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SINGULAR CAUCHY PROBLEM FOR THE EQUATION OF FLOW OF THIN VISCOUS FILMS WITH NONLINEAR CONVECTION

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For multidimensional equations of flow of thin capillary films with nonlinear diffusion and convection, we prove the existence of a strong nonnegative generalized solution of the Cauchy problem with initial function in the form of a nonnegative Radon measure with compact support. We determine the exact upper estimate (global in time) for the rate of propagation of the support of this solution. The cases where the degeneracy of the equation corresponds to the conditions of "strong" and "weak" slip are analyzed separately. In particular, in the case of "weak" slip, we establish the exact estimate of decrease in the L^2-norm of the gradient of solution. It is well known that this estimate is not true for the initial functions with noncompact supports.

1. Introduction

We study the Cauchy problem for a quasilinear degenerate fourth-order parabolic equation of the form

$$u_t + \operatorname{div}(a_0 u^n \nabla\Delta u - a_1 u^m \nabla u) = \overrightarrow{\chi} \cdot \nabla b(u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.1}$$

$$u(0,x) = \mu_0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

$$|b'(z)| \le c\,|z|^{\lambda-1} \quad \forall z \in \mathbb{R}^1, \quad b(0) = 0, \quad \lambda > 0, \quad c < \infty, \tag{1.3}$$

where $u = u(t,x)$, $n > 0$, $m \in \mathbb{R}^1$, $N \le 3$, $a_0 > 0$, $a_1 \ge 0$, $\overrightarrow{\chi} = \overrightarrow{(\chi_1, \ldots, \chi_N)} \in \mathbb{R}^N$, and μ_0 is a nonnegative Radon measure. Equations of the form (1.1) are encountered in simulating various physical processes in materials science and the theory of plasticity, in particular, in the description of flow of thin viscous films over the solid surface (see [1–8]). In the last model, the parameter $n > 0$ specifies the character of contact of liquid with the solid surface: for $n = 3$, this is contact contact without slip, for $n \in [2,3)$, we have "weak" slip, and the case $n \in (0,2)$ corresponds to "strong" slip.

Equation (1.1) does not belong to the class of quasilinear divergent high-order parabolic equations for which it is possible to apply general well-developed methods [e.g., methods of the theory of monotone operators (see [9])]. The construction of the theory of equations of the form (1.1) was originated in the well-known paper by Bernis and Friedman [10] devoted to the investigation of the mixed Cauchy–Neumann problem for a one-dimensional equation of the form

$$u_t + \operatorname{div}\left(|u|^n \nabla\Delta u\right) = 0. \tag{1.4}$$

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A generalized solution of this problem was determined and the existence of a nonnegative generalized solution for an arbitrary nonnegative initial function from H^1 was proved. The important property of nonnegativity of solutions selects the class of equations of the form (1.1) in the set of high-order parabolic equations. Numerous papers are devoted to the description of the qualitative properties of solutions depending on the parameter $n > 0$ and connected with the indicated nonnegativity of the constructed solutions which, in turn, is natural for specific physical applications (see, e.g., [11–15]). Thus, in particular, in [13], it is shown that, for $n \in (0,3)$, $N = 1$, the constructed nonnegative solution of Eq. (1.4) has optimal regularity. The regularity of this solution corresponds to the regularity of the source-type solution (with $u(0,x) = \delta_0$, where δ_0 is the Dirac function) constructed in [16] for $N = 1$ and in [17] for $N = 2, 3$. In the cited papers, it is also shown that, for $n \geq 3$, the required solution with finite mass does not exist. For the one-dimensional equation (1.1) with $b(u) = 0$ for $n > 0$ and $0 < m < 1$, the problems of solvability and qualitative behavior of solutions are investigated in [3, 18–20]. The problem of solvability and asymptotic behavior of the boundary of the support of the solution of the one-dimensional equation of thin films with nonlinear convection [(1.1) with $a_1 = 0$] is studied in [21].

The fourth-order degenerate equations (including thin-film and Cahn–Hilliard equations) were studied for the first time by Grün [6] and Elliot and Garke [5] who constructed nonnegative generalized solutions of the corresponding initial-boundary-value problems in multidimensional domains. Later, these investigations were continued for various classes of equations of the form (1.1) in [22–26]. Thus, the solutions of the Cauchy problem with finite nonnegative initial data from the space H^1 were constructed for Eq. (1.4) with $N \leq 3$ and $n \in (1/8, 3)$ in [11, 14, 23, 26], for Eq. (1.1) with $a_1 = 0$, $n \in (0,3)$, $N = 1,$ and $\lambda \in (\max\{3n/4 - 1, 1/8\}, 9/2)$ in [21], and for Eq. (1.1) with $m > 0$, $n \in \big(1/8, 2\big)$, and $\lambda \in (\max\{1, (3n-1)/4\}, (5N+8)/(4N) + \min\{n, 5/4\})$ if $N < 3$ or $\lambda \in (\max\{1, (3n-1)/4\}, 2 + \min\{n, 5/4\})$ if $N = 3$ in [25].

In [27], for Eq. (1.4) with $n \in (0,2)$ and $N = 1$, dal Passo and Garcke constructed (for the first time) a nonnegative generalized mass-preserving solution of problem (1.1)–(1.3) with initial function in the form of a Radon measure. In [24], this result was generalized to the case of multidimensional $(N \leq 3)$ equations (1.1) with $b(u) = 0$.

Remark 1.1. Note that the term "solution of the Cauchy problem" used in the cited papers is, to a certain extent, conventional. In fact, we study the solutions of a problem with unknown free boundary $\partial\mathrm{P}$ $(\mathrm{P} = \{(t,x)\colon u(t,x) > 0\})$ with the following boundary conditions imposed on this boundary:

$$u = \nabla u \cdot \vec{n} = (a_0 u^n \nabla\Delta u - a_1 u^m \nabla u - \vec{\chi} b(u)) \cdot \vec{n} = 0.$$

Up to now, the problem of description of the exact classes of functions in which the constructed solutions are unique remains open. Therefore, it is quite natural to admit the existence of solutions of nonfixed sign with higher smoothness than the solutions constructed above and, hence, having more reasons to be treated as solutions of the Cauchy problem.

In the present paper, for the case of "strong" slip $(0 < n < 2)$, on the basis of the results obtained in [24, 25, 27], we construct a global strong nonnegative solution of problem (1.1)–(1.3) with initial function in the form of a Radon measure with compact support and establish estimates for the rate of decay of this solution in various integral norms. In the fourth section, we study the case of "weak" slip $(2 \leq n < 3)$. The constructed solution is new even for the multidimensional $(N \leq 3)$ equation (1.4) $\big($i.e., for $a_1 = 0$ and $b(u) = 0\big)$. The established exact estimate (4.22) (independent of ε) of decrease (in t) in the norm of gradients $\int_{\mathbb{R}^N} |\nabla u_\varepsilon(t,x)|^2 dx$ of the approximating solutions $u_\varepsilon(t,x)$ characterized by the existence of the upper estimate of the rate of propagation of supports in t uniform in ε is the main result of this construction. This estimate is quite delicate. Thus, as shown in [27], the indicated estimate is, in principle, absent if the measure μ_0 has a noncompact support (the solutions $u(t,x)$ such that $\int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dx = \infty \ \ \forall t > 0$ can serve as an example).

2. Formulation of the Principal Results

Let $Q_{t_1}^{t_2} = (t_1, t_2)\times\mathbb{R}^N$, $B(0,r) = \{x \in \mathbb{R}^N : |x| < r\}$, and $D(0,r) = \{x \in \mathbb{R}^N : x_1 < r\}$. For an $N\times N$ matrix A and vectors $a, b \in \mathbb{R}^N$, we define $\langle a, A, b\rangle := \sum_{i,j=1}^{N} a_i A_{ij} b_j$. By χ_A we denote the characteristic function of the set A. For any measurable function $v(t,x)$, we define the set of positiveness $\mathrm{P} := \mathrm{P}(v) = \{v > 0\} = \{(t,x) \in \mathrm{Dom}(v) : v(t,x) > 0\}$, $C_c^k(Q) := \{v \in C^k(Q) : \operatorname{supp} v \subset Q\}$, $H^k(\mathbb{R}^N) := W_2^k(\mathbb{R}^N)$, $\|.\|_p := \|.\|_{L^p(\mathbb{R}^N)}$, and

$$\Psi_0(z) := \begin{cases} \dfrac{z^{m-n+2}}{(m-n+1)(m-n+2)} + \dfrac{R^{m-n+1}}{m-n+2} - \dfrac{R^{m-n+1}}{m-n+1} z, & m-n+2 \neq 0, 1, \\ -\ln z + z - 1, & m-n+2 = 0, \\ z \ln z - z + 1, & m-n+2 = 1, \end{cases} \tag{2.1}$$

where $R = 0$ for $m-n+1 > 0$ and $R = 1$ for $m-n+1 < 0$. In the cases where the region of integration is clear from the context, we omit the corresponding differentials.

Definition 2.1. *Suppose that $N \leq 3$, μ_0 is a nonnegative Radon measure in $\mathbb{R}^N$, $m > 0$, $n > 0$, and $\lambda > 0$. A nonnegative function $u(t,x) \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^N)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{loc}}(\mathbb{R}^N))$ is called a weak generalized solution of problem (1.1)–(1.3) if*

(i) $\chi_{\mathrm{P}} u^{n-2}|\nabla u|^3$, $\chi_{\mathrm{P}} u^{n-1}|\nabla u|^2$, $u^n|\nabla u|$, $u^m|\nabla u|$, and $b(u)$ belong to the space $L^1_{\mathrm{loc}}([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}^N))$, where $P = P(u)$;

(ii) for any function $\zeta \in C_c^3([0,\infty)\times\mathbb{R}^N)$, the following equality holds:

$$-\int_0^\infty\int_{\mathbb{R}^N} u\zeta_t - \int_{\mathbb{R}^N} \zeta(0,x)\, d\mu_0(x) + \int_0^\infty\int_{\mathbb{R}^N} \overrightarrow{\chi}\cdot\nabla\zeta\, b(u)$$

$$= \frac{n(n-1)}{2} a_0 \iint_{\mathrm{P}(u)} u^{n-2}|\nabla u|^2 \nabla u \nabla\zeta + \frac{n}{2} a_0 \iint_{\mathrm{P}(u)} u^{n-1}|\nabla u|^2 \Delta\zeta$$

$$+\ n a_0 \iint_{\mathrm{P}(u)} u^{n-1}\left\langle \nabla u, D^2\zeta, \nabla u\right\rangle + a_0 \int_0^\infty\int_{\mathbb{R}^N} u^n \nabla u \nabla\Delta\zeta -\ a_1 \int_0^\infty\int_{\mathbb{R}^N} u^m \nabla u\nabla\zeta. \tag{2.2}$$

Remark 2.1. The concept of weak solutions for multidimensional equations of thin-film type is proposed in [5, 6, 22, 24].

Theorem 2.1. *Assume that* $m > 0,\ 1/8 < n < 2,$

$$\max\left\{1, \frac{3n}{4}\right\} < \lambda < \frac{3nN+2}{4N} + \frac{1}{4}\max\left\{n + \frac{2}{N}, m\right\}, \tag{2.3}$$

and μ_0 *is a nonnegative Radon measure with finite mass such that* $\operatorname{supp}(\mu_0)$ *is compact. Then there exists a solution* $u(t,x)$ *of problem (1.1)–(1.3) specified by Definition 2.1 such that*

(i) for any $q' \in \Delta_1 := \left(\max\left\{1, \frac{2}{n+1}\right\}, \frac{4N}{2N+n(N-2)}\right)$ $(\Delta_1 := \{2\}$ *if* $N = 1)$, *there exists a vector function* $\overrightarrow{J} \in L^2_{\text{loc}}(\mathbb{R}^+; L^{q'}(\mathbb{R}^N;\mathbb{R}^N))\ \ \forall\, q' \in \Delta_1$ *such that*

$$u_t = -\operatorname{div}\overrightarrow{J} + \overrightarrow{\chi}\cdot\nabla b(u) \quad \textit{in} \quad L^2_{\text{loc}}(\mathbb{R}^+; (W^1_q(\mathbb{R}^N))^*), \quad q = \frac{q'}{q'-1}; \tag{2.4}$$

(ii) for any

$$\alpha \in \Delta_{n,\lambda} := \big((1/2-n)_+, \min\{(n+1)/3, 2-n\}\big) \tag{2.5}$$

satisfying, in addition, the inequality $\lambda > \max\left\{\frac{3(\alpha+n)}{4}, 1\right\}$ *[it is easy to see that, for* λ *from interval (2.3), the set of these* α *is nonempty], the following inclusions are true:*

$$u^{m-n+2} \in L^\infty_{\text{loc}}([0,\infty); L^1(\mathbb{R}^N)), \qquad u^{\frac{\alpha+n+1}{4}} \in L^4_{\text{loc}}([0,\infty); W^1_4(\mathbb{R}^N)),$$

$$u^{\frac{\alpha+n+1}{2}} \in L^2_{\text{loc}}([0,\infty); H^2(\mathbb{R}^N)), \qquad u^{\frac{\alpha+m+1}{2}} \in L^2_{\text{loc}}([0,\infty); H^1(\mathbb{R}^N));$$

(iii) $u(t,.) \underset{t\to 0}{\overset{*}{\rightharpoonup}} \mu_0$, *i.e.,* $\displaystyle\int_{\mathbb{R}^N} u(t,x)\varphi(x)\,dx \underset{t\to 0}{\to} \int_{\mathbb{R}^N}\varphi(x)\,d\mu_0(x)\ \ \forall\,\varphi \in C^0_c(\mathbb{R}^N)$;

(iv) $\displaystyle\int_{\mathbb{R}^N} u(t) = \int_{\mathbb{R}^N} d\mu_0$ *for any* $t > 0$.

Definition 2.2. *A weak solution* $u(t,x)$ *from Definition 2.1 satisfying, in addition, the regularity condition (ii) in Theorem 2.1 is called a strong solution of problem (1.1)–(1.3).*

Theorem 2.2. *Assume that* μ_0 *is a nonnegative Radon measure with compact support and* $\operatorname{supp}(\mu_0) \subset D(0,R_0),\ R_0 < \infty$. *Then the strong solution* $u(t,x)$ *constructed in Theorem 2.1 has compact support for all* $t > 0$ *and, in addition,* $\operatorname{supp} u(t,.)$ *propagates as a continuous function of* t *and there exists a nondecreasing continuous function* $\Gamma(t),\ \Gamma(0) = 0$, *such that* $\operatorname{supp} u(t,.) \subset D(0, R_0 + \Gamma(t))$ *and the following estimates hold:*

(a) universal estimates for the front of the support:

$$\Gamma(t) \le c_1 \max\left\{\Gamma_0(t), t^{1-\frac{N(\lambda-1)}{nN+4}}\right\} \quad \forall\, t > 0, \quad 1 < \lambda \le n + 1 + \frac{4}{N},$$

or

$$\Gamma(t) \le c_1 \max\left\{\Gamma_0(t), t^{1-\frac{N(\lambda-1)}{mN+2}}\right\} \quad \forall\, t > 0, \quad 1 < \lambda \le m + 1 + \frac{2}{N};$$

(b) estimates for the "slow" front corresponding to $\chi_1 \geq 0$ *and* $b(s) > 0$ *:*

(i) if $\chi_1 = 0$ *or* $\chi_1 > 0,\ b(u) \geq d_0\, u^{\lambda},\ d_0 > 0,$ *and* $\lambda < \min\{n+1, m+1\},$ *then*

$$\Gamma(t) \leq c_3 \Gamma_0(t) \quad \forall t > 0;$$

(ii) if $\chi_1 > 0,\ b(u) \geq d_0\, u^{\lambda},\ d_0 > 0,$ *and* $\lambda > \max\{n+1, m+1\},$ *then*

$$\Gamma(t) \leq c_2 \min\left\{\Gamma_0(t), \max\left\{t^{\frac{\lambda-n-1}{4(\lambda-1)-n}}, t^{\frac{\lambda-m-1}{2(\lambda-1)-m}}\right\}\right\} \quad \forall\, t > 0;$$

(iii) if $\chi_1 > 0,\ b(u) \geq d_0\, u^{\lambda},\ d_0 > 0,$ *and* $n+1 < \lambda < m+1$ *(or* $m+1 < \lambda < n+1$*), then*

$$\Gamma(t) \leq c_4 \Gamma_0(t)\ \forall\, t > 0,$$

where $\Gamma_0(t) = \max\left\{t^{1/(nN+4)},\ t^{1/(mN+2)}\right\}$ *and* $0 < c_i = c_i(n,\, m,\, \lambda,\, N,\, d_0,\, \|\mu_0\|_1)$.

Remark 2.2. By the corresponding linear change of coordinates, an arbitrary direction $\vec{\ell}$ can be transformed into the direction $(x_1, 0, 0)$ and, hence, we obtain the estimates of propagation of $\operatorname{supp} u(t,.)$ in the direction $\vec{\ell}$.

Theorem 2.3. *Assume that* μ_0 *is a nonnegative Radon measure with compact support and* $\operatorname{supp}(\mu_0) \subset B(0, R_0)$*. In addition, let*

$$n \in [2,3), \quad \lambda \in \left(1 + \frac{n}{4}, 2 + \frac{1}{N}\right), \quad a_1 = 0. \tag{2.6}$$

Then there exists a strong solution $u(t,x)$ *of problem (1.1)–(1.3) local in* $t \in (0, T_{\text{loc}})$ *and such that*

(i) for any $\alpha \in (-1, 2-n)$*, the following inclusions are true:*

$$u^{\frac{\alpha+n+1}{4}} \in L^4_{\text{loc}}((0,T_{\text{loc}}); W^1_4(\mathbb{R}^N)), \qquad u^{\frac{\alpha+n+1}{2}} \in L^2_{\text{loc}}((0,T_{\text{loc}}); H^2(\mathbb{R}^N));$$

(ii) $\operatorname{supp} u(t,.)$ *is compact for any* $t \in [0, T_{\text{loc}}]$ *and* $\operatorname{supp} u(t,.) \subset B(0, R_0 + \Gamma(t))$ *with* $\Gamma(t) \leq ct^{\frac{1}{N(n+1)+3}}$*, where* $0 < c = c(n,\ \lambda,\ N,\ \|\mu_0\|_1)$ *and* λ *from relation (2.6) additionally satisfies the condition* $\lambda > \dfrac{n+2}{2}$*;*

(iii) from Theorem 2.1;

(iv) from Theorem 2.1.

3. The Case of "Strong" Slip: $0 < n < 2$

Proof of Theorem 2.1. First, we regularize the initial function μ_0. We choose a sequence $\{u_{0\varepsilon}\}_{\varepsilon>0}$ of nonnegative functions from $H^1(\mathbb{R}^N) \cap L^{m-n+2}(\mathbb{R}^N)$ with compact support such that

$$u_{0\varepsilon} \xrightarrow{*} \mu_0, \int\limits_{\mathbb{R}^N} u_{0\varepsilon} = \int\limits_{\mathbb{R}^N} d\mu_0,\ \operatorname{supp}(u_{0\varepsilon}) \subset \operatorname{supp}(\mu_0) + B(0,\varepsilon). \tag{3.1}$$

Let $u_\varepsilon(t,x)$ be the solution of problem (1.1)–(1.3) with the initial function $u_{0\varepsilon}$ constructed in Theorem A.1. By the law of conservation of mass for $u_\varepsilon,$ we conclude that the sequence

$$\{u_\varepsilon\}_{\varepsilon>0} \quad \text{is uniformly bounded in} \quad L^\infty([0,\infty);L^1(\mathbb{R}^N)). \tag{3.2}$$

Since $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies estimates (v)–(vii) of Lemma A.3 with the right-hand side independent of $\varepsilon>0,$ we have

$$\{\nabla u_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in} \quad L^\infty_{\text{loc}}((0,\infty);L^2(\mathbb{R}^N)), \tag{3.3}$$

$$\{\Psi_0(u_\varepsilon)\}_{\varepsilon>0} \quad \text{is bounded in} \quad L^\infty_{\text{loc}}((0,\infty);L^1(\mathbb{R}^N)). \tag{3.4}$$

By virtue of the Nirenberg–Gagliardo inequality [29], assertion (3.3) implies the uniform boundedness of

$$\{u_\varepsilon\}_{\varepsilon>0} \quad \text{in} \quad L^\infty_{\text{loc}}((0,\infty);L^q(\mathbb{R}^N)) \quad \forall q<\infty, \quad N<3 \quad \text{and} \quad \forall q<6, \quad N=3. \tag{3.5}$$

By using inequality (A.2) (with $\zeta=1),$ for any α from (2.5), we conclude that

$$\left\{u_\varepsilon^{\frac{\alpha+n+1}{2}}\right\}_{\varepsilon>0} \quad \text{is bounded in} \quad L^2_{\text{loc}}((0,\infty);H^2(\mathbb{R}^N)), \tag{3.6}$$

$$\left\{u_\varepsilon^{\frac{\alpha+n+1}{4}}\right\}_{\varepsilon>0} \quad \text{is bounded in} \quad L^4_{\text{loc}}((0,\infty);W^1_4(\mathbb{R}^N)), \tag{3.7}$$

$$\left\{u_\varepsilon^{\frac{\alpha+m+1}{2}}\right\}_{\varepsilon>0} \quad \text{is bounded in} \quad L^2_{\text{loc}}((0,\infty);H^1(\mathbb{R}^N)). \tag{3.8}$$

By virtue of assertion (iv) in Theorem A.1, (A.3), estimates (v)–(vii) of Lemma A.3, and inequality (iii) in Lemma A.4, we see that

$$\left\{\overrightarrow{J}_\varepsilon\right\}_{\varepsilon>0} \quad \text{is uniformly bounded in} \quad L^2_{\text{loc}}((0,\infty);L^{q'}(\mathbb{R}^N)) \tag{3.9}$$

for any $q'\in\Delta_1$ (Δ_1 is taken from assertion (i) of Theorem 2.1) and λ satisfying inequality (2.3). In particular, it follows from (3.9) and inequality (ii) in Lemma A.4 that the sequence

$$\{\partial_t u_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in} \quad L^2_{\text{loc}}((0,\infty);(W^1_q(\Omega))^*), \quad q=\frac{q'}{q'-1}. \tag{3.10}$$

Integrating inequalities (A.3) with respect to time, we show that

$$\{u_\varepsilon\}_{\varepsilon>0} \quad \text{is uniformly bounded in} \quad L^q_{\text{loc}}([0,\infty);L^q(\mathbb{R}^N)) \tag{3.11}$$

for any $q\colon 1<q<1+\max\{n+4/N,\ m+2/N\}.$

By using the *a priori* estimates (3.2)–(3.11), by analogy with [24, 27], we perform the limit transition as $\varepsilon \to 0$. We now establish this type of convergence for a new, as compared with [24], term in (2.2) caused by the convective transport. Note that

$$\int\limits_0^\infty \int\limits_{\mathbb{R}^N} \overrightarrow{\chi} \cdot \nabla\zeta\, b(u_\varepsilon) = \int\limits_\sigma^\infty \int\limits_{\mathbb{R}^N} \overrightarrow{\chi} \cdot \nabla\zeta\, b(u_\varepsilon) + \int\limits_0^\sigma \int\limits_{\mathbb{R}^N} \overrightarrow{\chi} \cdot \nabla\zeta\, b(u_\varepsilon) =: I_1(\sigma) + I_2(\sigma) \quad \forall \sigma > 0.$$

First, we pass to the limit as $\varepsilon \to 0$ in the integrals $I_i(\sigma)$. It is clear that $b(u_\varepsilon) \underset{\varepsilon\to 0}{\to} b(u) \le c\, u^\lambda$ almost everywhere in $(0,\infty) \times \mathbb{R}^N$. Assertion (3.11) yields the uniform (in $\varepsilon > 0$) boundedness of the majorizing sequence $\{u_\varepsilon^\lambda\}_{\varepsilon>0}$ in $L^{1+\gamma}_{\text{loc}}((0,\infty)\times\mathbb{R}^N)$ for λ satisfying (2.3) and sufficiently small $\gamma > 0$. Thus, in view of the fact that $u_\varepsilon^\lambda \underset{\varepsilon\to 0}{\to} u^\lambda$ almost everywhere in $(0,\infty)\times\mathbb{R}^N$, by virtue of the Vitali theorem, we get

$$u_\varepsilon^\lambda \underset{\varepsilon\to 0}{\rightrightarrows} u^\lambda \quad \text{in} \quad L^1_{\text{loc}}((0,\infty)\times\mathbb{R}^N).$$

By using the generalized Lebesgue lemma, we conclude that

$$b(u_\varepsilon) \underset{\varepsilon\to 0}{\rightrightarrows} b(u) \quad \text{in} \quad L^1_{\text{loc}}((0,\infty)\times\mathbb{R}^N),$$

whence it follows that

$$I_1(\sigma) \underset{\varepsilon\to 0}{\to} \int\limits_\sigma^\infty \int\limits_{\mathbb{R}^N} \overrightarrow{\chi} \cdot \nabla\zeta\, b(u).$$

Inequality (i) in Lemma A.4 implies that $|I_2(\sigma)| \le \bar{\bar{o}}_\sigma$, where $\bar{\bar{o}}_\sigma$ is uniformly bounded in $\varepsilon > 0$ and $\bar{\bar{o}}_\sigma \underset{\sigma\to 0}{\to} 0$. Thus, due to the arbitrariness of the choice of $\sigma > 0$, we establish the required convergence.

Acting as in [27], we find a subsequence $u_\varepsilon(t,x)$ of solutions of problem (1.1)–(1.3) with initial functions $u_{0\varepsilon}$ convergent to the solution $u(t,x)$ of this problem with properties (i) and (ii) from Theorem 2.1.

We now show that the analyzed solution possesses property (iii). To do this, as a test function in identity (2.2), we take $\zeta = \varphi_{h,t}(\tau,x)$, where

$$\varphi_{h,t}(\tau,x) = \varphi(x)\xi_{h,t}(\tau), \qquad \varphi(x) \in C^3_c(\mathbb{R}^N),$$

$$\xi_{h,t}(\tau) = \begin{cases} 1 & \text{for } \tau \le t, \\ 1 - \dfrac{\tau - t}{h} & \text{for } t < \tau < t+h, \\ 0 & \text{for } \tau \ge t+h. \end{cases}$$

By virtue of (3.3), (3.10), and the Simon lemma on compactness (see, e.g., [5]), the solution $u(t,x) \in C^0_{\mathrm{loc}}((0,\infty);\ L^1_{\mathrm{loc}}(\mathbb{R}^N))$ and, hence,

$$-\lim_{h\to 0}\int_0^\infty\int_{\mathbb{R}^N} u(\tau,x)(\varphi_{h,t}(\tau,x))_\tau\,dxd\tau = \lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^N} u(\tau,x)\varphi(x)\,dxd\tau = \int_{\mathbb{R}^N} u(t,x)\varphi(x)\,dx$$

for all t. Since $\chi_{\mathrm{P}}u^{n-2}|\nabla u|^3,\ \chi_{\mathrm{P}}u^{n-1}|\nabla u|^2,\ u^n|\nabla u|,\ u^m|\nabla u|$, and $b(u)$ belong to the space $L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^N)$, as a result of the limit transition in equality (2.2) as $h\to 0$, we get

$$\int_{\mathbb{R}^N} u(t,x)\varphi(x)\,dx \underset{t\to 0}{\to} \int_{\mathbb{R}^N}\varphi(x)\,d\mu_0(x) \quad \forall\,\varphi\in C^3_c(\mathbb{R}^N).$$

Since $u(t,x)\in C^0_{\mathrm{loc}}((0,\infty);L^1_{\mathrm{loc}}(\mathbb{R}^N))$, it is easy to see that the test function in identity (2.2) can be chosen in the form $\zeta = \chi_{[0,\,t]}\varphi_R(x),\ R\ge 1$, where $\varphi_R(x)\in C^3(\mathbb{R}^N)$ is such that $\varphi_R(x)=1$ for $x\in B(0,R)$, $\varphi_R(x)=0$ for $x\in\mathbb{R}^N\backslash B(0,2R)$, and $0\le\varphi_R(x)\le 1$ for $x\in B(0,R)\backslash B(0,2R)$ and, in addition,

$$|\nabla\varphi_R|\le\frac{c}{R},\quad |D^2\varphi_R|\le\frac{c}{R^2},\quad \text{and}\quad |D^3\varphi_R|\le\frac{c}{R^3}.$$

Here, the constant c is independent of R and x. If we now pass in the deduced equality to the limit as $R\to\infty$, then, by analogy with [27], we establish property (iv).

Theorem 2.1 is proved.

Proof of Theorem 2.2. In [28], the evolution of the support is studied for the solution of problem (1.1)–(1.3) with an initial function from H^1 and estimates (a) and (ii) are established with constants depending only on the L^1-norm of the initial function and the parameters of the problem. In deducing these estimates, we essentially use the local *entropy* inequality (A.2). Thus, approximating the initial function μ_0 by the functions $u_{0\varepsilon}$ from (3.1) as in Theorem 2.1, passing to the limit as $\varepsilon\to 0$ in inequality (A.2), and using estimate (iv) from Lemma A.4 for the nonlinear convective term as in [28], we arrive at the required estimates (a) and (ii). Estimates (i) and (iii) independent of λ are established by analogy with [24] in view of the fact that, in the corresponding local entropy inequality, the term connected with convection appears on the left-hand side of the inequality with positive sign and, hence, can be omitted.

Similar estimates for the propagation of the front of support of the solution of problem (1.1)–(1.3) for the one-dimensional equation (1.1) with $a_1 = 0$ and an initial function from H^1 are obtained in [21].

4. The Case of "Weak" Slip: $2\le n<3$

Proof of Theorem 2.3. First, we prove the property of finite rate of propagation of the support of an arbitrary *strong* solution of the auxiliary Neumann problem for Eq. (1.1) with $a_1=0$ in the bounded domain Ω with smooth boundary and an arbitrary nonnegative initial function $u_0(x)\in H^1(\Omega)$ such that $\operatorname{supp}u_0(x)\subset B(0,R_0)\Subset\Omega$.

Following [23], we regularize Eq. (1.1) with $a_1=0$ (parameters $\varepsilon_2\to 0$ and $\sigma_1\to 0$) and introduce a "penalty" term in this equation (parameter $\varepsilon_1\to 0$) connected with the values of the solution from an arbitrary interval $[\delta,L],\ \delta>0,\ L<\infty$:

$$U_t + \operatorname{div}((|U|^n+\varepsilon_2)\nabla(\Delta U-\sigma_1 U_t)) + \frac{1}{\varepsilon_1}((U-L)_+ - (\delta-U)_+) = \overrightarrow{\chi}\nabla b(U), \quad (t,x)\in(0,T)\times\Omega. \quad (4.1)$$

The solution $U(t,x) = U_{\varepsilon_1,\varepsilon_2,\sigma_1,\delta,L}$ of the Neumann problem for Eq. (4.1) is understood in a sense of the integral identity

$$0 = \iint\limits_{Q_T} U_t\zeta + \frac{1}{\varepsilon_1}\iint\limits_{Q_T} ((U-L)_+ - (\delta - U)_+)\zeta$$

$$- \iint\limits_{Q_T} (|U|^n + \varepsilon_2)\nabla(\Delta U - \sigma_1 U_t)\nabla\zeta - \iint\limits_{Q_T} \overrightarrow{\chi} b'(U)\nabla U\zeta \quad \forall\zeta \in L^2(0,T;H^1(\Omega)). \tag{4.2}$$

In equality (4.2), we set $\zeta = -\mathrm{div}(\varphi^6\nabla U) + \sigma_1\varphi^6 U_t.$ By passing to the limit as $\varepsilon_1 \to 0,$ $\varepsilon_2 \to 0,$ and $\sigma_1 \to 0,$ by analogy with [23], we obtain the following relation for the limiting solution $\bar{u}(t,.) := U_{0,0,0,\delta,L}(t,.) \in K_{\delta,L} := \{v \in L^2(\Omega)\colon v \geq \delta$ and $v \leq L$ almost everywhere in $\Omega\}$:

$$\frac{1}{2}\int\limits_{\Omega} \varphi^6|\nabla\bar{u}(T)|^2\,dx + c^{-1}\iint\limits_{Q_T} \varphi^6\bar{u}^n|\nabla\Delta\bar{u}|^2$$

$$\leq \frac{1}{2}\int\limits_{\Omega} \varphi^6|\nabla u_0(x)|^2\,dx + c\Bigg\{\iint\limits_{Q_T} \varphi^2\bar{u}^n(\nabla\varphi\nabla\bar{u})^2|\nabla\varphi|^2$$

$$+ \iint\limits_{Q_T} \varphi^4\bar{u}^n|D^2\varphi\nabla u|^2 + \iint\limits_{Q_T} \varphi^4\bar{u}^n|D^2\bar{u}\nabla\varphi|^2$$

$$+ \iint\limits_{Q_T} \varphi^4\bar{u}^n|\Delta\bar{u}|^2|\nabla\varphi|^2 + \iint\limits_{Q_T} \varphi^6\bar{u}^{\lambda-1}|\nabla\bar{u}\Delta\bar{u}|$$

$$+ \iint\limits_{Q_T} \varphi^5\bar{u}^{\lambda}|D^2\bar{u}\nabla\varphi| + \iint\limits_{Q_T} \varphi^4\bar{u}^{\lambda}|D^2\varphi\nabla\bar{u}|\Bigg\}$$

$$=: \frac{1}{2}\int\limits_{\Omega} \varphi^6|\nabla u_0(x)|^2\,dx + \sum_{k=1}^{7} I_k, \tag{4.3}$$

where $\varphi \in C^2(\Omega)$ is an arbitrary nonnegative function such that the tangential component $\nabla\varphi$ is equal to zero on $\partial\Omega.$ By applying the Cauchy inequality and Lemma B.3 to the right-hand side of (4.3), we get

$$I_5 \le \varepsilon \iint\limits_{Q_T} \varphi^6 \bar{u}^{n-2} |D^2 \bar{u}|^2 |\nabla \bar{u}|^2 + c \iint\limits_{Q_T} \varphi^6 \bar{u}^{2\lambda - n}$$

$$\le c \iint\limits_{Q_T} \varphi^6 \bar{u}^{2\lambda-n} + \varepsilon \left\{ \iint\limits_{Q_T} \varphi^6 \bar{u}^n |\nabla \Delta \bar{u}|^2 + \iint\limits_{Q_T} \bar{u}^{n+2} |\nabla \varphi|^6 \right\} \quad \forall \varepsilon > 0,$$

$$I_6 \le \varepsilon \iint\limits_{Q_T} \varphi^4 \bar{u}^n |D^2 \bar{u} \nabla \varphi|^2 + c \iint\limits_{Q_T} \varphi^6 \bar{u}^{2\lambda-n}$$

$$\le c \iint\limits_{Q_T} \varphi^6 \bar{u}^{2\lambda-n} + \varepsilon \left\{ \iint\limits_{Q_T} \varphi^6 \bar{u}^n |\nabla \Delta \bar{u}|^2 + \iint\limits_{Q_T} \bar{u}^{n+2} |\nabla \varphi|^6 \right\} \quad \forall \varepsilon > 0,$$

$$I_7 \le \varepsilon \iint\limits_{Q_T} \varphi^4 \bar{u}^n |D^2 u \nabla \varphi|^2 + c \iint\limits_{Q_T} \varphi^6 \bar{u}^{2\lambda-n}$$

$$\le \varepsilon \iint\limits_{Q_T} \varphi^6 \bar{u}^{n-4} |\nabla \bar{u}|^6 + c \iint\limits_{Q_T} \varphi^6 \bar{u}^{2\lambda-n} + c \iint\limits_{Q_T} \bar{u}^{n+2} |\nabla \varphi|^6 \quad \forall \varepsilon > 0.$$

The remaining I_k are estimated as in [23]. The right-hand side of (4.3) is estimated by the method described above, and we choose sufficiently small $\varepsilon > 0$. By passing to the limit in the obtained inequality as $\delta \to 0$ and $L^{-1} \to 0$, we get the following *a priori energy* estimate for the limiting solution $u(t,x)$:

$$\frac{1}{2} \int\limits_{\Omega} \varphi^6 |\nabla u(t_2)|^2 \, dx + c^{-1} \int\limits_{t_1}^{t_2} \int\limits_{\Omega} \varphi^6 \left\{ |\nabla u^{\frac{n+2}{6}}|^6 + |\nabla \Delta u^{\frac{n+2}{2}}|^2 \right\}$$

$$\le \frac{1}{2} \int\limits_{\Omega} \varphi^6 |\nabla u(t_2)|^2 \, dx + c^{-1} \iint\limits_{\{u>0\}} \varphi^6 u^n |\nabla \Delta u|^2$$

$$\le \frac{1}{2} \int\limits_{\Omega} \varphi^6 |\nabla u(t_1)|^2 \, dx + c \int\limits_{t_1}^{t_2} \int\limits_{\{\varphi>0\}} \varphi^6 u^{2\lambda-n}$$

$$+ c \int\limits_{t_1}^{t_2} \int\limits_{\{\varphi>0\}} u^{n+2} \Big\{ |\nabla \varphi|^6 \varphi^2 |D^2 \varphi|^2 |\nabla \varphi|^2 + \varphi^3 |\Delta \varphi|^3 \Big\}, \quad 0 \le t_1 < t_2 \le T_{\mathrm{loc}}, \qquad (4.4)$$

where $t_1 = 0,\ t_2 = T_{\rm loc},\ n \in \left(2 - \sqrt{1 - \dfrac{N}{N+8}}, 3\right),\ \lambda > \dfrac{n}{2}\ \left(\lambda < \dfrac{n+6}{2}\ \text{for}\ N = 3\right)$, and $\varphi(x)$ is taken from (4.3). For arbitrary t_1 and t_2 such that $0 \le t_1 < t_2 \le T_{\rm loc}$, we establish inequality (4.4) in a similar way by setting $\zeta = -\chi_{[t_1,t_2]}{\rm div}(\varphi^6 \nabla U) + \sigma_1 \chi_{[t_1,t_2]} \varphi^6 U_t$ in relation (4.2).

Now let $\{\ell_\mu(t)\}_{\mu>0} \subset C_c^\infty(0,T)$ be such that $\ell_\mu \underset{\mu\to 0}{\to} \chi_{(0,T)}$. As a test function in the limiting integral identity for $u(t,.) := U_{0,0,0,0,0}(t,.)$ obtained from (4.2) as a result of the limit transition as $\varepsilon_1 \to 0,\ \varepsilon_2 \to 0,\ \sigma_1 \to 0,\ \delta \to 0$, and $L^{-1} \to 0$, we take $\zeta = -\ell_\mu(t)\varphi^4(u+\gamma)^\beta,\ \beta > \dfrac{1-n}{3},\ \forall \gamma > 0$. After simple transformations and the limit transition as $\mu \to 0$ and $\gamma \to 0$ (see, e.g., [21]), for $T \le T_{\rm loc}$, we obtain the following estimate:

$$\frac{1}{\beta+1}\int\limits_{\Omega} \varphi^4 u^{\beta+1}(T)\,dx \le \frac{1}{\beta+1}\int\limits_{\Omega} \varphi^4 u_0^{\beta+1}(x)\,dx + \varepsilon \iint\limits_{\{u>0\}} \varphi^6 u^n |\nabla \Delta u|^2 + \varepsilon \iint\limits_{Q_T} \varphi^6 |\nabla u^{\frac{n+2}{6}}|^6$$

$$+ c\int\limits_0^T \int\limits_{\{\varphi(t)>0\}} \Big\{ u^{n+2\beta}|\nabla \varphi|^2 + u^{\lambda+\beta}|\nabla \varphi^4| + u^{n+3\beta-1} \Big\}, \tag{4.5}$$

$$\beta > \frac{1-n}{3}, \quad \forall \varepsilon > 0,$$

where $n \in \left(2 - \sqrt{1 - \dfrac{N}{N+8}}, 3\right)$, $\lambda > 1$, and $\varphi \in C^1(\Omega)$ is an arbitrary nonnegative cutoff function.

Let $\Omega(s) = \Omega \backslash B(0, R_0 + s)$, let $Q_T(s) = (0,T) \times \Omega(s)$, and let ${\rm supp} u_0 \subset B(0,R_0) \Subset \Omega$. We introduce a nonnegative cutoff function $\xi(\tau)$ from the space $C^2(\mathbb{R}^1)$ with the following properties: $\xi = 0$ for $\tau \le 0$, $\xi = 1$ for $\tau \ge 1$, and $0 \le \xi(\tau) \le 1\ \ \forall \tau \in \mathbb{R}^1$. We define a family of test cutoff functions as follows:

$$\xi_{s,\delta}(x) = \xi\left(\frac{|x| - (R_0 + s)}{\delta}\right) \quad \forall s \in \mathbb{R}^1, \quad \delta > 0,$$

where $\delta > 0$ is such that $B(0, R_0 + s + \delta) \Subset \Omega$. The inequalities

$$|\nabla \xi_{s,\delta}| \le \frac{c}{\delta} \quad \text{and} \quad |\Delta \xi_{s,\delta}| \le \frac{c}{\delta^2}$$

hold for all x. If we now add inequalities (4.4) and (4.5) and set $\varphi(x) = \xi_{s,\delta}(x)$, then, after simple transformations, we get

$$\int\limits_{\Omega(s+\delta)} |\nabla u(T)|^2\,dx + \int\limits_{\Omega(s+\delta)} u^{\beta+1}(T)\,dx + c^{-1} \iint\limits_{Q_T(s+\delta)} \left\{ |\nabla u^{\frac{n+2}{6}}|^6 + |\nabla \Delta u^{\frac{n+2}{2}}|^2 \right\}$$

$$\leq c \left\{ \delta^{-6} \iint\limits_{Q_T(s)} u^{n+2} + \delta^{-2} \iint\limits_{Q_T(s)} u^{n+2\beta} + \delta^{-1} \iint\limits_{Q_T(s)} u^{\lambda+\beta} + \iint\limits_{Q_T(s)} u^{n+3\beta-1} + \iint\limits_{Q_T(s)} u^{2\lambda-n} \right\}$$

$$=: c \sum_{i=1}^{5} \delta^{-\alpha_i} \iint\limits_{Q_T(s)} u^{\xi_i}. \tag{4.6}$$

We apply the Nirenberg–Gagliardo interpolation inequality (Lemma B.1) in the region $\Omega(s+\delta)$ to a function $v := u^{\frac{n+2}{2}}$ with $a = \dfrac{2\xi_i}{n+2}$, $b = \dfrac{2(\beta+1)}{n+2}$, $d = 2$, $i = 0$, $j = 3$, and $\theta_i = \dfrac{N(n+2)(\xi_i - \beta - 1)}{\xi_i(N(n+2) + (6-N)(\beta+1))}$ under the condition

$$\beta < \xi_i - 1 \quad \text{for} \quad i = \overline{1,5}. \tag{4.7}$$

Integrating the obtained inequalities with respect to time and taking into account (4.6), we arrive at the following relations:

$$\iint\limits_{Q_T(s+\delta)} u^{\xi_i} \leq cT^{1-\frac{\theta_i \xi_i}{n+2}} \left(\sum_{i=1}^{5} \delta^{-\alpha_i} \iint\limits_{Q_T(s)} u^{\xi_i} \right)^{1+\nu_i} + cT \left(\sum_{i=1}^{5} \delta^{-\alpha_i} \iint\limits_{Q_T(s)} u^{\xi_i} \right)^{\frac{\xi_i}{\beta+1}},$$

where $\nu_i = \dfrac{6(\xi_i - \beta - 1)}{N(n+2) + (6-N)(\beta+1)}$. The obtained inequalities are true provided that

$$\frac{\theta_i \xi_i}{n+2} < 1 \Leftrightarrow \beta > \frac{N(\xi_i - n - 2)}{6} - 1. \tag{4.8}$$

Simple calculations show that inequalities (4.7) and (4.8) hold with some $\beta > \dfrac{1-n}{3}$ if and only if

$$n \in \left(2 - \sqrt{1 - \frac{N}{N+8}}, 3 \right) \quad \text{and} \quad \lambda \in \left(1 + \frac{n}{4}, n + 1 + \frac{3(n+2)}{N} \right). \tag{4.9}$$

Since all integrals on the right-hand sides of the deduced integral inequalities approach zero as $T \to 0$, it follows from Lemma B.2 with $s_1 = 0$ and sufficiently small T that

$$\operatorname{supp} u(t,.) \subset B(0, R_0 + \Gamma(t)) \Subset \Omega \quad \forall t \leq T_{\text{loc}} = T_{\text{loc}}(R_0), \tag{4.10}$$

i.e., the support of our solution propagates with a finite rate. Extending the function $u(t,x)$ by zero outside its support, we obtain the solution of problem (1.1)–(1.3) local in time.

We now establish the upper estimate for $\Gamma(t)$ in relation (4.10). Suppose that $\Omega(s) = \mathbb{R}^N \backslash B(0,s)$, $Q_T(s) = (0,T) \times \Omega(s)$ $\forall s > R_0$, $\operatorname{supp} u_0 \subset B(0,R_0)$, and $\Gamma(T) = R(T) - R_0$. Since the time interval in (4.10) is small, we can assume that $R(T) < 2R_0$. Thus, for all $s \in (R_0, 2R_0)$ in (4.4), we can take (up to regularization) $\varphi(x) = (|x| - s)_+$. As a result, we obtain

$$\frac{1}{2}\int\limits_{\Omega(s)} (|x|-s)_+^6 |\nabla u(T)|^2\,dx + c^{-1}\delta^6 \iint\limits_{Q_T(s+\delta)} |\nabla \Delta u^{\frac{n+2}{2}}|^2$$

$$\leq c \iint\limits_{Q_T(s)} \{u^{n+2} + (R(T)-s)_+^6 u^{2\lambda - n}\} \quad \forall\, T \leq T_{\rm loc}, \quad s \in (R_0, 2R_0). \tag{4.11}$$

By using the Hardy inequality

$$\int\limits_{\Omega(s)} (|x|-s)_+^4 u^2\,dx \leq c \int\limits_{\Omega(s)} (|x|-s)_+^6 |\nabla u|^2\,dx,$$

we establish the estimate

$$\int\limits_{\Omega(s+\delta)} u\,dx \leq \Bigg(\int\limits_{\Omega(s+\delta)} (|x|-s)_+^4 u^2\,dx\Bigg)^{\frac{1}{2}} \Bigg(\int\limits_{\Omega(s+\delta)} (|x|-s)_+^{-4}\,dx\Bigg)^{\frac{1}{2}}$$

$$\leq c \Bigg(\int\limits_{\Omega(s)} (|x|-s)_+^6 |\nabla u|^2\,dx\Bigg)^{\frac{1}{2}} \Bigg(\int\limits_{\Omega(s+\delta)} (|x|-s)_+^{-4}\,dx\Bigg)^{\frac{1}{2}}$$

$$\leq c(\delta^{N-4} + s^{N-1}\delta^{-3})^{\frac{1}{2}} \Bigg(\int\limits_{\Omega(s)} (|x|-s)_+^6 |\nabla u|^2\,dx\Bigg)^{\frac{1}{2}} \quad \forall\, \delta > 0, \quad s \in (R_0, 2R_0).$$

This yields

$$\Bigg(\int\limits_{\Omega(s+\delta)} u\,dx\Bigg)^2 \leq c f(\delta, s) \int\limits_{\Omega(s)} (|x|-s)_+^6 |\nabla u|^2\,dx, \quad f(\delta, s) := \delta^{N-4} + s^{N-1}\delta^{-3}. \tag{4.12}$$

Substituting the last estimate in inequality (4.11), we get

$$\sup_{t\in(0,T)}\left(\int\limits_{\Omega(s+\delta)} u\,dx\right)^2 + c\delta^6 f(\delta,s)\iint\limits_{Q_T(s+\delta)} |D^3 u^{\frac{n+2}{2}}|^2$$

$$\leq cf(\delta,s)\iint\limits_{Q_T(s)}\{u^{n+2}+\Gamma^6(T)u^{2\lambda-n}\}$$

$$=: cf(\delta,s)\sum_{i=1}^{2}\Gamma^{\eta_i}(T)\iint\limits_{Q_T(s)} u^{\xi_i}\quad \forall T\leq T_{\mathrm{loc}},\quad s\in(R_0,2R_0). \tag{4.13}$$

Further, by using the Nirenberg–Gagliardo, Hölder, and Young inequalities, after simple transformations, for $\varepsilon>0$, we get

$$\Gamma^{\eta_i}(T)\iint\limits_{Q_T(s)} u^{\xi_i} \leq \varepsilon\delta^6\iint\limits_{Q_T(s)} |D^3 u^{\frac{n+2}{2}}|^2 + c(\varepsilon)\frac{\Gamma^{\ell_i}(T)}{\delta^{\gamma_i}}\int\limits_0^T\left(\int\limits_{\Omega(s)} u\right)^{\frac{\xi_i(6-N)+N(n+2)}{N(n+2-\xi_i)+6}}.$$

Here, $\ell_i=\dfrac{\eta_i(nN+N+6)}{N(n+2-\xi_i)+6}$ and $\gamma_i=\dfrac{6N(\xi_i-1)}{N(n+2-\xi_i)+6}$ for $1<\xi_i<n+2+\dfrac{6}{N}$. The conditions imposed on ξ_i are equivalent to the inequalities

$$\frac{n+1}{2}<\lambda<n+1+\frac{3}{N}.$$

Substituting the estimates established above in (4.13) and performing the standard iterative procedure, we arrive at the inequality

$$\sup_{t\in(0,T)}\left(\int\limits_{\Omega(s+\delta)} u\,dx\right)^2 \leq cf(\delta,s)\sum_{i=1}^{2}\frac{G_T^{(i)}(s)}{\delta^{\gamma_i}},$$

$$G_T^{(i)}(s) := \Gamma^{\ell_i}(T)\int\limits_0^T\left(\int\limits_{\Omega(s)} u\right)^{\frac{\xi_i(6-N)+N(n+2)}{N(n+2-\xi_i)+6}}.$$

For $\delta\leq R_0$ $(\Rightarrow f(\delta,s)\delta^{-\gamma_i}\leq c\,\delta^{-\gamma_i-3}\ \ \forall s\in(R_0,2R_0))$, this yields

$$G_T^{(i)}(s+\delta)\leq cT\Gamma^{\ell_i}(T)\left(\sum_{i=1}^{2}\frac{G_T^{(i)}(s)}{\delta^{\alpha_i}}\right)^{\beta_i}\quad \forall s\in(R_0,2R_0),\quad s>\delta>0, \tag{4.14}$$

where $\beta_i = \dfrac{\xi_i(6-N)+N(n+2)}{2(N(n+2-\xi_i)+6)}$ and $\alpha_i = \gamma_i + 3 = \dfrac{3(\xi_i N + nN + 6)}{N(n+2-\xi_i)+6}$. By applying Lemma B.2 to relation (4.14), we get $G_T^{(i)}(s_0(T)) = 0,$ where

$$\Gamma(T) \le s_0(T) = c(T^{\frac{1}{N(n+1)+3}} + T^{\alpha_2}\Gamma^{\frac{\ell_2}{\alpha_2}}(T)), \quad c = c(n,\ \lambda, \quad N,\ \|u_0\|_1) > 0.$$

Since $\dfrac{\ell_2}{\alpha_2} > 1,$ for any $T \le T_{\rm loc},$ we obtain

$$\Gamma(T) \le cT^{\frac{1}{N(n+1)+3}}. \tag{4.15}$$

Remark 4.1. For $N = 1,$ inequality (4.15) coincides with the estimate of supports of the solutions of Eq. (1.4) obtained in [27] and with the estimate deduced for Eq. (1.1) with $a_1 = 0$ in [21]. For $N > 1,$ this estimate is new and, according to our hypothesis, it is exact and attained on a class of Radon measures concentrated on a sphere of radius R_0.

Further, as in the proof of Theorem 2.1, we approximate the solution of problem (1.1)–(1.3) by strong solutions of the Neumann problem whose supports propagate with a finite rate [property (4.10)]. We choose a sequence $\{u_{0\varepsilon}\}_{\varepsilon>0}$ of nonnegative functions in $H^1(\mathbb{R}^N)$ with compact support from (3.1). By virtue of (4.10) and (4.15), we conclude that $\operatorname{supp} u_\varepsilon(t,.) \subset B(0, R_{0\varepsilon} + \hat{c}t^{\frac{1}{N(n+1)+3}}),$ where $u_\varepsilon(t,x)$ is a solution (local in time) of the Cauchy problem for Eq. (1.1) with $a_1 = 0.$ The following estimate holds for any $\alpha \in (-1,0)$:

$$\int\limits_{\mathbb{R}^N} u_\varepsilon^{\alpha+1}(t,x)\,dx \le \|u_{0\varepsilon}\|_1^{\alpha+1} |\operatorname{supp} u_\varepsilon(t)|^{-\alpha} \le c\|u_{0\varepsilon}\|_1^{\alpha+1} R_0^{-\alpha N}(t), \tag{4.16}$$

$$R_0(t) := |R_0 + \hat{c}t^{\frac{1}{N(n+1)+3}}|.$$

Here, we have used the fact that the law of conservation of mass is true for all $t \le T_{\rm loc}$. By analogy with [23], we prove the existence of an *entropy* estimate of the form

$$\int\limits_0^T \int\limits_{\mathbb{R}^N} \left\{ \left| D^2 u_\varepsilon^{\frac{\alpha+n+1}{2}} \right|^2 + \left| \nabla u_\varepsilon^{\frac{\alpha+n+1}{4}} \right|^4 \right\} \le \frac{c}{\alpha(\alpha+1)} \int\limits_{\mathbb{R}^N} u_{0\varepsilon}^{\alpha+1}(x)dx - \frac{c}{\alpha(\alpha+1)} \int\limits_{\mathbb{R}^N} u_\varepsilon^{\alpha+1}(T,x)\,dx \tag{4.17}$$

for all $T \le T_{\rm loc}$ and $\alpha \in (-1, 2-n)$. By using inequalities (4.17), (4.16), the fact that the support of the solution is compact, and the law of conservation of mass, we establish property (i) of Theorem 2.3. For the solution $\bar{u}(t,.)$ from (4.3), for $n \in [2,3)$ and almost all $t \le T_{\rm loc},$ we have

$$\int\limits_\Omega |\nabla \bar{u}(t,x)|^2 \frac{\bar{u}^{\frac{n-4}{3}}}{\bar{u}^{\frac{n-4}{3}}} dx \le c(n) \left(\int\limits_\Omega |\nabla \bar{u}^{\frac{n+2}{6}}(t)|^6\,dx \right)^{\frac{1}{3}} \left(\int\limits_\Omega \bar{u}^{\frac{4-n}{2}}(t)\,dx \right)^{\frac{2}{3}}$$

$$\le c(n)\|u_{0\varepsilon}\|_1^{\frac{4-n}{3}} |\Omega|^{\frac{n-2}{3}} \left(\int\limits_\Omega |\nabla \bar{u}^{\frac{n+2}{6}}(t)|^6\,dx \right)^{\frac{1}{3}}.$$

We raise both sides of this inequality to the third power and integrate the result with respect to time from t_1 to t_2. By passing to the limit as $\delta \to 0$ and $L^{-1} \to 0,$ for the limiting solution $u_\varepsilon(t,x)$ with property (4.10), we obtain

$$c^{-1}(n)\|u_{0\varepsilon}\|_1^{n-4} R_0^{2-n}(T_{\text{loc}}) \int\limits_{t_1}^{t_2} \Bigg(\int\limits_{\mathbb{R}^N} |\nabla u_\varepsilon(t,x)|^2 dx \Bigg)^3 \le \iint\limits_{Q_{t_1}^{t_2}} |\nabla u_\varepsilon^{\frac{n+2}{6}}(t)|^6 \, dx. \tag{4.18}$$

For almost all $0 \le t_1 < t_2 \le T_{\text{loc}}$ and $\lambda > \dfrac{n}{2}$ $\left(\lambda < \dfrac{n+6}{2} \text{ for } N = 3\right),$ by using the imbedding theorems and Young's inequality, we get

$$\int\limits_{t_1}^{t_2} \int\limits_{\mathbb{R}^N} u_\varepsilon^{2\lambda - n} \le \gamma \int\limits_{t_1}^{t_2} \Bigg(\int\limits_{\mathbb{R}^N} |\nabla u_\varepsilon(t,x)|^2 dx \Bigg)^3 + c(\gamma)(t_2 - t_1) \quad \forall\, \gamma > 0. \tag{4.19}$$

Substituting (4.18) and (4.19) in (4.4) with $\varphi = 1,$ we find

$$\frac{1}{2} \int\limits_{\mathbb{R}^N} |\nabla u_\varepsilon(t_2)|^2 \, dx + (c^{-1}(n)\|u_{0\varepsilon}\|_1^{n-4} R_0^{2-n}(T_{\text{loc}}) - \gamma) \int\limits_{t_1}^{t_2} \Bigg(\int\limits_{\mathbb{R}^N} |\nabla u_\varepsilon(t,x)|^2 dx \Bigg)^3$$

$$\le \frac{1}{2} \int\limits_{\mathbb{R}^N} |\nabla u_\varepsilon(t_1)|^2 \, dx + c(\gamma)(t_2 - t_1), \quad t_1 < t_2 \le T_{\text{loc}}. \tag{4.20}$$

Let $F(t) := \displaystyle\int_{\mathbb{R}^N} |\nabla u_\varepsilon(t)|^2 \, dx - c(\gamma)t.$ In (4.20), we now choose $\gamma = \dfrac{1}{2} c^{-1}(n)\|u_{0\varepsilon}\|_1^{n-4} R_0^{2-n}(T_{\text{loc}})$ and set $t_1 = s$ and $t_2 = s + \tau.$ As a result, for the nonincreasing function $F(t),$ we obtain

$$c^{-1}(n)\|u_{0\varepsilon}\|_1^{n-4} R_0^{2-n}(T_{\text{loc}})\tau \, F^3(s+\tau) \le F(s)$$

and, hence,

$$F(s+\tau) \le c(n)\|u_{0\varepsilon}\|_1^{\frac{4-n}{3}} R_0^{\frac{n-2}{3}}(T_{\text{loc}}) \left(\frac{F(s)}{\tau}\right)^{\frac{1}{3}}. \tag{4.21}$$

Applying Lemma B.4 to (4.21), we obtain the required estimate for the law of decrease:

$$\int\limits_{\mathbb{R}^N} |\nabla u_\varepsilon(t,x)|^2 dx \le \tilde{c}(n) t^{-\frac{1}{2}} \|u_{0\varepsilon}\|_1^{\frac{4-n}{2}} R_0^{\frac{N(n-2)}{2}}(T_{\text{loc}}) \quad \forall t \in (0, T_{\text{loc}}]. \tag{4.22}$$

By using the Nirenberg–Gagliardo inequality (Lemma B.1), for all $p \in (1,\infty)$ if $N = 1, 2$ and $p \in (1,6)$ if $N = 3,$ by virtue of (4.22), we get

$$\|u_\varepsilon(t)\|_p \le \|\nabla u_\varepsilon(t)\|_2^{\frac{2N(p-1)}{p(N+2)}} \|u_{0\varepsilon}\|_1^{\frac{2N+p(2-N)}{p(N+2)}}$$

$$\le c\|u_{0\varepsilon}\|_1^{1-\frac{2nN(p-1)}{4p(N+2)}} t^{-\frac{N(p-1)}{2p(N+2)}} (R_0(T_{\rm loc}))^{\frac{N^2(p-1)(n-2)}{2p(N+2)}}. \tag{4.23}$$

Under conditions (2.6), in view of inequalities (4.16) and (4.23), for all $t \in (0, T_{\rm loc}]$, we establish the estimates of all nonlinear terms in (2.2) and (4.4) required for the limit transition as $\varepsilon \to 0$. As an example, we consider the estimate of the nonlinear term in (2.2) caused by the presence of convection. For $\lambda \in \left(\frac{1}{p}, 2 + \frac{1}{p} + \frac{4}{N}\right)$, $p > 1$, by virtue of (4.23), we can write

$$\int_0^t \|b(u_\varepsilon(\tau))\|_p \, d\tau \le c \int_0^t \|u_\varepsilon(\tau))\|_{\lambda p}^{\lambda} \, d\tau \le c\, t^{1-\frac{N(\lambda p-1)}{2p(N+2)}} \quad \forall t \in [0, T_{\rm loc}],$$

where $c = c(T_{\rm loc}, R_0, \|u_{0\varepsilon}\|_1, N, n) < \infty$. By passing to the limit as $\varepsilon \to 0$, by analogy with Theorem 2.1, we obtain the required solution of the Cauchy problem.

Appendix A

Theorem A.1 [25]. *Assume that* $N \le 3$, $m > 0$, $1/8 < n < 2$,

$$\max\left\{1, \frac{3n-1}{4}\right\} < \lambda < \frac{5N+8}{4N} + \min\left\{n, \frac{5}{4}\right\} \quad \textit{for} \quad N < 3,$$

$$\max\left\{1, \frac{3n-1}{4}\right\} < \lambda < 2 + \min\left\{n, \frac{5}{4}\right\} \quad \textit{for} \quad N = 3, \tag{A.1}$$

and $u_0(x) \in H^1(\mathbb{R}^N) \cap L^{m-n+2}(\mathbb{R}^N)$ *is a nonnegative function with* $\operatorname{supp} u_0 \subset B(0, R_0), R_0 < +\infty$. *Then there exists a solution* $u(t,x)$ *of problem (1.1)–(1.3) from Definition 2.1 such that*

- *(i)* $\operatorname{supp} u(t,.)$ *is compact for almost all* $t > 0$ *and there exists a nondecreasing continuous function* $\Gamma(t)$, $\Gamma(0) = 0$, *such that* $\operatorname{supp} u(t,.) \subset D(0, R_0 + \Gamma(t))$ $\forall t > 0$;
- *(ii) for any* α *from (2.5) satisfying the additional conditions*

$$\max\left\{\frac{3(\alpha+n)-1}{4}, 1\right\} < \lambda \le \frac{N+2}{N} + \frac{3(\alpha+n)}{4} \quad \textit{for} \quad N < 3,$$

$$\max\left\{\frac{3(\alpha+n)-1}{4}, 1\right\} < \lambda \le \frac{3(\alpha+n)+7}{4} \quad \textit{for} \quad N = 3$$

[it is easy to see that, under assumptions (A.1), the set of these α *is nonempty], the following inclusions are true:*

$$u^{m-n+2} \in L^\infty_{\rm loc}([0,\infty); L^1(\mathbb{R}^N)), \qquad u^{\frac{\alpha+n+1}{4}} \in L^4_{\rm loc}([0,\infty); W^1_4(\mathbb{R}^N)),$$

$$u^{\frac{\alpha+n+1}{2}} \in L^2_{\mathrm{loc}}([0,\infty); H^2(\mathbb{R}^N)), \qquad u^{\frac{\alpha+m+1}{2}} \in L^2_{\mathrm{loc}}([0,\infty); H^1(\mathbb{R}^N));$$

(iii) the following local entropy inequality holds for almost all $0 \le t_1 < t_2$ *and any nonnegative function* $\zeta \in C^2([t_1, t_2] \times \mathbb{R}^N)$ *:*

$$\frac{1}{\alpha(\alpha+1)} \int\limits_{\mathbb{R}^N} \zeta^4 u^{\alpha+1}(t_2, x)dx - \frac{1}{\alpha(\alpha+1)} \int\limits_{t_1}^{t_2}\int\limits_{\mathbb{R}^N} (\zeta^4)_t u^{\alpha+1}$$

$$+ \ c_3^{-1} \int\limits_{t_1}^{t_2}\int\limits_{\mathbb{R}^N} \zeta^4 \left\{ \left|\nabla u^{\frac{\alpha+m+1}{2}}\right|^2 + \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^4 + \left|D^2 u^{\frac{\alpha+n+1}{2}}\right|^2 \right\}$$

$$\le \frac{1}{\alpha(\alpha+1)} \int\limits_{\mathbb{R}^N} \zeta^4 u^{\alpha+1}(t_1, x)dx + c_3 \int\limits_{t_1}^{t_2} \int\limits_{\{\zeta(t)>0\}} u^{\alpha+m+1}(\zeta^2 |\nabla\zeta|^2 + \zeta^3 |\Delta\zeta|)$$

$$+ \ c_3 \int\limits_{t_1}^{t_2} \int\limits_{\{\zeta(t)>0\}} u^{\alpha+n+1}(|\nabla\zeta|^4 + \zeta^2 |\Delta\zeta|^2) - \int\limits_{t_1}^{t_2} \int\limits_{\{\zeta(t)>0\}} \overrightarrow{\chi} \mathcal{B}^{(\alpha)}(u) \nabla \zeta^4, \quad \text{(A.2)}$$

where $\mathcal{B}^{(\alpha)}(z) := \alpha^{-1} \int_0^z b'(\tau)\tau^\alpha \, d\tau$ *and* α *is taken from (ii);*

(iv) for almost all $0 \le t_1 < t_2$ $(t_1 = 0$ *for* $m - n + 2 \le 0)$ *and all* $q' \in \left(1, \frac{4N}{2N + n(N-2)}\right)$ $(q' = 2$ *for* $N = 1)$, *the flow* $\overrightarrow{J}$ *from relation (2.4) satisfies the inequality*

$$\int\limits_{t_1}^{t_2} \left\|\overrightarrow{J}(t)\right\|^2_{q'} dt \le \sup_{t \in (t_1, t_2)} \|u^n(t)\|_{\frac{q'}{2-q'}} \left(\mathcal{E}(u(t_1)) - \iint\limits_{Q_{t_1}^{t_2}} \overrightarrow{\chi} \chi_{\mathrm{P}} \, b'(u) \nabla u \Delta u \right),$$

where $\mathcal{E}(u(t)) := \frac{1}{2}\|\nabla u(t)\|_2^2 + \int_{\mathbb{R}^N} \Psi_0(u(t))dx$ *and* $\Psi_0(z)$ *is taken from (2.1);*

(v) $u(t, .) \underset{t\to 0}{\Rightarrow} u_0(.)$ *in* $L^2(\mathbb{R}^N)$.

Remark A.1. Note that, in [25], Theorem A.1 is proved for $t \in (0, T_{\mathrm{loc}})$ under weaker restrictions imposed on the parameter λ, namely,

$$1 < \lambda < \kappa + 1 + \max\{n + \kappa, m\} \quad \text{for} \quad N < 3,$$

$$1 < \lambda < \min\left\{\frac{4n+7}{3}, 4\right\} \quad \text{for} \quad N = 3; \quad \kappa := \frac{2}{N} \min\left\{\frac{n+4}{3}, 3 - n\right\}.$$

Lemma A.1 [11]. *Let* $f\colon [0,\infty) \to \mathbb{R}$ *be a function right continuous at* 0 *and such that* $f(t_2) - f(t_1) \le M(t_2 - t_1)^a t_1^{-b}$ $\forall t_2 > t_1 > 0,$ *where* $M > 0$ *and* $a > b > 0.$ *Then* $f(t) - f(0) \le \dfrac{M}{1 - 2^{b-a}} t^{a-b}$ $\forall t > 0.$

Lemma A.2 [24, 26]. *Let* $u(t,x)$ *be a strong solution of problem (1.1)–(1.3) constructed in Theorem A.1. Then, for any* $p \in (1, 3-n),$ *there exists a constant* $0 < c = c(n,\, m,\, p,\, N)$ *such that the following inequalities hold for all* $t > 0$:

$$\|u(t)\|_p \le c\|u_0\|_1^{\frac{nN+4p}{p(nN+4)}} t^{-\frac{N(p-1)}{p(nN+4)}} \quad \textit{and} \quad \|u(t)\|_p \le c\|u_0\|_1^{\frac{mN+2p}{p(mN+2)}} t^{-\frac{N(p-1)}{p(mN+2)}}. \tag{A.3}$$

Lemma A.3 [24]. *Let* $u(t,x)$ *be a strong solution of problem (1.1)–(1.3) constructed in Theorem A.1. Then the following estimates for the law of decrease are true for all* $t > 0$*:*

(i) $\displaystyle\int_0^t \|u(\tau)\|_{\hat p}^{\gamma}\, d\tau \le c_1 \|u_0\|_1^{\frac{\gamma(nN+4\hat p)}{\hat p(nN+4)}} t^{1-\frac{\gamma N(\hat p-1)}{\hat p(nN+4)}}$ $\forall \hat p > 1,$ $\gamma \in \left(0, \dfrac{\hat p(nN+4)}{N(\hat p -1)}\right);$

$\displaystyle\int_0^t \|u(\tau)\|_{\hat p}^{\gamma}\, d\tau \le c_1 \|u_0\|_1^{\frac{\gamma(mN+2\hat p)}{\hat p(mN+2)}} t^{1-\frac{\gamma N(\hat p-1)}{\hat p(mN+2)}}$ $\forall \hat p > 1$ $(\hat p < 3(m-n+3)$ *for* $N = 3),$ $\gamma \in \left(0, \dfrac{\hat p(mN+2)}{N(\hat p -1)}\right);$

(ii) $\displaystyle\int_0^t \|\chi_{\mathrm{P}} u^{n-2}(\tau)|\nabla u(\tau)|^3\|_p\, d\tau \le c_1 \|u_0\|_1^{\frac{nN+p(n+4)}{p(nN+4)}} t^{\frac{N-p(N-1)}{p(nN+4)}}$ $\forall p \in \left(\max\left\{\dfrac{4}{n+4}, \dfrac{8}{8n+5}\right\}, \dfrac{4}{3}\right);$

(iii) $\displaystyle\int_0^t \|\chi_{\mathrm{P}} u^{n-1}(\tau)|\nabla u(\tau)|^2\|_p\, d\tau \le c_1 \|u_0\|_1^{\frac{nN+p(2n+4)}{p(nN+4)}} t^{\frac{N-p(N-2)}{p(nN+4)}}$ $\forall p \in \left(\dfrac{3}{n+3}, \dfrac{3}{2}\right);$

(iv) $\displaystyle\int_0^t \|\chi_{\mathrm{P}} u^{n}(\tau)\nabla u(\tau)\|_p\, d\tau \le c_1 \|u_0\|_1^{\frac{nN+p(3n+4)}{p(nN+4)}} t^{\frac{N-p(N-3)}{p(nN+4)}}$ $\forall p \in \left(\dfrac{2}{n+2}, 2\right);$

$$\int_0^t \|u^{m}(\tau)\nabla u(\tau)\|_p\, d\tau \le c_2 \|u_0\|_1^{\frac{mN+p(m+2)}{p(mN+2)}} t^{\frac{N-p(N-1)}{p(mN+2)}}$$

$$\forall p \in \left(\max\left\{\frac{2}{m+2}, \frac{4}{2m+2n+3}\right\}, \min\left\{\frac{N}{N-1}, 2\right\}\right);$$

(v) $\mathcal{E}(u(t)) \le c_2\|u_0\|_1^{\frac{8+n(N-2)}{nN+4}} t^{-\frac{N+2}{nN+4}} + c_2\|u_0\|_1^{\frac{mN+2(m-n+2)}{mN+2}} t^{-\frac{N(m-n+1)}{mN+2}}$ *for* $m - n + 1 > 0;$

(vi) $\mathcal{E}(u(t)) \le c_2\|u_0\|_1^{\frac{8+n(N-2)}{nN+4}} t^{-\frac{N+2}{nN+4}} + c_2\|u_0\|_1^{\frac{mN+2s}{mN+2}} t^{-\frac{N(s-1)}{mN+2}}$ $\forall s \in (1, 3-n)$ *for* $m - n + 1 = 0;$

(vii) $\mathcal{E}(u(t)) \le c_2\|u_0\|_1^{\frac{8+n(N-2)}{nN+4}} t^{-\frac{N+2}{nN+4}}$ *for* $m - n + 1 < 0.$

Here, $0 < c_1 = c_1(n,\, m,\, p,\, N),$ $0 < c_2 = c_2(n,\, m,\, N)$ *[in addition, in (vii), the constant* c_2 *also depends on* s*], and* $\mathcal{E}(u(t))$ *is taken from assertion (iv) of Theorem A.1.*

Lemma A.4. *Let* $u(t,x)$ *be the strong solution of problem (1.1)–(1.3) constructed in Theorem A.1. Then the following estimates are true for all* $t > 0$*:*

(i) $\int_0^t \|b(u(\tau))\|_p \, d\tau \le c\|u_0\|_1^{\frac{nN+4p\lambda}{p(nN+4)}} t^{1-\frac{N(\lambda p-1)}{p(nN+4)}} \quad \forall\, \lambda \in \left(\frac{1}{p}, n+\frac{1}{p}+\frac{4}{N}\right), \quad p>1;$

$\int_0^t \|b(u(\tau))\|_p \, d\tau \le c\|u_0\|_1^{\frac{mN+2p\lambda}{p(mN+2)}} t^{1-\frac{N(\lambda p-1)}{p(mN+2)}} \quad \forall\, \lambda \in \left(\frac{1}{p}, m+\frac{1}{p}+\frac{2}{N}\right) \quad \left(\lambda < \frac{3}{p}(m-n+3) \text{ } \textit{for} \text{ } N=3\right), \ p>1;$

(ii) $\int_0^t \|b(u(\tau))\|_{q'}^2 \, d\tau \le c\|u_0\|_1^{\frac{2(nN+4\lambda q')}{q'(nN+4)}} t^{1-\frac{2N(\lambda q'-1)}{q'(nN+4)}} \quad \forall\, q' \in \Delta_1, \ \lambda \in \left(1, \frac{2N+8+n(3N-2)}{4N}\right);$

$\int_0^t \|b(u(\tau))\|_{q'}^2 \, d\tau \le c\|u_0\|_1^{\frac{2(mN+2\lambda q')}{q'(mN+2)}} t^{1-\frac{2N(\lambda q'-1)}{q'(mN+2)}} \quad \forall\, q' \in \Delta_1, \quad \lambda \in \left(1, \frac{2N+4+n(N-2)+2mN}{4N}\right) \quad \left(\lambda < \frac{(n+6)(m-n+3)}{4} \text{ } \textit{for} \text{ } N=3\right);$

(iii) $\iint_{Q_{t_1}^{t_2}} \overrightarrow{\chi}\chi_{\mathrm{P}}\, b'(u)\nabla u \Delta u \le c\|u_0\|_1^{1+\frac{4\lambda-3(\alpha+n)+\alpha}{nN+4}} t^{\frac{1+N(n-\lambda)}{nN+4}} \quad \forall\, \lambda \in \left(\frac{3(\alpha+n)}{4}, n+\frac{1}{N}\right);$

$\iint_{Q_{t_1}^{t_2}} \overrightarrow{\chi}\chi_{\mathrm{P}}\, b'(u)\nabla u \Delta u \le c\|u_0\|_1^{1+\frac{4\lambda-3(\alpha+n)+\alpha}{mN+2}} t^{\frac{2+N(m+3n-4\lambda)}{4(mN+2)}}$
$\forall\, \lambda \in \left(\frac{3(\alpha+n)}{4}, \frac{m+3n}{4}+\frac{1}{2N}\right) \quad \left(\lambda < \frac{3(\alpha+m)+8}{4} \text{ } \textit{for} \text{ } N=3\right);$

(iv) $\int_0^t \|\mathcal{B}^{(\alpha)}(u(\tau))\|_p \, d\tau \le c\|u_0\|_1^{\frac{nN+4p(\lambda+\alpha)}{p(nN+4)}} t^{1-\frac{N(p(\lambda+\alpha)-1)}{p(nN+4)}} \quad \forall \lambda \in \left(\frac{1}{p}-\alpha, n+\frac{1}{p}+\frac{4}{N}-\alpha\right), \ p>1;$

$\int_0^t \|\mathcal{B}^{(\alpha)}(u(\tau))\|_p \, d\tau \le c\|u_0\|_1^{\frac{mN+2p(\lambda+\alpha)}{p(mN+2)}} t^{1-\frac{N(p(\lambda+\alpha)-1)}{p(mN+2)}}$
$\forall\, \lambda \in \left(\frac{1}{p}-\alpha, m+\frac{1}{p}+\frac{2}{N}-\alpha\right) \quad \left(\lambda < \frac{3}{p}(m-n+3)-\alpha \text{ } \textit{for} \text{ } N=3\right)$ *and* $p>1.$

Here, $0<c=c(n,\, m,\, p,\, N)$, Δ_1 *is taken from assertion (i) of Theorem 2.1,* α *is taken from relation (2.5), and* $\mathcal{B}^{(\alpha)}(z)$ *is taken from (A.2).*

Proof of Lemma A.4. (i) Since $b(z) \le c\, z^\lambda$, we have $\int_0^t \|b(u(\tau))\|_p \, d\tau \le c \int_0^t \|u(\tau)\|_{\lambda p}^\lambda \, d\tau$. By applying estimate (i) from Lemma A.3 with $\hat{p} = \lambda p$ and $\gamma = \lambda$ to the right-hand side of this inequality, we obtain the required estimates.

(ii) By using the inequality $\int_0^t \|b(u(\tau))\|_{q'}^2 \, d\tau \le c \int_0^t \|u(\tau)\|_{\lambda q'}^{2\lambda} \, d\tau$ and estimate (i) from Lemma A.3 with $\hat{p} = \lambda q'$ and $\gamma = 2\lambda$, we arrive at the required result.

(iii) As in [25], we use the identity $u_{x_i x_j} = \gamma^{-1} u^{1-\gamma}(u^\gamma)_{x_i x_j} - (\gamma-1)u^{-1}u_{x_i}u_{x_j}$ with $\gamma = \frac{\alpha+n+1}{2}$ on positive approximating solutions. Then, after necessary limit transitions, we arrive at the following estimate for the integral $I := \iint_{Q_{t_1}^{t_2}} \overrightarrow{\chi}\chi_{\mathrm{P}}\, b'(u)\Delta u \nabla u$:

$$I \le c \iint\limits_{Q_{t_1}^{t_2}} u^{\frac{4\lambda-3(\alpha+n)+1}{4}} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right|^3 + c \iint\limits_{Q_{t_1}^{t_2}} u^{\frac{4\lambda-3(\alpha+n)+1}{4}} \left|\nabla u^{\frac{\alpha+n+1}{4}}\right| \left|\Delta u^{\frac{\alpha+n+1}{2}}\right|.$$

By applying the Hölder inequality with exponents 4 and $\frac{4}{3}$ to the first term and the Hölder inequality with exponents 4, 4, and 2 to the second term, we get

$$I \le c \left(\iint\limits_{Q_{t_1}^{t_2}} u^{4\lambda-3(\alpha+n)+1} \right)^{\frac{1}{4}} \left(\iint\limits_{Q_{t_1}^{t_2}} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 \right)^{\frac{3}{4}}$$

$$+ c \left(\iint\limits_{Q_{t_1}^{t_2}} \left| \nabla u^{\frac{\alpha+n+1}{4}} \right|^4 \right)^{\frac{1}{4}} \left(\iint\limits_{Q_{t_1}^{t_2}} u^{4\lambda-3(\alpha+n)+1} \right)^{\frac{1}{4}} \left(\iint\limits_{Q_{t_1}^{t_2}} \left| D^2 u^{\frac{\alpha+n+1}{2}} \right|^2 \right)^{\frac{1}{2}},$$

whence, by using the global *entropy* inequality (A.2) $(\zeta = 1)$, we obtain

$$I \le c \left(\int\limits_{\mathbb{R}^N} u^{\alpha+1}(t_1) \right)^{\frac{3}{4}} \left(\int\limits_{t_1}^{t_2} \|u(\tau)\|_{4\lambda-3(\alpha+n)+1}^{4\lambda-3(\alpha+n)+1} d\tau \right)^{\frac{1}{4}}.$$

Further, we apply inequality (A.3) with $p = \alpha + 1$ to the first factor and estimate (i) from Lemma A.3 with $\hat{p} = \gamma = 4\lambda - 3(\alpha + n) + 1$ to the second factor. As a result, we find

$$I \le c\|u_0\|_1^{1+\frac{4\lambda-3(\alpha+n)+\alpha}{nN+4}} t_1{}^{-b_1} (t_2 - t_1)^{a_1}, \ \frac{3(\alpha+n)}{4} < \lambda < \frac{3(\alpha+n)+n}{4} + \frac{1}{N};$$

$$I \le c\|u_0\|_1^{1+\frac{4\lambda-3(\alpha+n)+\alpha}{2(mN+2)}} t_1{}^{-b_2} (t_2 - t_1)^{a_2},$$

$$\frac{3(\alpha+n)}{4} < \lambda < \frac{3(\alpha+n)+m}{4} + \frac{1}{2N}, \ \lambda < \frac{3(\alpha+m)+8}{4} \quad \text{for} \quad N = 3,$$

where

$$a_1 = \frac{1}{4}\left(1 - \frac{N(4\lambda - 3(\alpha+n))}{nN+4}\right), \qquad a_2 = \frac{1}{4}\left(1 - \frac{N(4\lambda - 3(\alpha+n))}{mN+2}\right),$$

$$b_1 = \frac{3N\alpha}{4(nN+4)}, \quad \text{and} \quad b_2 = \frac{3N\alpha}{4(mN+2)}.$$

For $a_i - b_i > 0$, in view of Lemma A.1, this yields the required estimates.

(iv) Since $\mathcal{B}^{(\alpha)}(z) \le c\, z^{\lambda+\alpha}$, we have $\int_0^t \|\mathcal{B}^{(\alpha)}(u(\tau))\|_p \, d\tau \le c \int\limits_0^t \|u(\tau)\|_{p(\lambda+\alpha)}^{\lambda+\alpha} \, d\tau$. Applying estimate (i) from Lemma A.3 with $\hat{p} = p(\lambda + \alpha)$ and $\gamma = \lambda + \alpha$ to the right-hand side of this inequality, we arrive at the required estimates.

Appendix B

Lemma B.1 [29]. *If* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with piecewise smooth boundary,* $a > 1,$ $b \in (0, a),$ $d > 1,$ $0 \le i < j,$ i *and* $j \in \mathbb{N},$ *then there exist positive constants* d_1 *and* d_2 $(d_2 = 0$ *if* $\Omega = \mathbb{R}^N)$ *depending only on* $\Omega,$ $d,$ $j,$ $b,$ *and* N *and such that the following inequality holds for any function* $v(x) \in W^j_d(\Omega) \cap L^b(\Omega):$

$$\left\|D^i v\right\|_{L^a(\Omega)} \le d_1 \left\|D^j v\right\|^{\theta}_{L^d(\Omega)} \|v\|^{1-\theta}_{L^b(\Omega)} + d_2 \|v\|_{L^b(\Omega)},$$

where

$$\theta = \frac{\dfrac{1}{b} + \dfrac{i}{N} - \dfrac{1}{a}}{\dfrac{1}{b} + \dfrac{j}{N} - \dfrac{1}{d}} \in \left[\frac{i}{j}, 1\right).$$

Lemma B.2 [21]. *Let* $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m,$ $m \ge 1,$ *and let* $\beta = \prod_{j=1}^m \beta_j,$ $\overline{\beta_i} = \dfrac{\beta}{\beta_i} = \prod_{j=1, j\ne i}^m \beta_j.$ *Assume that* $G_i(s)$ *are nonnegative nonincreasing functions satisfying the conditions*

$$G_i(s+\delta) \le c_i \left(\sum_{i=1}^m \frac{G_i(s)}{\delta^{\alpha_i}}\right)^{\beta_i} \quad \forall s > 0, \quad \delta > 0, \quad i = \overline{1, m},$$

with real numbers $c_i > 0,$ $\beta_i > 1,$ *and* $\alpha_i \ge 0$ *for* $i = \overline{1, m}$ *or* $\alpha_i > 0$ *for* $i = \overline{1, \ell}.$ *Also let* $G(s) = \sum_{i=1}^m \overline{(c_i^{\overline{\beta}_i})}\,(G_i(s))^{\overline{\beta}_i}$ *and let the function* $H(s) = m^{\beta} \sum_{i=\ell+1}^m c_i^{\overline{\beta}_i} \overline{(c_i^{\overline{\beta}_i})}^{\,1-\beta_i} (G_i(s))^{\beta_i - 1}$ *be such that* $H(s_1) < 1$ *at a certain point* $s_1 \ge 0.$ *Then there exists a positive constant* $c > 1$ *depending on* $m,$ $\alpha_i,$ $\beta_i,$ $\ell,$ *and* $H(s_1)$ *such that* $G_i(s_0) \equiv 0$ *for all* $i = \overline{1, \ell},$ *where* $s_0 = s_1 + c \sum_{i=1}^{\ell} \left(c_i^{\overline{\beta}_i} \overline{(c_i^{\overline{\beta}_i})}^{\,1-\beta_i} (G(s_1))^{\beta_i - 1}\right)^{\frac{1}{\alpha_i \beta}}.$

Lemma B.3 [23]. *Let* $\Omega \subset \mathbb{R}^N,$ $N < 6,$ *be a bounded convex domain with smooth boundary and let* $n \in \left(2 - \sqrt{1 - \dfrac{N}{N+8}}, 3\right).$ *Then the following estimates hold for any strictly positive functions* $v \in H^2(\Omega)$ *such that* $\nabla v \cdot \vec{n} = 0$ *on* $\partial\Omega$ *and* $\displaystyle\int_\Omega v^n |\nabla \Delta v|^2 < \infty:$

$$\int_\Omega \varphi^6 \{v^{n-4}|\nabla v|^6 + v^{n-2}|D^2 v|^2 |\nabla v|^2\} \le c \left\{\int_\Omega \varphi^6 v^n |\nabla \Delta v|^2 + \int_{\{\varphi > 0\}} v^{n+2} |\nabla \varphi|^6\right\},$$

$$\int_\Omega \varphi^6 |\nabla \Delta v^{\frac{n+2}{2}}|^2 \le c \left\{\int_\Omega \varphi^6 v^n |\nabla \Delta v|^2 + \int_{\{\varphi > 0\}} v^{n+2} \{|\nabla \varphi|^6 + \varphi^2 |D^2 \varphi|^2 |\nabla \varphi|^2 + \varphi^3 |\Delta \varphi|^3\}\right\},$$

where $\varphi \in C^2(\Omega)$ *is an arbitrary nonnegative function such that the tangential component of* $\nabla\varphi$ *is equal to zero on* $\partial\Omega$ *and the constant* $c > 0$ *is independent of* $v.$

Lemma B.4 [30]. *Assume that a nonnegative nondecreasing function* $f(s)\colon [0,\infty) \to \mathbb{R}$ *satisfies the condition*

$$f(s+\delta) \le c_0 \left(\frac{f(s)}{\delta^{\alpha}}\right)^{\beta} \quad \forall\, s > 0, \quad \delta > 0,$$

with real numbers $c_0 > 0$, $0 < \beta < 1$, *and* $\alpha > 0$. *Then* $f(s) \le 2^{\frac{\alpha\beta}{1-\beta}} c_0^{\frac{1}{1-\beta}} s^{-\frac{\alpha\beta}{1-\beta}} \quad \forall s > 0$.

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