International Applied Mechanics, Vol. 44, No. 2, 2008

NECESSARY AND SUFFICIENT STABILITY CONDITIONS FOR INVARIANT SETS OF NONLINEAR IMPULSIVE SYSTEMS

R. I. Gladilina and A. O. Ignat'ev

UDC 539.36

Necessary and sufficient conditions for the uniform asymptotic stability of the invariant set of a nonlinear impulsive system are established

Keywords: stability, impulsive system, Lyapunov function, invariant set

Introduction. Systems of impulsive differential equations adequately model many real physical systems subjected to short-term forces at certain instants of time. Impulsive systems are of interest due primarily to their numerous applications, including those in mechanics. For example, such systems can be used in robotics, in space engineering, in modeling vibropercussion machines, buffer units, etc. Intensive development of the theory of impulsive systems and their successful application in applied problems require efficient stability criteria for solutions of such systems. The studies [1, 3, 4, 6, 7, 9–17, etc.] contributed a lot to the development of the theory of stability of impulsive systems. The Lyapunov-function method is one of the most efficient techniques for stability analysis of nonlinear impulsive systems.

The overwhelming majority of published studies that use the direct Lyapunov method analyze the sufficient stability conditions for solutions of impulsive systems, giving little attention to the issue of existence of Lyapunov functions. Whether Lyapunov functions exist is the key question in the direct Lyapunov method because answering it would indicate whether it is expedient to apply the Lyapunov-function method. The necessary conditions for the stability of solutions of impulsive systems with respect to all variables were established in [2, 11].

The purpose of this paper is to derive the necessary and sufficient conditions for the uniform asymptotic stability of the invariant set of a nonlinear impulsive system.

Problem Statement. Consider a system of differential equations with impulsive effect at fixed times:

$$\frac{dx}{dt} = f(t,x), \qquad t \neq \tau_i,$$

$$\Delta x = I_i(x), \qquad t = \tau_i \qquad (i \in N), \tag{1.1}$$

where $t \in R_+$, $x \in \Omega \subset R^n$, $f: R_+ \times \Omega \to R^n$, $I_i: \Omega \to R^n$, $f(t,0) \equiv 0$, and $I_i(0) \equiv 0$.

Denote the phase vector by $x = (y, z)$, $y \in R^m$, $z \in R^s$, $(m + s = n)$, and the vector functions $f(t,x)$ and $I_i(x)$ by $f = (Y, Z)$ and $I_i = (J_i, \bar{J}_i)$, respectively.

Let $x(t) = x(t, t_0, x_0) = (y(t, t_0, x_0), z(t, t_0, x_0))$ be a solution of system (1.1) satisfying the initial conditions $x(t_0) = x_0$. The solution of system (1.1) is assumed left continuous at the points $t = \tau_i$.

We will consider the problem in the domain

$$\Omega = \Omega_H = B_H^m \times R^s, \qquad (H > 0), \qquad B_H^m = \left\{ y \in R^m : \|y\| < H \right\}.$$

Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Donetsk, Ukraine. Translated from Prikladnaya Mekhanika, Vol. 44, No. 2, pp. 132–142, February 2008. Original article submitted September 5, 2006.

We adopt the following hypotheses for system (1.1).

(A1). The function $f(t,x)$ is continuous and has bounded partial derivatives in the domain $R_+ \times \Omega$:

$$\left|\frac{\partial f_j}{\partial x_k}(t,x)\right| \le K, \quad j=1,\dots,n, \quad k=1,\dots,n. \tag{1.2}$$

(A2). The function $I_i(x)$ is continuous and has bounded partial derivatives in the domain Ω:

$$\left|\frac{\partial I_j^i(x)}{\partial x_k}\right| \le K, \quad j=1,\dots,n, \quad k=1,\dots,n \quad (i \in N). \tag{1.3}$$

(A3). There exists a constant $h \in (0,H)$ such that if $x \in \Omega_h$, then

$$x + I_i(x) \in \Omega_H \quad (i \in N).$$

(A4). The sequence of times $\{\tau_i\}$ satisfies the following conditions:

$$0 = \tau_0 < \tau_1 < \tau_2 < \dots, \quad \tau_i \to \infty \text{ as } i \to \infty; \quad \tau_i - \tau_{i-1} \ge \theta > 0 \quad (i \in N).$$

(A5). The solution of system (1.1) is *z*-continuable, which means [8] that any solution $x(t)$ is defined for all $t > t_0$ such that $\| y(t,t_0,x_0)\| < H$.

(A6). There exists a constant $\mu > 0$ such that the inequality

$$\| y^+(\tau_i,x)\| \ge \mu \| y(\tau_i,x)\| \quad (i \in N) \tag{1.4}$$

holds at $t = \tau_i$, where $y^+(\tau_i, x) = y(\tau_i + 0, x)$.

Let the set

$$M = \{x : y = 0\} \tag{1.5}$$

be the invariant set of system (1.1), which means [8] that if $t_0 \ge 0, x_0 \in M$, then $x(t,t_0,x_0) \in M$ for all $t \ge t_0$.

Let us introduce the following definitions.

Definition 1.1. The set (1.5) is called

(i) stable for any $\varepsilon > 0$ and $t_0 \in R_+$ if there exists $\delta(\varepsilon, t_0) > 0$ such that $\| y(t,t_0,x_0)\| < \varepsilon$ follows from $\| y_0 \| < \delta$ and $\| z_0 \| < \infty$ for all $t \ge t_0$;

(ii) uniformly stable if there exists a number $\delta(\varepsilon) > 0$ independent of t_0.

Definition 1.2. The set (1.5) is called

(i) attracting if for any $t_0 \in R_+$ there exists a number $\lambda(t_0) > 0$ such that each solution $x(t,t_0,x_0)$ with $\| y_0 \| < \lambda$, $\| z_0 \| < \infty$ is defined for all $t \ge t_0$ and satisfies the condition

$$\lim_{t\to\infty} \| y(t,t_0,x_0)\| = 0, \tag{1.6}$$

(ii) uniformly attracting if there exists a number $\lambda > 0$ independent of t_0 and such that condition (1.6) is satisfied uniformly in (t_0, x_0) from the domain

$$t_0 \in R_+, \quad \| y_0 \| < \lambda, \quad \| z_0 \| < \infty, \tag{1.7}$$

i.e., for any $e > 0$ there exists a $\delta(e) > 0$ such that $\| y(t,t_0,x_0)\| < \varepsilon$ for (t_0, x_0) from the domain (1.7) for all $t \ge t_0 + \sigma$.

Definition 1.3. The set (1.5) is called

(i) asymptotically stable if it is stable and attracting;

(ii) uniformly asymptotically stable if it is uniformly stable and uniformly attracting.

Let us introduce the notation

$$G_i = \{(t,x) \in R^{n+1} : \tau_{i-1} < t < \tau_i, x \in B_H\}, \quad G = \bigcup_{i=1}^{\infty} G_i,$$

and the concepts of piecewise continuous and piecewise differentiable Lyapunov functions [11].

Definition 1.4. A function V: $R_+ \times B_H \to R$ is said to belong to the class V if the function V is continuous and differentiable on each of the sets G_i and $V(t, 0) \equiv 0$ for $t_0 \in R_+$, and is left continuous at the points τ_i.

For $(t,x) \in G = \bigcup_{k=1}^{\infty} G_k$, let us define a derivative of the function V by virtue of (1.1):

$$\dot{V}_{(1.1)}(t,x) = \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x).$$

Denote a discontinuity of the function V for (1.1) at $t = \tau_i$ by

$$\Delta V_i = V(\tau_i + 0, x + I_i(x)) - V(t,x) \quad (i \in N).$$

In this case, the Lyapunov function along the solution $x(t, t_0, x_0)$ is denoted by $v(t) = V(t, x(t, t_0, x_0))$.

In what follows, we will use the Khan class of K-continuous, strictly increasing functions $a: R_+ \to R_+$ that satisfy the condition $a(0) = 0$.

2. Stability of Invariant Sets. The theorems below are valid for system (1.1).

Theorem 2.1. Let a function $V \in V$ satisfying the conditions below exist for system (1.1):

$$a(||y||) \le V(t,x) \quad \text{for} \quad (t,x) \in R_+ \times \Omega, \quad a \in K, \tag{2.1}$$

$$V(t,x) \le b(||y||) \quad \text{for} \quad (t,x) \in R_+ \times \Omega, \quad b \in K, \tag{2.2}$$

$$\dot{V}_{(1.1)}(t,x) \le 0 \quad \text{for} \quad (t,x) \in G, \tag{2.3}$$

$$V(\tau_i + 0, x + I_i(x)) - V(\tau_i, x) \le 0, \quad i \in N. \tag{2.4}$$

Then the set (1.5) is invariant and uniformly stable.

Proof. For any $\varepsilon > 0$, put $\delta(\varepsilon) = b^{-1}(a(\varepsilon))$. It follows from inequality (2.2) that

$$V(t_0, x_0) \le b(||y_0||) < b(b^{-1}(a(\varepsilon))) = a(\varepsilon)$$

for any $||y_0|| < \delta$, $||z_0|| < +\infty$ and $t_0 \ge 0$.

By virtue of (2.3)–(2.4), the function $V(t, x)$ is not increasing along the solution. With (2.1), we get

$$a(||y(t, t_0, x_0)||) \le V(t, x(t, t_0, x_0)) \le V(t_0, x_0) < a(\varepsilon),$$

whence follows the inequality $||y(t, t_0, x_0)|| < \varepsilon$ for all $t \ge t_0$, i.e., the set (1.5) is uniformly stable.

Let us prove that it is invariant. Consider the solution $x(t, t_0, x_0)$ for $y_0 = 0$ and arbitrary $t_0 \ge 0$ and z_0. With (2.1)–(2.2), we have $V(t_0, 0, z_0) = 0$. Since the function $V(t, x)$ is not increasing along any solution, $V(t, x(t, t_0, 0, z_0)) \equiv 0$, whence (2.1) leads to $y(t, t_0, 0, z_0) \equiv 0$, which means that the set (1.5) is invariant.

Theorem 2.2. Let a function $V \in V$ satisfying conditions (2.1), (2.2), and (2.4) and the condition

$$\dot{V}_{(1.1)}(t,x) \le -c(||y||) \quad \text{for} \quad (t,x) \in G, \quad c \in K, \tag{2.5}$$

exist for system (1.1). Then the set (1.5) is uniformly asymptotically stable.

Proof. As follows from Theorem 2.1, the set (1.5) is invariant and uniformly stable. Let us prove that it is uniformly attracting. Let $0 < \rho < H$ and $\lambda = b^{-1}(a(\rho))$. If $t_0 \ge 0$ and $||y_0|| < \lambda$, then

$$a(||y(t,t_0,x_0)||) \le V(t, x(t,t_0,x_0)) \le V(t_0,x_0) \le b(||y_0||) < b(b^{-1}(a(\rho))) = a(\rho)$$

follows from inequalities (2.1) and (2.2) from the fact that, according to (2.4) and (2.5), the function $V(t, x)$ is not increasing along any solution.

Thus, $\|y(t, t_0, x_0)\| < \rho$ for $t \geq t_0$.

For any given $\varepsilon > 0$, we choose $\delta = b^{-1}(a(\varepsilon))$ and set $\sigma = a(\rho)/c(\delta)$. Let us show that there exists a time $t_1 \in [t_0, t_0 + \sigma]$ at which $\|y(t_1, t_0, x_0)\| < \delta$. Assume by contradiction that $\|y(t, t_0, x_0)\| \geq \delta$ and $c(\|y(t, t_0, x_0)\|) \geq c(\delta)$ for all $t \in [t_0, t_0 + \sigma]$.

With (2.4), we obtain the estimate

$$v(t) \leq v(t_0) + \int_{t_0}^{t} v'(t)dt$$

for all $t \in [t_0, t_0 + \sigma]$.

Considering conditions (2.1) and (2.5), we obtain

$$0 < a(\| y(t, t_0, x_0)\|) \leq V(t, x(t, t_0, x_0)) \leq v(t_0) + \int_{t_0}^{t} v(t)dt \leq v(t_0) - \int_{t_0}^{t} c(\delta)dt < a(\rho) - c(\delta)\sigma = 0$$

at $t = t_0 + \sigma$. This contradiction means that there exists a time $t_1 \in [t_0, t_0 + \sigma]$ such that $\|y(t_1, t_0, x_0)\| < \delta$ and $b(\|y(t_1, t_0, x_0)\|) < b(\delta) = a(\varepsilon)$. Since the function V is not increasing along the solution for all $t \geq t_0 + \sigma > t_1$, we have

$$a(\| y(t, t_0, x_0)\|) \leq V(t, x(t, t_0, x_0)) \leq V(t_1, x(t_1, t_0, x_0)) \leq b(\| y(t_1, t_0, x_0)\|) < a(\varepsilon)$$

whence it follows that $\|y(t, t_0, x_0)\| < \varepsilon$, which means that the limit (1.6) holds uniformly in t_0 and x_0 from the domain (1.7). Therefore, the attraction is uniform (Definition 1.2). Since the set (1.5) is uniformly stable and uniformly attracting, it is uniformly asymptotically stable. The theorem is proved.

3. Existence of Lyapunov Functions. Whether Lyapunov functions exist is the key question to answer in using the direct Lyapunov method. Also of interest is to establish whether the behavior of trajectories defined by a Lyapunov function with certain properties is sufficient to ascertain the existence of a function with such properties.

The problem of existence of Lyapunov functions naturally arises in analyzing the stability of impulsive systems with respect to certain variables.

Theorem 2.2 provides sufficient conditions for the asymptotic stability of the invariant set (1.5) of the impulsive system (1.1). Let us prove that this theorem is reversible. To this end, we will first prove the following lemma.

Lemma 3.1. Let conditions (A1)–(A5) be satisfied. Then the estimates

$$\left\| \frac{\partial y(t_0 + \tau, t_0, x_0)}{\partial x_{0k}} \right\| < M(\tau), \quad k = \overline{1, n}, \tag{3.1}$$

are valid for $\tau > 0$, where $M(\tau)$ is a monotonically increasing, positive, continuous function.

Proof. According to [1; 11, p. 51], the derivatives $u(t) = \dfrac{\partial x(t, t_0, x_0)}{\partial x_{0k}}$ $(1 \leq k \leq n)$ satisfy the following system of variational equations with respect to the initial data:

$$\frac{du}{dt} = A(t)u, \quad t \neq \tau_i,$$

$$\Delta u = B_i u, \quad t = \tau_i, \quad i \in N, \tag{3.2}$$

where $A(t) = \dfrac{\partial f(t, x(t, t_0, x_0))}{\partial x}$, $B_i = \left.\dfrac{\partial I_i(x(t, t_0, x_0))}{\partial x}\right|_{t=\tau_i}$ with the initial conditions

$$u(t_0) = e_k, \tag{3.3}$$

where e_k is the kth unit vector of the space R^n, i.e., the vector with the kth component equal to unity and the remaining component equal to zero.

The solution $u(t) = u(t, t_0, x_0)$ of this system can be represented as

$$u(t) = u_0 + \int_{t_0}^{t} A(\tau)u(\tau)d\tau + \sum_{t_0 \le \tau_i < t} B_i u(\tau_i).$$

Then we have the inequality

$$||u(t)|| \le ||u_0|| + \int_{t_0}^{t} ||A(\tau)u(\tau)||d\tau + \sum_{t_0 \le \tau_i < t} ||B_i u(\tau_i)||.$$

Using the Cauchy–Bunyakovsky inequality, we get

$$||u(t)|| \le ||u_0|| + \int_{t_0}^{t} ||A(\tau)|| \, ||u(\tau)||d\tau + \sum_{t_0 \le \tau_i < t} ||B_i|| \, ||u(\tau_i)||.$$

Considering (1.2) and (1.3) for the elements of the matrices A and B_i, we obtain

$$||u(t)|| \le ||u_0|| + n^2 K \int_{t_0}^{t} ||u(\tau)||d\tau + n^2 K \sum_{t_0 \le \tau_i < t} ||u(\tau_i)||.$$

Applying Lemma 2.2 [9, p. 17] when $C = ||u_0||$ and $\beta = \gamma = n^2K$ yields the estimate

$$||u(t)|| \le ||u_0|| (1 + L_1)^p e^{L_1 (t - t_0)},$$

where $L_1 = n^2K$ and p is the number of points τ_i on the interval $[t_0, t_0 + T)$.

Note that $||u_0|| = 1$ by virtue of (3.3). Since the expression $(1 + L_1)^p$ monotonically increases with increase in $\tau = t - t_0$, the above inequality means that the estimate (3.1) is valid.

Since system (3.2) is the same for all the groups of derivatives, the estimate is valid for any $u(t) = \dfrac{\partial x(t, t_0, x_0)}{\partial x_{0k}}, k = \overline{1, n}$.

Theorem 3.1. Let conditions (A1)–(A6) be satisfied, the set (1.5) be invariant and uniformly asymptotically stable, and the domain Ω_ρ $(0 < \rho < H)$ fall into the domain of attraction of this set. Then there exists a constant $P > 0$, functions $a, b, c \in K$, and a function V: $R_+ \times \Omega_\rho \to R_+$ that satisfy the conditions

$$\left\| \frac{\partial V}{\partial x}(t, x) \right\| \le P \quad \text{for} \quad (t, x) \in R_+ \times \Omega_\rho, \quad t \ne \tau_i, \tag{3.4}$$

$$a(||y||) \le V(t, x) \quad \text{for} \quad (t, x) \in R_+ \times \Omega_\rho, \quad a \in K, \tag{3.5}$$

$$V(t, x) \le b(||y||) \quad \text{for} \quad (t, x) \in R_+ \times \Omega_\rho, \quad b \in K, \tag{3.6}$$

$$\dot{V}_{(1.1)}(t, x) \le -c(||y||) \quad \text{for} \quad (t, x) \in R_+ \times \Omega_\rho, \quad t \ne \tau_i, \quad c \in K, \tag{3.7}$$

$$V(\tau_i + 0, x + I_i(x)) - V(\tau_i, x) \le 0, \quad x \in \Omega_\rho \quad (i \in N). \tag{3.8}$$

If, moreover, system (1.1) is periodic with period ω, then the function $V(t, x)$ can also be chosen periodic with period ω.

Proof. Since the set (1.5) is uniformly asymptotically stable, $||y(t, t_0, x_0)|| \to 0$ as $t \to \infty$ uniformly in $t_0 \ge 0$, $x_0 \in \Omega_\rho$. Therefore, the inequality

$$||y(t_0 + s, t_0, x_0)||^2 < \varphi(s) \tag{3.9}$$

holds in this domain. Here $\varphi(s)$ is a monotonically decreasing, continuous, scalar function that satisfies the condition $\lim_{s\to\infty}\varphi(s)=0$. Indeed, if we take an infinite sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ $(\varepsilon_i>0)$ decreasing and converging to zero, then for any ε_i from this sequence there exists a number $\sigma_i(\varepsilon_i)$ such that $\|y(t,t_0,x_0)\|<\varepsilon_i$ for all $t>t_0+\sigma_i(\varepsilon_i)$. The sequence σ_i is diverging, i.e., $\sigma_{i+1}>\sigma_i$. Let us consider a monotonically decreasing positive function $\varphi(s)$ such that $\varphi(\sigma_{i+1})=e_i^2$ $(i\in N)$. Such a function will satisfy all the conditions required.

Let M: $R_+\to R_+$ be a monotonically increasing continuous function such that $\lim_{t\to\infty}M(t)=+\infty$. As shown in [5, p. 314–315], there exists a continuously differentiable function $g=g(\varphi)$ such that $g\in K$, $g'\in K$, and

$$\int_0^{\infty} g(\varphi(s))ds<N_1<+\infty \quad (N_1>0), \tag{3.10}$$

$$\int_0^{\infty} g'(\varphi(s))M(s)ds<N_2<+\infty \quad (N_2>0). \tag{3.11}$$

Let us define the function V as follows:

$$V(t,x)=\int_t^{\infty} g(\|y(s,t,x)\|^2)ds\equiv\int_0^{\infty} g(\|y(t+s,t,x)\|^2)ds \quad \text{for} \quad (t,x)\in R_+\times\Omega_\rho, \quad t\neq\tau_i, \tag{3.12}$$

$$V(\tau_i,x)=V(\tau_i-0,x) \quad \text{for} \quad t=\tau_i, \quad x\in\Omega_\rho \quad (i\in N) \tag{3.13}$$

and show that it satisfies all the conditions of the theorem.

Estimates (3.9) and (3.10) yield

$$V(t,x)=\int_0^{\infty} g(\|y(t+s,t,x)\|^2)ds\le\int_0^{\infty} g(\varphi(s))ds<N_1 \quad \text{for} \quad (t,x)\in R_+\times\Omega_\rho, \quad t\neq\tau_i.$$

Hence, the integral (3.12) converges. Next, there exists

$$\lim_{t\to\tau_i-0}\int_t^{\infty} g(\|y(s,t,x)\|^2)ds=V(\tau_i-0,x)=V(\tau_i,x).$$

Hence, the function $V(t,x)$ is defined and uniformly bounded in the domain $R_+\times\Omega_\rho$, is continuous in this domain at $t\neq\tau_i$, and is left continuous at $t=\tau_i$.

Let us prove property (3.4). Find the derivatives $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial t}$ at $t\neq\tau_i$

$$\frac{\partial V}{\partial x_k}=\int_t^{\infty} g'(\|y(s,t,x)\|^2)\frac{\partial(\|y(s,t,x)\|^2)}{\partial x_k}ds, \quad k=\overline{1,n}.$$

Determine estimates for $\frac{\partial(\|y(s,t,x)\|^2)}{\partial x_k}$:

$$\left|\frac{\partial(\|y(s,t,x)\|^2)}{\partial x_k}\right|=\left|\frac{\partial}{\partial x_k}(y_1^2+\ldots+y_m^2)\right|=2\left|\left(y_1\frac{\partial y_1}{\partial x_k}+\ldots+y_m\frac{\partial y_m}{\partial x_k}\right)\right|=2\left|y\frac{\partial y}{\partial x_k}\right|\le2\|y\|\left\|\frac{\partial y}{\partial x_k}\right\|.$$

Since $\|y\|<\rho$, we finally obtain

$$\left|\frac{\partial(\|y(s,t,x)\|^2)}{\partial x_k}\right| \le 2\|y\|\left\|\frac{\partial y}{\partial x_k}\right\| < 2\rho M(s) \equiv \overline{M}(s), \quad k=\overline{1,n}, \tag{3.14}$$

according to (3.1).

Considering estimates (3.11) and (3.14), we get

$$\left|\frac{\partial V}{\partial x_k}\right| \le \left|\int_t^\infty g'(\|y(s,t,x)\|^2)\frac{\partial(\|y(s,t,x)\|^2)}{\partial x_k}ds\right| < \int_t^\infty g'(\varphi(s))\overline{M}(s)ds < N_2 = P, \quad k=\overline{1,n}.$$

Property (3.4) is proved.

Let us prove property (3.5). Let $t_0 \in (\tau_{i-1}, \tau_i)$. The solution $x(t) = x(t, t_0, x_0)$ of system (1.1) at $t \in [t_0, \tau_i]$ coincides with one of the solutions of the system of ordinary differential equations

$$\frac{dx}{dt} = f(t,x).$$

Let us consider the first m equations of this system

$$\frac{dy_1}{dt} = Y_1(t,x), \ \ldots, \ \frac{dy_m}{dt} = Y_m(t,x).$$

Multiplying the kth equation by y_k ($k = 1, \ldots, m$) and then summing these equations over k, we obtain

$$y_1\frac{dy_1}{dt} = y_1 Y_1(t,x), \ \ldots, \ y_m\frac{dy_m}{dt} = y_m Y_m(t,x),$$

$$\frac{1}{2}\frac{d}{dt}\sum_{k=1}^m y_k^2 = \sum_{k=1}^m y_k Y_k = yY,$$

whence

$$\frac{1}{2}\frac{d\|y\|^2}{dt} \le |yY| \le \|y\| \, \|Y\|. \tag{3.15}$$

Since the set (1.5) is invariant, the identity $Y(t, 0, z) \equiv 0$ holds. Therefore, sequentially applying the mean-value theorem, we get

$$\|Y(t,y,z)\| \equiv \|Y(t,y_1,\ldots,y_m,z)\| \le \|Y(t,y_1,\ldots,y_m,z) - Y(t,0,y_2,\ldots,y_m,z)\|$$

$$+\|Y(t,0,y_2,\ldots,y_m,z) - Y(t,0,0,y_3,\ldots,y_m,z)\| + \ldots + \|Y(t,0,0,\ldots,y_m,z) - Y(t,0,0,\ldots,0,z)\|$$

$$= \left\|\frac{\partial Y}{\partial y_1}(t,\xi_1 y_1, y_2,\ldots,y_m,z)y_1\right\| + \left\|\frac{\partial Y}{\partial y_2}(t,0,\xi_2 y_2, y_3,\ldots,y_m,z)y_2\right\| + \ldots + \left\|\frac{\partial Y}{\partial y_m}(t,0,0,\ldots,\xi_m y_m,z)y_m\right\|$$

$$\le \left\|\frac{\partial Y}{\partial y_1}(t,\xi_1 y_1, y_2,\ldots,y_m,z)\right\||y_1| + \left\|\frac{\partial Y}{\partial y_2}(t,0,\xi_2 y_2, y_3,\ldots,y_m,z)\right\||y_2| + \ldots + \left\|\frac{\partial Y}{\partial y_m}(t,0,0,\ldots,\xi_m y_m,z)\right\||y_m|,$$

where $\xi_k \in (0,1)$ $(k = 1,\ldots,m)$.

Considering condition (1.2), we obtain

$$\left\|\frac{\partial Y}{\partial y_k}(t,0,\ldots,\xi_k y_k,\ldots,y_m,z)\right\| = \sqrt{\sum_{j=1}^m\left(\frac{\partial Y_j}{\partial y_k}\right)^2} \le \sqrt{\sum_{j=1}^m K^2} = \sqrt{m}K \qquad (k=1,\ldots,m).$$

Hence,

$$||Y(t,y,z)|| \le \sum_{k=1}^{m} \left\| \frac{\partial Y}{\partial y_k} \right\| |y_k| \le \sqrt{m}K \sum_{k=1}^{m} |y_k| \le m\sqrt{m}K ||y||.$$

Substituting this into (3.15) yields the estimate

$$\frac{d||y||^2}{dt} \le 2L_2 ||y||^2,$$

where $L_2 = m\sqrt{m}K$.

Separating variables and integrating yield

$$||y_0|| e^{-L_2(t-t_0)} \le ||y(t,t_0,x_0)|| \le ||y_0|| e^{L_2(t-t_0)}.$$

Thus, the estimate

$$||y(t,t_0,x_0)|| \ge ||y_0|| e^{-L_2(t-t_0)}$$

is valid for $\tau_{i-1} < t_0 \le \tau_i$. By virtue of (1.4), we have

$$||y(\tau_i+0,t_0,x_0)|| \ge \mu ||y(\tau_i,t_0,x_0)|| \ge \mu ||y_0|| e^{-L_2(\tau_i - t_0)}.$$

The interval $[t_0, t_0+\theta]$ contains no more than one point τ_i by virtue of condition (A4). Therefore,

$$||y(t,t_0,x_0)|| \ge \min(1,\mu) ||y_0|| e^{-L_2(t-t_0)} \quad \text{for} \quad t \in [t_0, t_0+\theta].$$

Denoting $\gamma = \min(1,\mu)$ yields

$$V(t,x) \ge \int_0^\theta g(||y(t+s,t,x)||^2)ds \ge \int_0^\theta g(||y(t)||^2 \gamma^2 e^{-2L_2 s})ds \ge g(||y(t)||^2 \gamma^2 e^{-2L_2\theta})\theta \equiv a(||y||).$$

Condition (3.5) is satisfied.

Let us prove property (3.6). Considering that the set (1.5) is invariant and using formulas (3.12) and (3.13), we get

$$V(t, 0, z) \equiv 0.$$

Using the mean-value theorem and the proved fact that $\left|\frac{\partial V}{\partial x_k}\right| < P, k = \overline{1,n}$, we obtain

$$V(t,x) \equiv V(t,y_1,\ldots,y_m,z) - V(t,0,y_2,\ldots,y_m,z)$$

$$+V(t,0,y_2,\ldots,y_m,z) - V(t,0,0,y_3,\ldots,y_m,z) + \ldots + V(t,0,0,\ldots,0,y_m,z) - V(t,0,0,0,\ldots,z)$$

$$\le \left|\frac{\partial V}{\partial y_1}(t,\xi_1 y_1, y_2,\ldots,y_m,z)y_1\right| + \left|\frac{\partial V}{\partial y_2}(t,0,\xi_2 y_2, y_3,\ldots,y_m,z)y_2\right| + \ldots + \left|\frac{\partial V}{\partial y_m}(t,0,0,\ldots,\xi_m y_m,z)y_m\right|,$$

where $\xi_k \in (0,1)$ $(k = 1, \ldots, m)$.

Next,

$$V(t,x) \le \sum_{k=1}^{m} \left|\frac{\partial V}{\partial y_k}\right| |y_k| \le \sum_{k=1}^{m} P|y_k| \le Pm||y|| \equiv b(||y||).$$

Condition (3.6) is satisfied.

Let us set up an expression for the total derivative $\dot{V}_{(1.1)}(t,x)$ by virtue of system (1.1)

$$\dot{V}_{(1.1)}(t,x)=\left[\frac{dV}{d\tau}(\tau,x(\tau,t,x))\right]_{\tau=t}.$$

Since the solution $x(s,\tau,x(\tau,t,x))=x(s,t,x)$ is unique, we have

$$\dot{V}_{(1.1)}(t,x)=\left[\frac{d}{d\tau}\left(\int_{\tau}^{\infty}g(\|y(s,\tau,x(\tau,t,x))\|^2)ds\right)\right]_{\tau=t}=\left[\frac{d}{d\tau}\left(\int_{\tau}^{\infty}g(\|y(s,t,x)\|^2)ds\right)\right]_{\tau=t}$$

$$=-g(\|y(t,t,x)\|^2)\equiv -c(\|y\|) \quad \text{for} \quad (t,x)\in R_+\times\Omega_\rho, \quad t\neq\tau_i,$$

i.e., condition (3.7) is satisfied.

Property (3.8) follows from $y(\tau_i+s,\tau-0,x)=y(\tau_i+s,\tau_i+0,x+I_i(x))$ and from (3.12) and (3.13).

Assume that system (1.1) is periodic with period ω. Let us show that the function $V(t, x)$ defined by (3.12) and (3.13) is periodic with period ω, i.e.,

$$V(t+\omega,x)\equiv V(t,x).$$

Indeed,

$$V(t+\omega,x)=\int_{t+\omega}^{\infty}g(\|y(s,t+\omega,x)\|^2)ds \quad \text{for} \quad (t,x)\in R_+\times\Omega_\rho.$$

Changing the variable $s=\tau+\omega$ in the integral yields

$$V(t+\omega,x)=\int_{t}^{\infty}g(\|y(\tau+\omega,t+\omega,x)\|^2)d\tau.$$

Using the property of solutions of periodic systems

$$y(t+\omega,t_0+\omega,x_0)\equiv y(t,t_0,x_0),$$

we obtain

$$V(t+\omega,x)\equiv V(t,x).$$

The theorem is proved.

4. Conclusions. The existence theorems for the Lyapunov function prove that the Lyapunov method is universal. This implies with some certainty that Lyapunov functions are a suitable choice for solving relevant problems. It follows from our study that the behavior of trajectories defined by a Lyapunov function is, according to Theorem 2, not only necessary but also sufficient condition for the existence of such a function.

REFERENCES

1. M. U. Akhmetov and N. A. Perestyuk, "Differentiable dependence of the solutions of impulse systems on initial data," *Ukr. Math. J.*, **41**, No. 8, 878–882 (1989).
2. R. I. Gladilina and A. O. Ignatyev, "On necessary and sufficient conditions for the asymptotic stability of impulsive systems," *Ukr. Math. J.*, **55**, No. 8, 1254–1264 (2003).
3. R. I. Gladilina and A. O. Ignat'ev, "Stability of periodic systems with impulsive effect," *Mat. Zametki*, **76**, No. 1, 44–51 (2004).

4. S. I. Gurgula and N. A. Perestyuk, "Second Lyapunov's method in systems with impulsive effect," *Dokl. AN USSR*, *Ser. A*, No. 10, 11–14 (1982).
5. I. G. Malkin, *Theory of Stability of Motion* [in Russian], Nauka, Moscow (1966).
6. A. A. Martynyuk and V. I. Slyn'ko, "Stability of a nonlinear impulsive system," *Int. Appl. Mech*., **40**, No. 2, 231–239 (2004).
7. M. O. Perestyuk and O. S. Chernikova, "On stability of integral sets of impulsive differential systems," *Ukr. Math. J*., **54**, No. 2, 306–315 (2002).
8. V. V. Rumyantsev and A. S. Oziraner, *Stability and Stabilization of Motion with Respect to a Part of Variables* [in Russian], Nauka, Moscow (1987).
9. A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations* [in Russian], Vyshcha Shkola, Kyiv (1987).
10. M. U. Akhmetov and A. Zafer, "Stability of the zero solution of impulsive differential equations by the Lyapunov second method," *J. Math. Anal. Appl*., **248**, 69–82 (2000).
11. D. D. Bainov and P. S. Simeonov, *System with Impulse Effect: Stability*, *Theory and Applications*, Ellis Horwood, Chichester (1989).
12. A. Dishliev and D. Stoykov, "Stability of limiting equations of impulsive differential equations," *Int. J. Differ. Equat. Appl*., **6**, No. 4, 389–410 (2002).
13. A. I. Dvirnyi, "Sufficient conditions for the practical and technical stability of quazilinear impulsive systems," *Int. Appl. Mech*. **42**, No. 1, 104–110 (2005).
14. S. Kaul, V. Lakshmikantham, and S. Leela, "Extremal solutions, comparison principle and stability criteria for impulsive differential equations with variable times," *Nonlinear Analysis. Theory*, *Methods and Applications*, **22**, No. 10, 1263–1270 (1994).
15. V. B. Larin, "Special cases in a control problem," *Int. Appl. Mech*., **42**, No. 2, 221–227 (2006).
16. V. B. Larin, "On static output-feedback stabilization of a periodic system," *Int. Appl. Mech*., **42**, No. 3, 357–363 (2006).
17. V. B. Larin and A. A. Tunik, "Dynamic output feedback compensation of external disturbances," *Int. Appl. Mech*., **42**, No. 5, 606–616 (2006).
18. X. Liu, "Progress in stability of impulsive system with applications to population growth models," in: A. A. Martynyuk (ed.), *Advances in Stability Theory*, **13**, Taylor and Francis, London (2003), pp. 321–338.
19. A. A. Martynyuk and I. P. Stavroulakis, "Stability analysis with respect to two measures of impulsive system under structural perturbations," *Ukr. Math. J*., **51**, No. 11, 1476–1484 (1999).
20. A. M. Samoilenko and N. A. Perestuk, *Impulsive Differential Equations*, World Scientific, Singapure (1995).
21. P. S. Simeonov and D. D. Bainov, "Stability with respect to part of the variables in system with impulse effect," *J. Math. Anal. Appl*., **124**, 547–560 (1987).
22. T. V. Zavrazhina, "Control of the spatial motions of a gantry manipulator with elastic links," *Int. Appl. Mech*., **42**, No. 2, 228–234 (2006).