

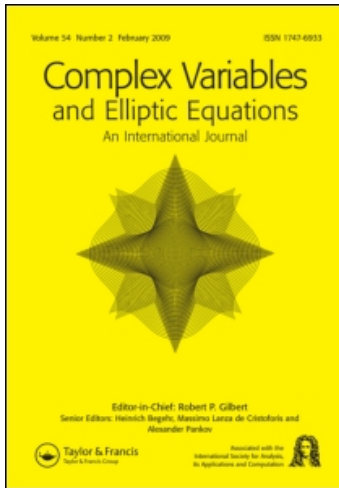
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### ***ACL* and differentiability of the open discrete ring mappings**

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## *ACL* and differentiability of the open discrete ring mappings†

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We study the so-called ring  $Q$ -mappings which are the natural generalization of quasiregular mappings. It is proved that open discrete ring  $Q$ -mappings are differentiable a.e. and belong to the class  $ACL$  in  $\mathbb{R}^n$ ,  $n \geq 2$ ; furthermore,  $f \in W_{loc}^{1,1}$  provided that  $Q \in L_{loc}^1$ .

**Keywords:** open discrete mappings;  $ACL$  and differentiability; modulus and capacity

**AMS Subject Classifications:** Primary 30C65; Secondary 30C75

### 1. Introduction

Let  $D$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and  $f: D \rightarrow \mathbb{R}^n$  be a  $Q$ -quasiconformal mapping. Then necessarily

$$M(f\Gamma) \leq \int_D K_I(x, f) \cdot \rho^n(x) dm(x) \quad (1.1)$$

for every family  $\Gamma$  of paths in  $D$  and every admissible function  $\rho$  for  $\Gamma$ , see e.g. [1], where  $K_I(x, f)$  stands for the well-known inner or outer dilatation of  $f$  at  $x$ . One can replace the above necessary condition with the following, equivalent by Gehring's result [2], inequality

$$M(f(\Gamma(S_1, S_2, A))) \leq \int_{A(r_1, r_2, x_0)} K_I(x, f) \cdot \rho^n(|x - x_0|) dm(x) \quad (1.2)$$

for every point  $x_0 \in D$  and every  $r_1, r_2$ , such that  $0 < r_1 < r_2 < r_0 = \text{dist}(x_0, \partial D)$ , where  $A = A(r_1, r_2, x_0) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$  and  $\Gamma(S_1, S_2, A)$  is a family of all paths joining the spheres  $S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}$ ,  $i = 1, 2$ , in  $A(r_1, r_2, x_0)$ . The above inequalities together with the modulus technique are the powerful tools for the study of quasiconformal (quasiregular) mappings in the plane and in space, see e.g. [3–6]. In order to extend as much as possible the set of maps for the study of which the well-developed modulus technique can be also applied, we replace in (1.1)

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†Dedicated to Professor Andreian Cazacu on the occasion of her 80th birthday.

(in (1.2)) the dilatation  $K_f(x, f)$  with a measurable function  $Q(x)$ , say of the class  $L^1_{loc}(D)$ , and then declare the inequality

$$M(f\Gamma) \leq \int_D Q(x) \cdot \rho^n(x) dm(x) \quad (1.3)$$

or

$$M(f(\Gamma(S_1, S_2, A))) \leq \int_A Q(x) \cdot \eta^n(|x - x_0|) dm(x), \quad (1.4)$$

due to [7], as the necessary condition for the mapping  $f: D \rightarrow \mathbb{R}^n$  to belong to the class of the  $Q$ -homeomorphisms, or the ring  $Q$ -homeomorphisms, respectively, etc. See also the conception of the weighted modulus, [8,9], and applications of the  $Q$ -homeomorphisms, cf. [10–14].

Note that, if  $f$  is homeomorphism, the inequality (1.4) holds at every point  $x_0 \in D$  and  $Q(x) \leq K$  a.e., the definitions of  $Q$ -homeomorphism and ring  $Q$ -homeomorphism are equivalent, see [2], and give that  $f$  is  $K$ -quasiconformal mapping. Moreover, every  $K$ -quasiconformal (or  $K$ -quasiregular) mapping satisfies to (1.4) and (1.3) with  $Q(x) \equiv K$ . Our article is devoted to the study of mappings having unbounded  $Q(x)$  in above definitions.

Recall that a mapping  $f: D \rightarrow \mathbb{R}^n$  is said to be absolutely continuous on lines, write  $f \in ACL$ , if all coordinate functions  $f = (f_1, \dots, f_n)$  are absolutely continuous on almost all straight lines parallel to the coordinate axes for any  $n$ -dimensional parallelepiped  $P$  with edges parallel to the coordinate axes and such that  $\bar{P} \subset D$ .

It is well-known that quasiconformal and quasiregular mappings are absolutely continuous on lines, see e.g. Corollary 31.4 in [4], Lemma 4.11 and Theorem 4.13 in [15] and differentiable a.e., see e.g. Corollary 32.2 in [4], Theorem 2.1 Ch. I in [6] and Theorem 4 in [16]. Moreover, in the plane case, every  $ACL$ -homeomorphism is differentiable a.e., see [3]. However, above results did not give any information about differentiability (or  $ACL$ ) for more general mappings having non-bounded dilatation. The first step in this direction was made in the work of one of the authors, see [17]. More detail, it has been shown that  $Q$ -homeomorphisms are differentiable a.e. and belong to the class  $ACL$  provided that a function  $Q$  is locally integrable. In the present article we extend these results to open discrete mappings satisfying the conditions of the type (1.4).

Thus, the goal of the present article is to prove the following:

- (I) Open discrete ring  $Q$ -mappings  $f: D \rightarrow \overline{\mathbb{R}^n}$  with  $Q \in L^1_{loc}$  are differentiable a.e. in  $D$ .
- (II) Open discrete ring  $Q$ -mappings  $f: D \rightarrow \overline{\mathbb{R}^n}$  with  $Q \in L^1_{loc}$  belong to the class  $ACL$  in  $D$ .
- (III) Open discrete ring  $Q$ -mappings  $f: D \rightarrow \overline{\mathbb{R}^n}$  with  $Q \in L^1_{loc}$  have  $(N^{-1})$ -condition, which is inverse to the well-known Luzin  $(N)$ -property, have non-degenerate Jacobian  $J(x, f)$  a.e., belong to the Sobolev class  $W^{1,1}_{loc}$  and satisfy the inequality

$$\|f'(x)\|^n \leq C_n \cdot |J(x, f)| Q^{n-1}(x)$$

a.e. where a constant  $C_n$  depends only on  $n$ .

**2. Preliminaries**

Let  $D$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ . A mapping  $f: D \rightarrow \mathbb{R}^n$  is said to be discrete if the preimage  $f^{-1}(y)$  of every point  $y \in \mathbb{R}^n$  consists of isolated points, and an open if the image of every open set  $U \subseteq D$  is open in  $\mathbb{R}^n$ . The notation  $G \Subset D$  means that  $\overline{G}$  is a compact subset of  $D$ . We suppose that  $f: D \rightarrow \mathbb{R}^n$  is continuous and sense-preserving, i.e. a topological index  $\mu(y, f, G) > 0$  for any  $G \Subset D$  and  $y \in f(G) \setminus f(\partial G)$ . A neighbourhood of a point  $x$  or a set  $A$  is an open set containing  $x$  or  $A$ , correspondingly. Suppose that  $x \in D$  has a connected neighbourhood  $G$  such that  $\overline{G} \cap f^{-1}(f(x)) = \{x\}$ . Then  $\mu(f(x), f, G)$  is well-defined and independent of the choice of  $G$  for discrete open  $f$  and denoted by  $i(x, f)$ . For  $f: D \rightarrow \mathbb{R}^n$  and  $E \subset D$ , we use the multiplicity functions

$$N(y, f, E) = \text{card}\{x \in E : f(x) = y\},$$

$$N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E).$$

In what follows, we also use the notations  $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$  and  $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ . The above definitions can be extended in a natural way to mappings  $f: D \rightarrow \overline{\mathbb{R}^n}$ .

The following notion is motivated by the Gehring ring definition of quasi-conformality, see [2], and generalizes a notion of ring  $Q$ -homeomorphism, see [18].

Given a domain  $D$  and two sets  $E$  and  $F$  in  $\overline{\mathbb{R}^n}$ ,  $n \geq 2$ ,  $\Gamma(E, F, D)$  denotes the family of all paths  $\gamma: [a, b] \rightarrow \overline{\mathbb{R}^n}$  which join  $E$  and  $F$  in  $D$ , i.e.  $\gamma(a) \in E$ ,  $\gamma(b) \in F$  and  $\gamma(t) \in D$  for  $a < t < b$ . We set  $\Gamma(E, F) = \Gamma(E, F, \overline{\mathbb{R}^n})$  if  $D = \overline{\mathbb{R}^n}$ . Let  $r_0 = \text{dist}(x_0, \partial D)$  and  $Q: D \rightarrow [0, \infty]$  is a measurable function. Set

$$A(r_1, r_2, x_0) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\},$$

$$S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}, \quad i = 1, 2.$$

A homeomorphism  $f: D \rightarrow \overline{\mathbb{R}^n}$  is said to be a ring  $Q$ -homeomorphism at a point  $x_0 \in D$ , if

$$M(f(\Gamma(S_1, S_2, A))) \leq \int_A Q(x) \cdot \eta^n(|x - x_0|) dm(x) \tag{2.1}$$

holds for every annulus  $A = A(r_1, r_2, x_0)$ ,  $0 < r_1 < r_2 < r_0$  and every measurable function  $\eta: (r_1, r_2) \rightarrow [0, \infty]$  such that

$$\int_{r_1}^{r_2} \eta(r) dr \geq 1.$$

If (2.1) holds for every  $x_0 \in D$ ,  $f$  is said to be a ring  $Q$ -homeomorphism. In general case, every  $Q$ -homeomorphism  $f: D \rightarrow \overline{\mathbb{R}^n}$  is a ring  $Q$ -homeomorphism, but the inverse conclusion, generally speaking, is not true. In [18] there are examples of ring  $Q$ -homeomorphisms in a fixed point  $x_0$  such that  $Q(x) \in (0, 1)$  on some set for which  $x_0$  is a density point. We will not discuss here these connections in more details.

Let  $D \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a domain and  $Q: D \rightarrow [0, \infty]$  be a measurable function. We say that a continuous sense-preserving mapping  $f: D \rightarrow \overline{\mathbb{R}^n}$  is a ring  $Q$ -mapping in  $D$  if (2.1) holds for every  $x_0 \in D$ . Note that corresponding to these definitions the

class of the so-called  $Q$ -mappings which consists of the continuous sense-preserving mappings satisfying the condition (1.3) is included in the class of ring  $Q$ -mappings. Thus, all results for ring  $Q$ -mappings formulated below hold, in particular, for  $Q$ -mappings.

A condenser is a pair  $E=(A, C)$  where  $A \subset \mathbb{R}^n$  is open and  $C$  is non-empty compact set contained in  $A$ . A condenser  $E=(A, C)$  is said to be in a domain  $G$  if  $A \subset G$ . For a given condenser  $E=(A, C)$ , we set

$$\text{cap } E = \text{cap } (A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^n dm(x) \tag{2.2}$$

where  $W_0(E) = W_0(A, C)$  is the family of non-negative functions  $u: A \rightarrow \mathbb{R}^1$  such that (1)  $u$  is continuous and finite on  $A$ , (2)  $u(x) \geq 1$  for  $x \in C$ , and (3)  $u$  is  $ACL$ . In the above formula

$$|\nabla u| = \left( \sum_{i=1}^n (\partial_i u)^2 \right)^{1/2}.$$

The quantity  $\text{cap } E$  is called the capacity of the condenser  $E$ .

We say that a family of curves  $\Gamma_1$  is minorized by a family  $\Gamma_2$ , denoted by  $\Gamma_1 > \Gamma_2$ , if for every curve  $\gamma \in \Gamma_1$  there is a subcurve that belongs to the family  $\Gamma_2$ . It is known that  $M(\Gamma_1) \leq M(\Gamma_2)$  as  $\Gamma_1 > \Gamma_2$ , see Theorem 6.4 in [4].

A following statement is well-known as the Rademacher–Stepanov theorem, see e.g. [19, p. 311], and section 3.1.9 in [20].

**PROPOSITION 2.3** *Let  $A \subset B \subset \mathbb{R}^n$ , a set  $B$  be an open,  $f: B \rightarrow \mathbb{R}^n$  and*

$$\limsup_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x - a|} < \infty$$

*for every  $a \in A$ . Then  $f$  is differentiable a.e. on  $A$ .*

### 3. Differentiability

Let  $f: D \rightarrow \mathbb{R}^n$  be a discrete open mapping. Let  $\beta: [a, b] \rightarrow \mathbb{R}^n$  be a path and  $x \in f^{-1}(\beta(a))$ . A path  $\alpha: [a, c] \rightarrow D$  is called a maximal  $f$ -lifting of  $\beta$  starting at  $x$  if (1)  $\alpha(a) = x$ ; (2)  $f \circ \alpha = \beta|_{[a,c]}$ ; (3) if  $c < c' \leq b$ , then there is no path  $\alpha': [a, c'] \rightarrow D$  such that  $\alpha = \alpha'|_{[a,c]}$  and  $f \circ \alpha' = \beta|_{[a,c']}$ . If  $f$  is a discrete open mapping, then every path  $\beta$  with  $x \in f^{-1}(\beta(a))$  has a maximal  $f$ -lifting starting at a point  $x$ , see Corollary 3.3 Ch. II in [6]. We need the following statement, see Proposition 10.2 Ch. II in [6].

**LEMMA 3.1** *Let  $E=(A, C)$  be a condenser in  $\mathbb{R}^n$  and  $\Gamma_E$  be the family of all paths of the form  $\gamma: [a, b] \rightarrow A$  with  $\gamma(a) \in C$  and  $|\gamma| \cap (A \setminus F) \neq \emptyset$  for every compact  $F \subset A$ . Then  $\text{cap } E = M(\Gamma_E)$ .*

**THEOREM 3.2** *Let  $D$  be a domain in  $\mathbb{R}^n, n \geq 2$ , and  $f: D \rightarrow \overline{\mathbb{R}^n}$  be a ring  $Q$ -mapping with  $Q \in L^1_{loc}$ . Suppose that  $f$  is discrete and open. Then  $f$  is differentiable a.e. in  $D$ .*

*Proof* Without loss of generality we may assume that  $\infty \notin D' = f(D)$ . Let us consider the set function  $\Phi(B) = m(f(B))$  defined over the algebra of all the Borel sets  $B$  in  $D$ . By sections 2.2, 2.3 and 2.12 in [15]

$$\varphi(x) = \limsup_{\varepsilon \rightarrow 0} \frac{\Phi(B(x, \varepsilon))}{\Omega_n \varepsilon^n} < \infty \tag{3.1}$$

for a.e.  $x \in D$ . Consider the spherical ring  $R_\varepsilon(x) = \{y : \varepsilon < |x - y| < 2\varepsilon\}$ ,  $x \in D$ , with  $\varepsilon > 0$  such that  $B(x, 2\varepsilon) \subset D$ . Note that  $E = (B(x, 2\varepsilon), \overline{B(x, \varepsilon)})$  is a condenser in  $D$  and  $fE = (fB(x, 2\varepsilon), \overline{fB(x, \varepsilon)})$  is a condenser in  $D'$ . Let  $\Gamma_E$  and  $\Gamma_{fE}$  be path families from Lemma 3.1. Then

$$\text{cap}(fB(x, 2\varepsilon), \overline{fB(x, \varepsilon)}) = M(\Gamma_{fE}). \tag{3.2}$$

Let  $\Gamma^*$  be a family of maximal  $f$ -liftings of  $\Gamma_{fE}$  starting at  $\overline{B(x, \varepsilon)}$ . We show that  $\Gamma^* \subset \Gamma_E$ . Suppose the contrary. Then there is a path  $\beta : [a, b] \rightarrow \mathbb{R}^n$  of  $\Gamma_{fE}$  such that the corresponding maximal  $f$ -lifting  $\alpha : [a, c] \rightarrow B(x, 2\varepsilon)$  is contained in some compact  $K$  inside of  $B(x, 2\varepsilon)$ . Thus  $\overline{\alpha}$  is a compactum in  $B(x, 2\varepsilon)$ , see Theorem 2 of section II, paragraph 41 of chapter 4 in [21]. Remark that  $c \neq b$ . Indeed, in the contrary case  $\overline{\beta}$  is a compact in  $f(A)$  that contradicts to the condition  $\beta \in \Gamma_{fE}$ . Consider the set

$$G = \left\{ x \in \mathbb{R}^n : x = \lim_{k \rightarrow \infty} \alpha(t_k) \right\}, \quad t_k \in [a, c], \quad \lim_{k \rightarrow \infty} t_k = c.$$

Without loss of generality, we may assume that  $t_k$  is the monotone sequence. By continuity of  $f$ , for  $x \in G, f(\alpha(t_k)) \rightarrow f(x)$  as  $k \rightarrow \infty$  where  $t_k \in [a, c], t_k \rightarrow c$  as  $k \rightarrow \infty$ . However,  $f(\alpha(t_k)) = \beta(t_k) \rightarrow \beta(c)$  as  $k \rightarrow \infty$ . Thus,  $f$  is a constant in  $G \subset B(x, 2\varepsilon)$ . On the other hand, from the Cantor condition on the compact  $\overline{\alpha}$ ,

$$G = \bigcap_{k=1}^{\infty} \overline{\alpha([t_k, c])} = \limsup_{k \rightarrow \infty} \alpha([t_k, c]) = \liminf_{k \rightarrow \infty} \alpha([t_k, c]) \neq \emptyset$$

by monotonicity of the sequences of connected sets  $\alpha([t_k, c])$ , see [21]. Thus,  $G$  is connected by I(9.12) in [22]. Consequently,  $G$  is a single point by discreteness of  $f$ . So a path  $\alpha : [a, c] \rightarrow B(x, 2\varepsilon)$  can be extended to  $\alpha : [a, c] \rightarrow K \subset B(x, 2\varepsilon)$  and  $f(\alpha(c)) = \beta(c)$ . By Corollary 3.3 of Ch. II in [6] we can construct a maximal  $f$ -lifting  $\alpha'$  of  $\beta|_{[c, b]}$  started at  $\alpha(c)$ . Uniting the liftings  $\alpha$  and  $\alpha'$ , we have a new  $f$ -lifting  $\alpha''$  of  $\beta$  defined on  $[a, c']$ ,  $c' \in (c, b)$ , that contradicts to the maximality of  $f$ -lifting  $\alpha$ . Thus  $\Gamma^* \subset \Gamma_E$ . Remark that  $\Gamma_{fE} \supset f\Gamma^*$  and, consequently,

$$M(\Gamma_{fE}) \leq M(f\Gamma^*) \leq M(\Gamma_E).$$

Let  $\{r_i\}_{i=1}^{\infty}$  be an arbitrary sequence of numbers with  $\varepsilon < r_i < 2\varepsilon$  such that  $r_i \rightarrow 2\varepsilon - 0$ . Denote by  $\Gamma_i$  a family of paths joining the spheres  $|x| = \varepsilon$  and  $|x| = r_i$  in a ring  $\varepsilon < |x| < r_i$ . Then  $\Gamma_E \supset \Gamma_i$  for every  $i \in \mathbb{N}$ . Consider the family of functions

$$\eta_{i,\varepsilon}(t) = \begin{cases} \frac{1}{r_i - \varepsilon}, & t \in (\varepsilon, r_i), \\ 0, & t \in \mathbb{R} \setminus (\varepsilon, r_i). \end{cases}$$

By definition of a ring  $Q$ -mapping

$$M(f\Gamma_E) \leq M(f\Gamma_i) \leq \frac{1}{(r_i - \varepsilon)^n} \int_{\varepsilon < |x| < r_i} Q(x) dm(x) \leq \frac{1}{(r_i - \varepsilon)^n} \int_{B(x, 2\varepsilon)} Q(x) dm(x). \quad (3.3)$$

Letting to the limit in (3.3) as  $i \rightarrow \infty$ , we obtain

$$M(f\Gamma_E) \leq \frac{1}{\varepsilon^n} \int_{B(x, 2\varepsilon)} Q(x) dm(x). \quad (3.4)$$

From (3.2) and (3.4)

$$\text{cap}(fB(x, 2\varepsilon), f(\overline{B(x, \varepsilon)})) \leq \frac{2^n \Omega_n}{m(B(x, 2\varepsilon))} \int_{B(x, 2\varepsilon)} Q(y) dm(y). \quad (3.5)$$

On the other hand, by Lemma 5.9 in [15]

$$\text{cap}(fB(x, 2\varepsilon), f(\overline{B(x, \varepsilon)})) \geq \left( C_n \frac{d^n(fB(x, \varepsilon))}{m(fB(x, 2\varepsilon))} \right)^{\frac{1}{n-1}} \quad (3.6)$$

where  $C_n$  depends only on  $n$ ,  $d(A)$  is a diameter of the set  $A$  and  $m(A)$  is the Lebesgue measure of  $A$ . Combining (3.5) and (3.6), we obtain that

$$\frac{d(fB(x, \varepsilon))}{\varepsilon} \leq \gamma_n \left( \frac{m(fB(x, 2\varepsilon))}{m(B(x, 2\varepsilon))} \right)^{1/n} \left( \frac{1}{m(B(x, 2\varepsilon))} \int_{B(x, 2\varepsilon)} Q(y) dm(y) \right)^{(n-1)/n}$$

and hence

$$L(x, f) \leq \limsup_{\varepsilon \rightarrow 0} \frac{d(fB(x, \varepsilon))}{\varepsilon} \leq \gamma_n \varphi^{1/n}(x) Q^{(n-1)/n}(x)$$

where

$$L(x, f) = \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|}, \quad (3.7)$$

and  $\varphi$  is a function which is defined by (3.1). Thus,  $L(x, f) < \infty$  a.e. in  $D$ . Applying Proposition 2.3, we conclude that  $f$  is differentiable a.e. in  $D$ . ■

**COROLLARY 3.3** *Let  $D$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and  $f: D \rightarrow \overline{\mathbb{R}^n}$  be a ring  $Q$ -mapping with  $Q \in L^1_{loc}$ . Suppose that  $f$  is discrete and open. Then the partial derivatives of  $f$  are locally integrable.*

*Proof* Given a compact set  $V \subset D$ , we have

$$\int_V L(x, f) dx \leq \gamma_n \int_V \varphi^{1/n}(x) Q^{(n-1)/n}(x) dm(x).$$

Applying the Hölder inequality as  $p = n$ ,  $q = n/(n - 1)$ , see (17.3) in [23], we obtain

$$\int_V \varphi^{1/n}(x) Q^{(n-1)/n}(x) dm(x) \leq \left( \int_V \varphi(x) dm(x) \right)^{1/n} \left( \int_V Q(x) dm(x) \right)^{(n-1)/n}$$

and since  $Q \in L^1_{loc}$

$$\int_V L(x, f) dm(x) \leq \gamma_n N(f, V)^{2/n} \left( \int_V Q(x) dm(x) \right)^{(n-1)/n} < \infty,$$

see Lemma 2.3 in [15].

**COROLLARY 3.4** *Let  $D$  be a domain in  $\mathbb{R}^n, n \geq 2$ , and  $f: D \rightarrow \overline{\mathbb{R}^n}$  be a ring  $Q$ -mapping with  $Q \in L^1_{loc}$ . Suppose that  $f$  is discrete and open. Then*

$$\|f'(x)\|^n \leq C_n \cdot |J(x, f)| Q^{n-1}(x)$$

*a.e. where a constant  $C_n$  depends only on  $n$ .*

**4. On the ACL property of discrete open  $Q$ -mappings**

**THEOREM. 4.1** *Let  $D$  be a domain in  $\mathbb{R}^n, n \geq 2$ , and  $f: D \rightarrow \overline{\mathbb{R}^n}$  be a ring  $Q$ -mapping with  $Q \in L^1_{loc}$ . Suppose that  $f$  is discrete and open. Then  $f \in ACL$ .*

*Proof* Without loss of generality, we may assume that  $\infty \notin D' = f(D)$ . Let  $I = \{x \in \mathbb{R}^n : a_i < x_i < b_i, i = 1, \dots, n\}$  be an  $n$ -dimensional interval in  $\mathbb{R}^n$  such that  $\bar{I} \subset D$ . Then  $I = I_0 \times J$  where  $I_0$  is an  $(n - 1)$ -dimensional interval in  $\mathbb{R}^{n-1}$  and  $J$  is an open segment of the axis  $x_n, J = (a, b)$ . Next we identify  $\mathbb{R}^{n-1} \times \mathbb{R}$  with  $\mathbb{R}^n$ . We prove that for almost everywhere segments  $J_z = \{z\} \times J, z \in I_0$ , the mapping  $f|_{J_z}$  is absolutely continuous.

Consider the set function  $\Phi(B) = m(f(B \times J))$  defined over the algebra of all Borel sets  $B$  in  $I_0$ . By sections 2.2, 2.3 and 2.12 in [15]

$$\varphi(z) = \limsup_{r \rightarrow 0} \frac{\Phi(B(z, r))}{\Omega_{n-1} r^{n-1}} < \infty \tag{4.1}$$

for a.e.  $z \in I_0$  where  $B(z, r)$  is a ball in  $\mathbb{R}^{n-1}$  centred at the point  $z \in I_0$  of the radius  $r$  and  $\Omega_{n-1}$  is a volume of the unit ball in  $\mathbb{R}^{n-1}$ .

Let  $\Delta_i, i = 1, 2, \dots$ , be some enumeration  $S$  of all intervals in  $J$  such that  $\overline{\Delta_i} \subset J$  and the ends of  $\Delta_i$  are the rational numbers. Set

$$\varphi_i(z) := \int_{\Delta_i} Q(z, x_n) dx_n.$$

Then by the Fubini theorem, see, e.g. III 8.1 in [19], the functions  $\varphi_i(z)$  are a.e. finite and integrable in  $z \in I_0$ . In addition, by the Lebesgue theorem on differentiability of the indefinite integral there is a.e. a finite limit

$$\lim_{r \rightarrow 0} \frac{\Phi_i(B^{n-1}(z, r))}{\Omega_{n-1} r^{n-1}} = \varphi_i(z) \tag{4.2}$$

where  $\Phi_i$  for a fixed  $i = 1, 2, \dots$  is the set function

$$\Phi_i(B) = \int_B \varphi_i(\zeta) d\zeta$$

given over the algebra of all Borel sets  $B$  in  $I_0$ .

Let us show that the mapping  $f$  is absolutely continuous on each segment  $J_z, z \in I_0$ , where the finite limits (4.1) and (4.2) exist. Fix one of such a point  $z$ . We have to prove that the sum of diameters of the images of an arbitrary finite collection of mutually disjoint segments in  $J_z = \{z\} \times J$  tends to zero together with the total length of the segments. In view of the continuity of the mapping  $f$ , it is sufficient to verify this fact only for mutually disjoint segments with rational ends in  $J_z$ . So, let  $\Delta_i^* = \{z\} \times \overline{\Delta_i} \subset J_z$  where  $\Delta_i \in S, i = 1, \dots, k$ , under the corresponding

re-enumeration of  $S$  are mutually disjoint intervals. Without loss of generality, we may assume that  $\overline{\Delta}_i, i = 1, \dots, k$  are also mutually disjoint.

Let  $\delta > 0$  be an arbitrary rational number which is less than half of the minimum of the distances between  $\Delta_i^*, i = 1, \dots, k$ , and also less than their distances to the end-points of the interval  $J_z$ . Let  $\Delta_i^* = \{z\} \times [\alpha_i, \beta_i]$  and  $A_i = A_i(r) = B^{n-1}(z, r) \times (\alpha_i - \delta, \beta_i + \delta), i = 1, \dots, k$  where  $B^{n-1}(z, r)$  is an open ball in  $I_0 \subset \mathbb{R}^{n-1}$  centred at the point  $z$  of the radius  $r > 0$ .

For small  $r > 0, E_i = (A_i, \Delta_i^*), i = 1, \dots, k$  are condensers in  $I$  and hence,  $fE_i = (fA_i, f\Delta_i^*), i = 1, \dots, k$  are condensers in  $D'$ . By Lemma 3.1,

$$\text{cap}(fA_i, f\Delta_i^*) = M(\Gamma_{fE_i}).$$

Denoting by  $\Gamma_{E_i^*}$  a family of maximal  $f$ -liftings of  $\Gamma_{fE_i}$  starting at  $\Delta_i^*$ , we obtain  $\Gamma_{E_i^*} \subset \Gamma_{E_i}$  and

$$\text{cap}(fA_i, f\Delta_i^*) \leq M(f\Gamma_{E_i}). \tag{4.3}$$

Let  $m$  be a natural number such that  $1/m < \min\{\delta, z\}$ . Consider the ring  $\varepsilon_1 < |x - z_0| < \varepsilon_2$  where  $z_0 = (z, \frac{\alpha_i + \beta_i}{2}), \varepsilon_1 = \frac{\beta_i - \alpha_i}{2}, \varepsilon_2 = \frac{\beta_i - \alpha_i}{2} + \delta - 1/m$ . Let  $\Gamma_{i,m}$  be a path family joining the spheres  $S_1 = \{|x - z_0| = \varepsilon_1\}$  and  $S_2 = \{|x - z_0| = \varepsilon_2\}$  in  $\mathbb{R}^n$ . Note that  $\Gamma_{i,m} \subset \Gamma_{E_i}$  and by (4.3)

$$\text{cap}(fA_i, f\Delta_i^*) \leq M(f\Gamma_{i,m}). \tag{4.4}$$

Consider the family of functions

$$\eta_{i,m}(t) = \begin{cases} \frac{1}{r - 1/m}, & t \in \left(\frac{\beta_i - \alpha_i}{2}, \frac{\beta_i - \alpha_i}{2} + \delta - 1/m\right), \\ 0, & t \in \mathbb{R} \setminus \left(\frac{\beta_i - \alpha_i}{2}, \frac{\beta_i - \alpha_i}{2} + \delta - 1/m\right), \end{cases}$$

as  $r < \delta$ . By definition of ring  $Q$ -mappings, from (4.4) we have

$$\text{cap}(fA_i, f\Delta_i^*) \leq \frac{1}{(r - 1/m)^n} \int_{A_i} Q(x) dm(x). \tag{4.5}$$

Letting into the limit in (4.6) as  $m \rightarrow \infty$ , we obtain

$$\text{cap}(fA_i, f\Delta_i^*) \leq \frac{1}{r^n} \int_{A_i} Q(x) dm(x). \tag{4.6}$$

On the other hand, by Lemma 5.9 in [15],

$$\text{cap}(fA_i, f\Delta_i^*) \geq \left(C_n \frac{d_i^n}{m_i}\right)^{\frac{1}{n-1}} \tag{4.7}$$

where  $d_i$  is a diameter of the set  $f\Delta_i^*, m_i$  is a volume of  $fA_i$  and  $C_n$  is a constant depending only on  $n$ .

Combining (4.6) and (4.7), we have

$$\left(\frac{d_i^n}{m_i}\right)^{\frac{1}{n-1}} \leq \frac{c_n}{r^n} \int_{A_i} Q(x) dm(x) \tag{4.8}$$

with a constant  $c_n$  depending only on  $n$  and all  $i = 1, \dots, k$ .

By the discrete Hölder inequality with  $p = n/(n - 1)$ ,  $q = n$ ,  $x_k = d_k/m_k^{1/n}$  and  $y_k = m_k^{1/n}$ , see, e.g. Equation (17.3) in [23], we obtain

$$\sum_{i=1}^k d_i \leq \left( \sum_{i=1}^k \left( \frac{d_i^n}{m_i} \right)^{\frac{1}{n-1}} \right)^{\frac{n-1}{n}} \left( \sum_{i=1}^k m_i \right)^{\frac{1}{n}},$$

i.e.

$$\left( \sum_{i=1}^k d_i \right)^n \leq \left( \sum_{i=1}^k \left( \frac{d_i^n}{m_i} \right)^{\frac{1}{n-1}} \right)^{n-1} \Phi(B(z, r)).$$

By (4.8)

$$\left( \sum_{i=1}^k d_i \right)^n \leq \gamma_n \frac{\Phi(B(z, r))}{\Omega_{n-1} r^{n-1}} \left( \sum_{i=1}^k \int_{A_i} Q(x) dm(x) \right)^{n-1}$$

where  $\gamma_n$  depends only on  $n$ . Passing to the limit first as  $r \rightarrow 0$  and then as  $\delta \rightarrow 0$ , we obtain

$$\left( \sum_{i=1}^k d_i \right)^n \leq \gamma_n \varphi(z) \left( \sum_{i=1}^k \varphi_i(z) \right)^{n-1}. \tag{4.9}$$

Finally, in view of (4.9), the absolute continuity of the indefinite integral of  $Q$  over the segment  $J_z$  implies the absolute continuity of the mapping  $f$  over the same segment. Hence  $f \in ACL$ . ■

Combining Theorem 4.1 and Corollary 3.3, we obtain a following conclusion, see also [24].

**COROLLARY 4.2** *Let  $D$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and  $f: D \rightarrow \overline{\mathbb{R}^n}$  be a ring  $Q$ -mapping with  $Q \in L^1_{loc}$ . Suppose that  $f$  is discrete and open. Then  $f \in W^{1,1}_{loc}$ .*

Recall that a mapping  $f: D \rightarrow \mathbb{R}^n$  has  $(N^{-1})$ -property if  $m(E) = 0$  whenever  $m(f(E)) = 0$  where  $m(E)$  is the Lebesgue measure of  $E$  in  $\mathbb{R}^n$ .

**COROLLARY 4.12** *Let  $D$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and  $f: D \rightarrow \overline{\mathbb{R}^n}$  be a ring  $Q$ -mapping with  $Q \in L^1_{loc}$ . Suppose that  $f$  is discrete and open. Then  $f$  has  $(N^{-1})$ -property.*

*Proof* is based on one result of the work [25, Theorem 1.2]. By Corollaries 3.4 and 4.2, it is sufficient to show that  $Q^{n-1} \in L^{n'-1}_{loc}$  where  $1/n' + 1/n = 1$ . We have  $n' = n/(n - 1)$  and  $n' - 1 = \frac{1}{n-1}$ . Since  $Q \in L^1_{loc}$ ,  $Q^{n-1} \in L^{\frac{1}{n-1}}_{loc}$  and the proof is complete. ■

**COROLLARY 4.4** *Let  $D$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and  $f: D \rightarrow \overline{\mathbb{R}^n}$  be a ring  $Q$ -mapping with  $Q \in L^1_{loc}$ . Suppose that  $f$  is discrete and open. Then  $J(x, f) \neq 0$  a.e.*

Corollary 4.4 follows immediately from Corollary 4.3 by the S.P. Ponomarev result, see e.g. [26].

**Remark 4.5** It is known that every mapping  $f: D \rightarrow \mathbb{R}^n$  of the class  $W^{1,n}_{loc}$  such that  $\|f'(x)\|^n \leq K(x) \cdot J(x, f)$  is open and discrete provided that  $K(x) \in L^p_{loc}$  for  $p > n - 1$ ,

see [27,28] and, consequently, all results of the present article are applicable to this Sobolev class, see also Remark 4.10 in [29].

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