \quad h = 0, 1, 2, \ldots,$$

where $a_1(h) = \overline{s}_h$, $a_2(h) = s_{h+1}$, $a_3(h) = t_{h+1}$, $a_4(h) = \overline{t}_h$.

Due to Hölder's inequality and the embedding theorem defined by lemma 7.1, we have

$$J_{h+1} := \int\limits_{A_{h+1}} (u_R(x) - k_{h+1})^p dx \leq \int\limits_{A_{k_{h+1}}(\overline{s}_h, \overline{t}_h)} (u_R(x) - k_{h+1})^p \zeta^p_{h+1}(x) dx \leq$$

$$\leq \left(\text{mes } A_{k_{h+1}}(\overline{s}_h, \overline{t}_h)\right)^{\frac{p}{n}} \left(\int\limits_{A_{k_{h+1}}(\overline{s}_h, \overline{t}_h)} (u_R(x) - k_{h+1})^{\frac{np}{n-p}} \zeta^{\frac{pn}{n-p}}_{h+1}(x) dx \right)^{\frac{n-p}{n}} \leq$$

$$\leq \left(\text{mes } A_{k_{h+1}}(\overline{s}_h, \overline{t}_h)\right)^{\frac{p}{n}} \times$$

$$\times \prod_{i=1}^{n} \left(\int\limits_{A_{k_{h+1}}(\overline{s}_h, \overline{t}_h)} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i} \zeta^{p_i}_{h+1}(x) + (u_R(x) - k_{h+1})^{p_i} \left|\frac{\partial \zeta_{h+1}}{\partial x_i}\right|^{p_i} \right) dx \right)^{\frac{p}{np_i}}. \tag{4.14}$$

Using the embedding theorem with $q = \frac{2n^2}{\delta}$, we get such estimate for the case $p = n$.

It follows from Lemma 4.1, that there holds

$$\begin{aligned} R_0 := \int\limits_{A_{k_{h+1}}(\overline{s}_h, \overline{t}_h)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \zeta^{p_i}_{h+1} dx \leq \\ \leq c_1 \sigma_0^{-p_n} \rho^{-\frac{p(n-p)}{p-1} - \delta} 2^{hp_n} \left(\text{mes } A_{k_{h+1}}(s_h, t_h)\right)^{1 - \frac{p}{n} + \frac{\delta}{n}} \end{aligned} \tag{4.15}$$

In virtue of property (4.1) and assumption (2.6) we have also

$$\begin{aligned} R_1 := \int\limits_{A_{k_{h+1}}(\overline{s}_h, \overline{t}_h)} (u_R - k_{h+1})^{p_i} \left|\frac{\partial \zeta_{h+1}}{\partial x_i}\right|^{p_i} dx \leq \\ \leq c_2 \sigma_0^{-p_n} 2^{p_i h} \rho^{-\frac{p_i}{b_i}} M^{p_i}(\overline{s}_h) \text{mes } A_{k_{h+1}}(s_h, t_h) \leq \\ \leq c_3 \sigma_0^{-p_n} 2^{p_n h} \rho^{-\frac{p(n-p)}{p-1} - \delta} \left(\text{mes } A_{k_{h+1}}(s_h, t_h)\right)^{1 - \frac{p}{n} + \frac{\delta}{n}}. \end{aligned} \tag{4.16}$$

Using estimates (4.15), (4.16), we deduce from (4.14) that

$$J_{h+1} \le c_4 2^{p_n h} \rho^{-\left(\frac{p(n-p)}{p-1}+\delta\right)} \left(\text{mes } A_{k_{h+1}}(s_h,t_h)\right)^{1+\frac{\delta}{n}}. \tag{4.17}$$

Estimate mes $A_{k_{h+1}}(s_h,t_h)$ from above as follows

$$J_h \ge \int_{A_{k_{h+1}}(s_h,t_h)} (u_R-k_h)^p dx \ge (k_{h+1}-k_h)^p \text{mes } \left(A_{k_{h+1}}(s_h,t_h)\right) =$$

$$= 2^{-p(h+1)} k_0^p \text{mes } \left(A_{k_{h+1}}(s_h,t_h)\right).$$

By the last estimate, inequality (4.17) implies that

$$J_{h+1} \le c_5 2^{p\left(2+\frac{\delta}{n}\right)h} k_0^{-p\left(1+\frac{\delta}{n}\right)} \rho^{-\left(\frac{p(n-p)}{p-1}+\delta\right)} J_h^{1+\frac{\delta}{n}}, \quad h=0,1,\dots \tag{4.18}$$

Due to Lemma 7.3 it follows from (4.18) that

$$J_h \to 0 \quad \text{as} \quad h\to\infty, \tag{4.19}$$

if J_0 satisfies the smallness condition

$$J_0 \le \left[c_5 k_0^{-p\left(1+\frac{\delta}{n}\right)} \rho^{-\frac{p(n-p)}{p-1}+\delta}\right]^{-\frac{n}{\delta}} 2^{-\frac{p(2n+\delta)n}{\delta^2}} \tag{4.20}$$

or, equivalently,

$$k_0^{p\left(1+\frac{n}{\delta}\right)} \ge c_5^{\frac{n}{\delta}} 2^{\frac{2(2n+\delta)n}{\delta^2}} \rho^{-\left(n+\frac{n(n-p)p}{(n-1)\delta}\right)} J_0. \tag{4.21}$$

Estimate (4.12) follows immediately from (4.19), (4.21). To conclude the proof, it remains to note that using the same arguments we obtain (4.13) for $p=n$. The smallness condition for this case is the following:

$$J_0 \le \left[C_6 k_0^{-n(1+\frac{\delta}{2n})} \rho^{-\delta} (\ln\rho^{-1})^n\right]^{-\frac{2n}{\delta}} 2^{-\frac{2n^2}{\delta^2}(4n+\delta)}.$$

Now we are ready to prove Proposition 2.1, which is a main step on the way to the proof of Theorem 2.1.

Proof of Proposition 2.1. Since $\Omega\left(\frac{3}{2}\rho\right) \setminus \Omega\left(\frac{\rho}{2}\right) \subset E\left(\frac{\rho}{2},R\right)$, by the Hölder inequality, we obtain

$$I_i = \int\limits_{\Omega\left(\frac{3}{2}\rho\right)\setminus\Omega\left(\frac{\rho}{2}\right)} u_R^p(x)dx \le c\rho^p \left(\int\limits_{E\left(\frac{\rho}{2},R\right)} u_R^{\frac{np}{n-p}}(x)dx\right)^{\frac{n-p}{n}}.$$

Applying embedding Lemma 7.1 with $\alpha_1=\alpha_2=\dots=\alpha_n=0$, inequality (3.40) from Corollary 3.1, and assumption (2.6), we deduce from the last inequality that

$$I_1 \le c_1\rho^p \sum_{i=1}^n \int\limits_{E\left(\frac{\rho}{2},R\right)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \le c_2\left(M(\rho)\rho^{p+\frac{\delta}{2}}+\rho^p\right) \le c_3\rho^{-\frac{n-p}{p-1}+p+\frac{\delta}{2}}.$$

Inserting this estimate into inequality (4.12), we obtain after simple computations

$$\operatorname*{vraimax}_{\Omega\left(\frac{3\rho}{2}(1-\sigma_0)\right)\setminus\Omega\left(\frac{\rho}{2}(1+\sigma_0)\right)} u(x) \leq c_4\rho^{-\frac{n-p}{p-1}+\frac{\delta}{2p\left(1+\frac{np}{\delta q}\right)}}. \tag{4.22}$$

From this estimate necessity of inequality (2.7) follows immediately. Proposition 2.1 is proved in the case $1 < p < n$.

5 Proof of Theorem 2.1

Define

$$\lambda = \min\left\{\frac{1}{2}\cdot\frac{(p_1-1)(p-1)}{n-p}\gamma, \frac{n-1}{n-p}p - p_n, p_1 - 1\right\}, \tag{5.1}$$

where γ is from (2.7).

Lemma 5.1. *Let all conditions of theorem 2.1 are satisfied, then there exists positive constant* C_8 *such that*

$$\sum_{i=1}^{n}\int\limits_{A_k}(u-k)^{\lambda-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx \leq C_8\left(k^{p+\lambda-1}+1\right) \text{ mes } A_k^{1-\frac{p}{n}+\frac{\delta}{n}}, \quad 1<p<n, \tag{5.2}$$

for every $k > \tilde{k}$. *Here* $\tilde{k}$ *depends on known parameters of the problem only.*

Proof. It is easy to check the following property

$$v_k(x) := (u-k)_+^{\lambda}\psi_r^{l-1}(x) \in \overset{\circ}{W}{}^1_{(\overline{p})}(\Omega(R_0)) \;\forall k > M\left(\frac{R_0}{2}\right), \quad l \geq p_n, \tag{5.3}$$

where $\psi_r(x)$ is the cutoff function from (3.5). Due to property (5.3) we can substitute into the integral identity (2.2) the test functions:

$$\varphi(x) = v_k(x), \quad \psi = \psi_r(x), \quad k > M\left(\frac{R_0}{2}\right).$$

Using structural conditions (1.2), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\int\limits_{A_k}(u-k)^{\lambda-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^l(x)dx \leq c_1\sum_{i=1}^{n}\int\limits_{A_k}(u-k)^{p_i-1+\lambda}\left|\frac{\partial \psi_r}{\partial x_i}\right|^{p_i}dx+\\
&+c_1\sum_{i=1}^{n}\int\limits_{A_k}(u-k)^{\lambda}\left[g_2(x)u^{p\left(1-\frac{1}{p_i}\right)}(x)+f_2(x)\right]\left|\frac{\partial\psi_r}{\partial x_i}\right|dx+\\
&+c_1\int\limits_{A_k}\Big[\sum_{i=1}^{n}h_i^p(x)(u-k)^{p-1+\lambda}+g_3(x)(u-k)_+^{\lambda}u^{p-1}+\\
&+f_3(x)(u-k)^{\lambda}\Big]\psi_r^l(x)dx = \sum_{j=1}^{3}I_j,
\end{aligned} \tag{5.4}$$

where $A_k = \{x \in \Omega(R_0) : u(x) > k\}$. Property (2.7), the Hölder inequality, and estimate (3.7) imply that

$$I_1 \le c_2 r^{\frac{\gamma}{2}(p_1-1)\left(1+\frac{2\lambda}{p_1-1}\right)}, \quad I_2 \le c_3 r^{\frac{\gamma}{2}\left(1-\frac{1}{p_i}\right)\left(1+\frac{2\delta}{\gamma}\right)}, \tag{5.5}$$

$$\begin{aligned} I_3 \le c_4 \left(\int_{A_k} (u-k)^{\frac{n(p-1+\lambda}{n-p}} \psi_r(x)^{\frac{ln}{n-p}} dx \right)^{\frac{n-p}{n}} (\text{mes } A_k)^{\frac{\delta}{n}} + \\ + c_4 \left(k^{p-1+\lambda} + 1\right) (\text{mes } A_k)^{1-\frac{p}{n}+\frac{\delta}{n}} . \end{aligned} \tag{5.6}$$

Applying Lemma 7.2 with $\alpha_i = 1-\lambda$, $i = \overline{1,n}$, $q = \dfrac{np}{(n-p)}\left(1+\frac{\lambda-1}{p}\right)$, we deduce

$$\begin{aligned} &\left(\int\limits_{A_k} \left[(u-k)\psi_r(x)^{\frac{l}{p-1+\lambda}} \right]^{\frac{n(p-1+\lambda)}{n-p}} dx \right)^{\frac{n-p}{n}} \le \\ &\le c_5 \sum_{i=1}^n \int\limits_{A_k} (u-k)^{\lambda-1} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \psi_r(x)^{\frac{l(p_i+\lambda-1)}{p+\lambda-1}} dx + \\ &+ c_5 \sum_{i=1}^n \int\limits_{A_k} (u-k)^{p_i-1+\lambda} \psi_r(x)^{\frac{l(p_i+\lambda-1)}{p+\lambda-1}-p_i} \left| \frac{\partial \psi}{\partial x_i} \right|^{p_i} dx := I_3^{(1)} + I_3^{(2)}. \end{aligned} \tag{5.7}$$

Similarly to (5.5) we estimate $I_3^{(2)}$. Therefore from inequality (5.4), due to (5.5)–(5.7), it follows

$$\begin{aligned} &\sum_{i=1}^n \int_{A_k} (u-k)^{\lambda-1} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \psi_r^l(x) dx \le \\ &\le c_6 \left(\sum_{i=1}^n \int_{A_k} (u-k)^{\lambda-1} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \psi_r^{\frac{l(p_i+\lambda-1)}{p+\lambda-1}} dx \right) \times \\ &\times (\text{mes} A_k)^{\frac{\delta}{n}} + c_6 r^{\frac{\gamma}{2}(p_1-1)\left(1+\frac{2\lambda}{p_1-1}\right)} + c_6 r^{\frac{\gamma}{2}\frac{p_1-1}{p_1}\left(1+\frac{2\delta}{\gamma}\right)} + \\ &+ c_6 \left(k^{p+\lambda-1} + 1\right) (\text{mes} A_k)^{1-\frac{p}{n}+\frac{\delta}{n}} . \end{aligned} \tag{5.8}$$

This inequality yields to

$$\begin{aligned} &I_k(2r) := \sum_{i=1}^n \int\limits_{A_k \cap (\Omega \setminus \Omega(2r))} (u-k)^{\lambda-1} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \le \\ &\le c_6 (\text{mes} A_k)^{\frac{\delta}{n}} I_k(r) + c_6 \mu(r) + c_6 g(k), \\ &g(k) := \left(k^{p+\lambda-1} + 1\right) (\text{mes} A_k)^{1-\frac{p}{n}+\frac{\delta}{n}} , \\ &\mu(r) := r^{\frac{\gamma}{2}(p_1-1)\left(1+\frac{2\lambda}{p_1-1}\right)} + r^{\frac{\gamma}{2}\frac{(p_1-1)}{p_1}\left(1+\frac{2\delta}{\gamma}\right)} \to 0 \text{ as } r \to 0. \end{aligned} \tag{5.9}$$

Starting from this estimate, we can use the same scheme as in Lemma 3.1. But we propose here another way of proof.

Now we come back to the term I_3 from (5.4) and estimate it using inequality (2.7). As result, using additional condition (2.1), we obtain

$$\begin{aligned} I_3 &\le c_7 M(r)^\lambda \left(1 + M(r)^{p-1}\right) (\mathrm{mes} A_k)^{1-\frac{p}{n}+\frac{\delta}{n}} \le \\ &\le c_8 \left(r^{-\left(\frac{n-p}{p-1}-\gamma\right)(p+\lambda-1)} + 1\right) (\mathrm{mes} A_k)^{1-\frac{p}{n}+\frac{\delta}{n}} . \end{aligned} \tag{5.10}$$

Using now estimates (5.5), (5.10), we derive from inequality (5.4)

$$I_k(2r) \le c_9 \left(r^{-\nu} + 1\right) \ \forall r > 0, \quad \nu = \left(\frac{n-p}{(p-1)} - \gamma\right)(p+\lambda-1) > 0. \tag{5.11}$$

Thus, nondecreasing function $I_k(r)$ satisfies both inequalities (5.9) and (5.11). We shall prove that there exist number $\tilde{k} > 0$ and sequence $r_i \to 0$ as $i \to \infty$, such that

$$I_k(r_i) \le \left(\mathrm{mes} A_{\tilde{k}}\right)^{-\frac{\delta}{n}} (\mu(r_i) + g(k)) \ \ \forall k \ge \tilde{k}. \tag{5.12}$$

The proof of (5.12) corresponds to scheme of analysis of nonhomogeneous functional inequalities, (see Lemmas 5, 6, [10]). Let us suppose that (5.12) fails and, consequently,

$$\mu(r) + g(k) \le \left(\mathrm{mes} A_{\tilde{k}}\right)^{\frac{\delta}{n}} I_k(r) \ \forall r : 0 < r < r_0. \tag{5.13}$$

Using (5.13) and (5.9) yields

$$I_k(2r) \le 2c_6 \left(\mathrm{mes} A_{\tilde{k}}\right)^{\frac{\delta}{n}} I_k(r) \ \forall k > \tilde{k}, \ \forall r < r_0. \tag{5.14}$$

Iterating this relationship, we deduce easily the following estimate

$$I_k(r) \ge (2c_6)^{-1} \left(\mathrm{mes} A_{\tilde{k}}\right)^{-\frac{\delta}{n}} I_k(r_0) \left(\frac{r}{r_0}\right)^{-h(\tilde{k})} \quad \forall r \le r_0, \tag{5.15}$$

where $h(\tilde{k}) = (\ln 2)^{-1} \left(-\ln(2c_6) - \frac{\delta}{n} \ln \left(\mathrm{mes} A_{\tilde{k}}\right)\right)$. Let $\tilde{k}$ be a fixed number such that the following inequality holds

$$h(\tilde{k}) = (\ln 2)^{-1} \left(\frac{\delta}{n} \ln \left(\left(\mathrm{mes} A_{\tilde{k}}\right)^{-1}\right) - \ln(2c_6)\right) > \nu, \tag{5.16}$$

where ν is from (5.11). It is clear that estimate (5.15) contradicts to estimate (5.11) for such $\tilde{k}$. Therefore our assumption (5.13) is not true and estimate (5.12) is proved for $\tilde{k}$, satisfying condition (5.16). Substituting estimate (5.12) in the right-hand side of (5.9) and passing to the limit as $r_i \to 0$, we obtain (5.2). Lemma 5.1 is proved in the case $1 < p < n$.

Proof of Theorem 2.1. Due to Lemma 7.2 with $\alpha = 1 - \lambda$, $q = \dfrac{n}{n-p}(p + \lambda - 1)$, we deduce from (5.2) that

$$\left(\int\limits_{A_k} (u-k)_+^{\frac{n(p+\lambda-1)}{n-p}} dx\right)^{\frac{n-p}{n}} \le c_1 k^{p+\lambda-1} (\mathrm{mes} A_k)^{1-\frac{p}{n}+\frac{\delta}{n}} . \tag{5.17}$$

Using the Hölder inequality, we have additionally

$$\int\limits_{A_k}(u-k)_+dx \le \left(\int\limits_{A_k}(u-k)_+^{\frac{n(p+\lambda-1)}{n-p}}dx\right)^{\frac{n-p}{n(p+\lambda-1)}}(\text{mes}A_k)^{1-\frac{n-p}{n(p+\lambda-1)}}. \tag{5.18}$$

Combining (5.17) and (5.18), we obtain

$$\int\limits_{A_k}(u-k)_+dx \le k\,(\text{mes}A_k)^{1+\frac{\delta}{n(p+\lambda-1)}} \quad \forall k>\tilde{k}. \tag{5.19}$$

The boundedness of a solution $u(x)$ in domain $\Omega(R_0)$ follows from (5.19) (see Lemma 5.1, [2]). Using mentioned boundedness and property (3.7), we can pass to the limit in inequality (3.11) as $r\to 0$. Then

$$\int\limits_{\Omega}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx < c. \tag{5.20}$$

Let us take in the integral identity (2.2) as a test function $\psi(x)=\psi_r(x)$. Then due to boundedness of solution and property (5.20), it is easy pass to the limit in (2.2) as $r\to 0$. Thus, we obtain validity of integral identity with an arbitrary $\varphi\in \overset{\circ}{W}{}^1_{(\overline{p})}(\Omega)$ and $\psi(x)\equiv 1$. Theorem 2.1 is proved in the case $1<p<n$. These proofs results can be repeated for the case $p=n$, using appropriate estimates from Section 3.

6 Point-wise estimates of source-type solution

Proof of Theorem 2.2. Let us introduce the family of cutoff functions

$$\overline{\psi}_r(x)=1-\psi_r(x)\ \forall r>0, \tag{6.1}$$

where $\psi_r(x)$ is from (3.5). We fix now an arbitrary $\rho>0$ and substitute into the integral identity (2.11) the test function $\varphi=\overline{\psi}^{\,l}_{2\rho}(x)$, $l>np$. Using the Hölder inequality, we get

$$1=l\sum_{i=1}^{n}\int\limits_{\Omega(4\rho)\setminus\Omega(2\rho)}\left|\frac{\partial U}{\partial x_i}\right|^{p_i-2}\frac{\partial U}{\partial x_i}\overline{\psi}^{\,l-1}_{2\rho}(x)\frac{\partial\overline{\psi}_{2\rho}(x)}{\partial x_i}dx \le$$

$$\le c_1\sum_{i=1}^{n}\left(\int\limits_{\Omega(4\rho)\setminus\Omega(2\rho)}\left|\frac{\partial U}{\partial x_i}\right|^{p_i}U^{-\lambda}dx\right)^{\frac{p_i-1}{p_i}}\left(\int\limits_{\Omega(4\rho)\setminus\Omega(2\rho)}U^{\lambda(p_i-1)}\left|\frac{\partial\overline{\psi}_{2\rho}}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}, \tag{6.2}$$

where $0<\lambda<1$. Let $\chi_\rho(x)$ be the cut-off function from (2.3) with

$$a_1=\rho,\quad a_2=2\rho,\quad a_3=4\rho,\quad a_4=6\rho.$$

Substituting into the integral identity (2.11) the test function $\varphi(x) := U^{1-\lambda}(x)\chi_\rho^l(x)$ yields

$$\sum_{j=1}^n \int\limits_{\Omega(4\rho)\setminus\Omega(2\rho)} \left|\frac{\partial U}{\partial x_j}\right|^{p_j} U^{-\lambda}(x)dx \le c_2 \sum_{j=1}^n \int\limits_{\Omega(6\rho)\setminus\Omega(\rho)} U^{p_j-\lambda}(x)\left|\frac{\partial \chi_\rho(x)}{\partial x_j}\right|^{p_j} dx. \quad (6.3)$$

Estimating the first term in the right-hand side of (6.2) by (6.3) and using property (3.7), we obtain

$$\begin{aligned} 1 &\le c_3 \sum_{i=1}^n \left(\sum_{j=1}^n M(\rho)^{p_j-\lambda}\rho^{\frac{(n-p)(p_j-1)}{p-1}}\right)^{\frac{p_i-1}{p_i}} \left(M(2\rho)^{\lambda(p_i-1)}\rho^{\frac{(n-p)(p_i-1)}{p-1}}\right)^{\frac{1}{p_i}} \le \\ &\le c_4 \sum_{i=1}^n \left[\sum_{j=1}^n \left(M(\rho)\rho^{\frac{n-p}{p-1}}\right)^{p_j}\right]^{\frac{p_i-1}{p_i}}. \end{aligned} \quad (6.4)$$

From (6.4) the first estimate in (2.13) follows immediately. Now we shall prove the second inequality from (2.13). Let us introduce the following function $v(x) \in \overset{\circ}{W}{}^1_{(\overline{p})}(\Omega(R_0))$, which was firstly used by J.Serrin, [6]:

$$v(x) = \begin{cases} 0, & u(x) \le M(R) \\ u(x) - M(R), & M(R) < u(x) \le m(\rho) \\ m(\rho) - M(R), & u(x) > m(\rho). \end{cases}$$

Substitute into the integral identity (2.11) the test function $\varphi(x) = v(x)$. As result we obtain

$$\sum_{i=1}^n \int\limits_{M(R)<u(x)<m(\rho)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx = m(\rho) - M(R). \quad (6.5)$$

Due to Lemma 7.1 and (6.5) we have

$$\left(\int_{\{u(x)\ge M(R)\}} |v(x)|^q dx\right)^{\frac{1}{q}} \le c(m(\rho) - M(\rho))^{\frac{1}{p}}, \quad q = \frac{np}{n-p}. \quad (6.6)$$

It is easy to see that

$$\int\limits_{\{u(x)\ge M(R)\}} |v(x)|^q dx \ge \int\limits_{u(x)\ge m(\rho)} |v(x)|^q dx \ge (m(\rho) - M(\rho))^q \rho^n. \quad (6.7)$$

Combining (6.6) and (6.7) we obtain (2.13). Theorem 2.2 is proved in the case $1 < p < n$. It is not hard to prove the same result in the case $p = n$ by the same arguments.

7 Appendix

Lemma 7.1 ([1]). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, function $v(x) \in \overset{\circ}{W}{}_1^1(\Omega)$, and*

$$\sum_{i=1}^{n} \int_{\Omega} |v(x)|^{\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx < \infty, \ \alpha_i \geq 0, \quad p_i \geq 1.$$

If $1 < p < n$, p is defined by (1.3), then $v(x) \in L_q(\Omega)$, $q = \dfrac{np}{n-p}\left(1 + \dfrac{1}{n}\sum\limits_{i=1}^{n} \dfrac{\alpha_i}{p_i}\right)$ and the following inequality holds

$$\|v\|_{L_q(\Omega)} \leq K_3 \prod_{i=1}^{n} \left(\int_{\Omega} |v(x)|^{\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{np_i\left(1+\frac{1}{n}\sum\limits_{k=1}^{n}\frac{\alpha_k}{p_k}\right)}}, \tag{7.1}$$

where the constant K_3 depends on n, α_i, p_i, $i = 1, \ldots, n$ only. If $p = n$, then $v(x) \in L_q(\Omega)$ for an arbitrary $q > 1$ and inequality (7.1) holds with the constant K_3 depending on n, α_i, p_i, q, $i = 1, \ldots, n$ only.

It is easy to prove the following statement by obvious computations using Lemma 7.1.

Lemma 7.2. *Let $v(x) \in \overset{\circ}{W}{}_1^1(\Omega)$ and*

$$\sum_{i=1}^{n} \int_{\Omega} |v|^{-\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx < \infty, \quad \alpha_i \geq 0, \quad p_i \geq 1, \quad \alpha_i < p_i, \quad i = \overline{1,n}.$$

If $1 < p < n$, then $v(x) \in L_q(\Omega)$, $q = \dfrac{np}{n-p}\left(1 - \dfrac{1}{n}\sum\limits_{i=1}^{n} \dfrac{\alpha_i}{p_i}\right)$ and the following inequality holds

$$\|v\|_{L_q(\Omega)} \leq \overline{K}_3 \prod_{i=1}^{n} \left(\int_{\Omega} |v(x)|^{-\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{np_i\left(1-\frac{1}{n}\sum\limits_{k=1}^{n}\frac{\alpha_k}{p_k}\right)}} \tag{7.2}$$

with some positive constant $\overline{K}_3$ depending on $n, \alpha_i, p_i, i = \overline{1,n}$ only.

If $p = n$, then $u(x) \in L_q(\Omega)$ for any $q > 1$ and inequality (7.2) holds with the constant $\overline{K}_3$ depending on $n, \alpha_i, p_i, q, \Omega, i = \overline{1,n}$.

Lemma 7.3 ([2]). *Let sequence y_l, $l = 0, 1, 2, \ldots$ of nonnegative numbers satisfies the following relationship*

$$y_{l+1} \leq c b^l y_l^{1+\varepsilon}, \quad l = 0, 1, \ldots$$

where positive constants $c > 0, \varepsilon > 0, b > 1$ do not depend on l. Then the following estimate is true

$$y_l \leq c^{\frac{(1+\varepsilon)^l - 1}{\varepsilon}} b^{\frac{(1+\varepsilon)^l - 1}{\varepsilon^2} - \frac{1}{\varepsilon}}.$$

Particularly, if

$$y_0 \leq \theta := c^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}$$

then $y_l \leq \theta b^{-\frac{1}{\varepsilon}}$ *and, consequently,* $y_l \to 0$ *as* $l \to \infty$.

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