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ON THE STABILITY OF EQUILIBRIUM FOR ALMOST PERIODIC SYSTEMS

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1. INTRODUCTION

Consider a system of ordinary differential equations

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

where $x=(x_1, \dots, x_n)$, $t \in R$; $f : R \times B_H \to R^n$ is a continuous function; $B_H = \{x \in R^n : \|x\| \leq H\}$; $f(t, 0) \equiv 0$.

The classical criterion of asymptotic stability of zero solution of equations (1), which was obtained by Lyapunov [1], assumes existence of positive definite function V and negative definite function dV/dt. In applications one can constract a positive definite function V, which derivative is not negative definite but is less than or equal to zero. Exactly for such cases Barbashin and Krasovskii [2] created the effective asymptotic stability criterion if function f in equation (1) is autonomous (f does not depend on t). Then Krasovskii [3] extended this criterion for the case where function f is periodic in t. For general case of nonautonomous function f Matrosov [4] proved that Barbashin–Krasovskii's theorem is not right, hence he needed to introduce the second auxiliary function. The goal of this paper is to prove Barbashin–Krasovskii's criterion for the case of almost periodic function f which is a particular case of the class of nonautonomous functions.

2. DEFINITIONS AND PRELIMINARY RESULTS

Definition 1. A continuous function $F(t) : R \to R^n$ is called almost periodic if for every $\epsilon > 0$ there exists $l = l(\epsilon) > 0$ such that any segment $[\alpha; \alpha + l]$, $\alpha \in R$ contains at least one number τ such that $\|F(t + \tau) - F(t)\| < \epsilon$ for every $t \in R$. A number τ is called an ϵ-almost period of F.

Definition 2. A continuous function $F(t, x) : R \times B_r \to R^n (0 < r < \infty)$ is called uniformly almost periodic in t if for every $\epsilon > 0$ there exists $l = l(\epsilon) > 0$ such that any segment $[\alpha; \alpha + l]$, $\alpha \in R$ contains at least one number τ such that $\|F(t + \tau, x) - F(t, x)\| < \epsilon$ for every $t \in R, x \in B_r$.

Remark. A continuous function $F(t)$, which satisfies Definition 1, in papers [5–8] is called uniformly almost periodic, so Definitions 1 and 2 are some different from corresponding Definitions in [5–8].

LEMMA 1. [7] Let $F_1(t), \dots, F_N(t) : R \to R^n$ be almost periodic functions. Then for every $\epsilon > 0$ there exists $l = l(\epsilon) > 0$ such that any segment $[\alpha; \alpha + l]$, $\alpha \in R$ contains a number τ such that

$$\|F_i(t+\tau) - F_i(t)\| < \epsilon, \quad i = 1, 2, \dots, N; \; t \in R.$$

LEMMA 2. If the function $F(t, x) : R \times B_H \to R^n$ is Lipschitzian in x and almost periodic in t for every fixed $x \in B_H$, then it is uniformly almost periodic in t.

Proof. Since the function $F(t, x)$ satisfies Lipschitz conditions in x, then

$$\|F(t, x) - F(t, y)\| \le L\|x - y\| \tag{2}$$

where L is Lipschitz constant.

Let $\epsilon > 0$ be any real number. B_H is a compact set, hence there is a finite set of points $x_1, \dots, x_N$ such that $x_j \in B_H$ $(j = 1, \dots, N)$ and for each $x \in B_H$ there exists such number i $(1 \le i \le N)$ that

$$\|x - x_i\| < \frac{\epsilon}{3L}. \tag{3}$$

From Lemma 1 it follows that there exists $l > 0$ such, that in any segment $[\alpha; \alpha + l]$ there exists such number τ, that

$$\|F(t, x_i) - F(t+\tau, x_i)\| < \frac{\epsilon}{3} \tag{4}$$

for each $t \in R$, $i = 1, \dots, N$.

Now we will show that for every $x \in B_H$ each number τ, which satisfies inequality (4), is an ϵ-almost period of the function $F(t, x)$. Let x_k be the same element of the set $x_1, \dots, x_N$ for which $\|x - x_k\| < \epsilon/(3L)$. Then by (2)–(4) we obtain

$$\begin{aligned}\|F(t+\tau, x) - F(t, x)\| &\le \|F(t+\tau, x) - F(t+\tau, x_k)\| + \|F(t+\tau, x_k) \\ &\quad - F(t, x_k)\| + \|F(t, x_k) - F(t, x)\| \\ &< \frac{\epsilon}{3} + 2L \cdot \frac{\epsilon}{3L} = \epsilon.\end{aligned}$$

The proof of Lemma 2 is complete. ■

3. MAIN RESULTS

In this section of paper we consider the system of ordinary differential equations (1) under assumptions above. Besides we assume that the function $f(t, x)$ is Lipschitzian in x and almost periodic in t for every fixed $x \in B_H$. Let

$$x(t, t_0, x_0) \quad (t_0 < t < \infty) \tag{5}$$

be the semitrajectory of the system (1) which satisfies the boundary condition $x_0 = x(t_0, t_0, x_0)$.

LEMMA 3. Consider the semitrajectory (5) which belongs to B_h $(0 < h < H)$. Let $\{\epsilon_k\}$ be monotonically approaching zero sequence of positive numbers and $\{\tau_k\}$ be some sequence of ϵ_k-almost periods of $f(t, x)$ (for every ϵ_k there corresponds ϵ_k-almost period τ_k).Then the limit relation

$$\lim_{k\to\infty} \|x(t^*, t_0, x_k) - x(t^* + \tau_k, t_0, x_0)\| = 0 \tag{6}$$

holds, where $x_k = x(t_0 + \tau_k, t_0, x_0)$ and t^* is some fixed moment of time which is more than t_0 $(t^* > t_0)$.

Proof. Consider movements of representative points in the phase space along trajectories

$$x(t, t_0, x_k) \tag{7}$$

and

$$x(t, t_0 + \tau_k, x_k). \tag{8}$$

From position x_k for the time $\Delta t = t^* - t_0$ the representative point reaches position $x(t^*, t_0, x_k)$ along the trajectory (7) and it reaches position

$$x(t^* + \tau_k, t_0 + \tau_k, x_k) = x(t^* + \tau_k, t_0, x_0)$$

along the trajectory (8). Trajectory (8) of equations (1) with initial boundary value problem $x_k = x(t_0 + \tau_k)$ may be interpreted as one of the system

$$\frac{dx}{dt} = f(t + \tau_k, x) \tag{9}$$

with initial position x_k and initial moment of time t_0. But according to Lemma 2 a right-hand part of the system (1) is uniformly almost periodic in t on the set $R \times B_H$, therefore right-hand parts of the systems (1),(9) differ from each other on no matter how small if k is large enough natural number. Hence it follows the limit relation (6). ■

THEOREM 1. Let differential equations (1) satisfy above conditions and there exists continuously differentiable function $V(t, x) : R \times B_H \to R$ such that the following conditions are fulfilled on the set $R \times B_H$:

(i) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$, where $a, b \in K$; K is class of Hahn's functions [1];

(ii) $V(t, x)$ is almost periodic in t for each fixed $x \in B_H$;

(iii) $dV/dt \leq 0$, dV/dt may be equal to zero only on the set $M \subset R \times B_H$ which does not contain entire semitrajectories $(t, x(t, t_0, x_0))$ $(t \geq t_0)$ of the system (1).

Then the solution

$$x = 0 \tag{10}$$

of differential equations (1) is asymptotically stable.

Proof. From conditions (i),(iii) it follows that solution (1) is uniformly stable [1]. Let $\epsilon \in (0; H)$ be any positive number. Denote by $t_0 \in R$ the initial moment of time. By the stability of the zero solution there exists $\delta > 0$ such that if $x_0 \in B_\delta$, then $x(t, t_0, x_0) \in B_\epsilon$ for every $t \geq t_0$. Choose such $\delta > 0$ and show that any solution $x(t) = x(t, t_0, x_0)$ with $x_0 \in B_\delta$ tends to zero as $t \to \infty$. Suppose that this is not true, i.e. there exist $\eta > 0$ and $x_0 \in B_\delta$ such that $\|x(t, t_0, x_0)\| > \eta > 0$ as $t \geq t_0$.

The function $V(t, x(t, t_0, x_0))$ is monotonically nonincreasing because $dV/dt \leq 0$. Hence there exists limit

$$\lim_{t \to \infty} V(t, x(t, t_0, x_0)) = V_0 \geq a(\eta) > 0$$

and it easy to see that $V(t, x(t, t_0, x_0)) \geq V_0$ for $t \in [t_0; \infty)$.

Consider some monotonically approaching zero sequence ϵ_k of positive numbers, where ϵ_1 is sufficiently small. By Lemma 2 for every ϵ_i there exists such sequence of ϵ_i-almost periods $\tau_{i,1}, \tau_{i,2}, \dots, \tau_{i,n}, \dots \to \infty$ for functions $f(t, x)$ and $V(t, x)$, that inequalities

$$|V(t + \tau_{i,n}, x) - V(t, x)| < \epsilon_i,$$

$$\|f(t + \tau_{i,n}, x) - f(t, x)\| < \epsilon_i$$

hold for each $t \in R, x \in B_\epsilon$. Without loss of generality one can suppose $\tau_{i,n} < \tau_{i+1,n}$ for every i, n. Designate $\tau_k = \tau_{k,k}$.

Consider the sequence of points $x_k = x(t_0 + \tau_k, t_0, x_0)$ $(k = 1, 2, \dots)$. It is bounded because $x_k \in B_\epsilon$, therefore there is a limit point x^* of this sequence. Without loss of generality one can assume the sequence x_k itself converges to x^*. Because of continuity and almost periodicity of the function $V(t, x)$

$$\begin{aligned} V(t_0, x^*) &= \lim_{n\to\infty} V(t_0, x_n) \\ &= \lim_{k\to\infty}\lim_{n\to\infty} V(t_0 + \tau_k, x_n) = \lim_{n\to\infty} V(t_0 + \tau_n, x_n) \\ &= \lim_{n\to\infty} V(t_0 + \tau_n, x(t_0 + \tau_n, t_0, x_0)) = V_0. \end{aligned}$$

Now consider the semitrajectory $x(t, t_0, x^*)$, $t_0 < t < \infty$. From condition (iii) of the theorem it follows the existence of such moment of time t^* $(t^* > t_0)$, when inequality

$$V(t^*, x(t^*, t_0, x^*) = V_1 < V_0$$

holds.

Solutions of differential equations (1) are continuous in initial data, so one can write down

$$x(t^*, t_0, x^*) = \lim_{k\to\infty} x(t^*, t_0, x_k)$$

because $\{x_k\} \to x^*$ as $k \to \infty$. Hence it follows

$$\lim_{k\to\infty} V(t^*, x(t^*, t_0, x_k)) = V_1. \tag{11}$$

Using uniform almost periodicity property of $f(t, x)$ and the limit relation (6), we obtain inequality

$$\|x(t^*, t_0, x_k) - x(t^* + \tau_k, t_0, x_0)\| \le \gamma_k \tag{12}$$

where $\gamma_k \to 0$ as $k \to \infty$. Because of uniform almost periodicity property of $V(t, x)$ we have

$$|V(t^*, x) - V(t^* + \tau_k, x)| < \epsilon_k \tag{13}$$

and from conditions (11),(12) it follows that

$$|V(t^*, x(t^* + \tau_k, t_0, x_0)) - V_1| < \eta_k, \tag{14}$$

where $\eta_k \to 0$ as $k \to \infty$.

From (13) we obtain

$$|V(t^*, x(t^* + \tau_k, t_0, x_0)) - V(t^* + \tau_k, x(t^* + \tau_k, t_0, x_0))| < \epsilon_k. \tag{15}$$

From (14),(15) we have

$$|V(t^* + \tau_k, x(t^* + \tau_k, t_0, x_0)) - V_1| < \eta_k + \epsilon_k, \tag{16}$$

where $\eta_k + \epsilon_k \to 0$ as $k \to \infty$.

On the other hand

$$\lim_{k\to\infty} V(t^* + \tau_k, x(t^* + \tau_k, x(t^* + \tau_k, t_0, x_0)) = V_0. \tag{17}$$

The relations (16),(17) are in contradiction to the inequality $V_1 < V_0$. The contradiction obtained proves Theorem 1. ∎

The Instability Theorem of Barbashin–Krasovskii may be extended on the case of almost periodic functions analogously.

THEOREM 2. Let the right-hand side of differential equations (1) be such, that there exists continuously differentiable function $V(t, x) : R \times B_H \to R$ such that the following conditions are fulfilled on the set $R \times B_H$:

(i) $|V(t, x)| \leq b(\|x\|), \quad b \in K$;

(ii) $V(t, x)$ is almost periodic in t for each fixed $x \in B_H$;

(iii) for every $t \in R$ and for every $\delta > 0$ there exists such $x \in B_\delta$,that $V(t, x) > 0$;

(iv) $\mathrm{d}V/\mathrm{d}t \geq 0$; $\mathrm{d}V/\mathrm{d}t$ may be equal to zero only on the set $M \subset R \times B_H$ which does not contain entire semitrajectories $(t, x(t, t_0, x_0))$ $(t \geq t_0)$ of the system (1).

Then the solution (10) of differential equations (1) is unstable.

Proof. Let $\epsilon \in (0; \infty)$. We will take arbitrary $t_0 \in R$ and arbitrary small $\delta > 0$. Let us choose $x \in B_\delta$ such, that $V(t_0, x_0) > 0$. We can do it by condition (iii) of the Theorem. By condition (i) there exists $\eta > 0$ such, that $|V(t, x)| < V(t_0, x_0)$ for every $x \in B_\eta$. The function $V(t, x(t, t_0, x_0))$ is nondecreasing, i.e. $V(t, x(t, t_0, x_0)) \geq V(t_0, x_0)$ for $t \geq t_0$. It means that $\|x(t, t_0, x_0)\| \geq \eta$ for each $t \geq t_0$. We will show that there exists such moment of time t_1 $(t_1 > t_0)$,that $\|x(t_1, t_0, x_0)\| > \epsilon$. Suppose that this is not true, i.e. inequalities

$$\eta \leq \|x(t, t_0, x_0)\| \leq \epsilon. \tag{18}$$

hold for each $t > t_0$.

Using inequalities (18) and the condition (iv) of the Theorem, we obtain a contradiction by means of the same way as by the proof of Theorem 1. We pass the literal repetition of these reasonings. The contradiction prooves the semitrajectory to leave B_ϵ. The proof is complete.

4. EXAMPLES

Example 1. Consider the system of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\lambda y - x(x^2 + y^2)(\sin^2 t + \sin^2 \sqrt{2}\, t), \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \lambda x, \ \lambda \in R. \tag{19}$$

Using $V = \frac{1}{2}(x^2 + y^2)$ we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -x^2(x^2 + y^2)(\sin^2 t + \sin^2 \sqrt{2}\, t) \leq 0.$$

$\mathrm{d}V/\mathrm{d}t$ may be equal to zero only on the set $x = 0$ which does not include entire semitrajectories of (19). Therefore by Theorem 1 the solution

$$x = 0, y = 0$$

of the system (19) is asymptotically stable.

Example 2. Consider the equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\lambda y + x(x^2 + y^2)(\sin^2 t + \sin^2 \sqrt{2}\, t), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \lambda x, \ \lambda \in R. \tag{20}$$

By Theorem 2, using $V = \frac{1}{2}(x^2 + y^2)$ one can show that the zero solution of the system (20) is unstable.

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