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ON NECESSARY AND SUFFICIENT CONDITIONS FOR THE ASYMPTOTIC STABILITY OF IMPULSIVE SYSTEMS

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We prove that the sufficient conditions for the asymptotic stability of impulsive systems obtained by Gurgula and Perestyuk are also necessary conditions.

The majority of physical systems are simulated by differential or difference equations. At the same time, the theory of differential equations with pulse influence, in fact, deals with a combination of differential and difference systems and describes continuous evolutions and discrete events that take place in real models of physical systems. The foundations of this theory are presented in [1]. In recent years, this theory has been extensively developed in many directions [2–5] to meet the needs of modern technology. One of the most important directions is the theory of stability of such systems [6–10] and, in particular, the method of Lyapunov functions [11–13]. In [8, 10], it was shown that if there exists a Lyapunov function with special properties, then the trivial solution of a system with pulse influence is uniformly asymptotically stable. The aim of the present paper is to prove the converse statement, i.e., to prove that if the trivial solution of a system of differential equations with pulse influence is uniformly asymptotically stable, then there exists a Lyapunov function with corresponding properties.

Consider the following system of differential equations with pulse influence at fixed moments of time:

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \quad t \neq \tau_i, \quad t \in R_+, \\ \Delta x &= J_i(x), \quad t = \tau_i, \quad i \in N, \end{aligned} \tag{1}$$

where $t \in R_+ = [0, \infty)$, $x \in B_H \subset R^n$, $B_H = \{x \in R^n \colon \|x\| \leq H\}$, and $\|\cdot\|$ denotes the Euclidean norm. Let $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$. We also assume that the following hypotheses concerning system (1) are true:

(H$_1$) in the domain $R_+ \times B_H$, the function $f(t,x)$ is continuous and satisfies the Lipschitz condition uniformly in $t \in R_+$:

$$\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|, \quad f(t,0) \equiv 0;$$

(H$_2$) the functions $J_i(x)$ are continuous and satisfy the Lipschitz condition: $\|J_i(x_1) - J_i(x_2)\| \leq L\|x_1 - x_2\|$ for $x_1 \in B_H$ and $x_2 \in B_H$, $i \in N$; moreover, $J_i(0) = 0$, $i \in N$;

(H$_3$) $\tau_{i+1} - \tau_i \geq \theta > 0$ for $i \in N$.

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Below, we use the following notation: $E = \bigcup_{i=1}^{\infty} (\tau_{i-1}, \tau_i)$ and $x(t) = F(t, t_0, x_0)$ is the solution of system (1) that satisfies the initial condition $F(t_0, t_0, x_0) = x_0$, $x_0 \in B_H$, if $t_0 \in E$. In the case where $t_0 = \tau_i$, $i \in N$, the expression $F(t, t_0, x_0)$ for $t > t_0$ denotes the solution of system (1) with the initial condition $F(t_0 + 0, t_0 + 0, x_0 + J_i(x_0)) = x_0 + J_i(x_0)$. According to [1, 10], a solution $x(t)$ is assumed to be left-continuous at the points τ_i, i.e., $x(\tau_i) = x(\tau_i - 0)$, $i \in N$. We introduce the following definitions:

Definition 1. *The trivial solution of system (1) is called asymptotically stable if, for a given* $\varepsilon > 0$, *one can find* $\delta = \delta(\varepsilon) > 0$ *such that* $\|F(t, t_0, x_0)\| < \varepsilon$ *for any* $t_0 \in R_+$, $x_0 \in B_\delta$, *and* $t \geq t_0$.

Definition 2. *The solution* $x = 0$ *of system (1) is called uniformly attracting if, for some* $\eta > 0$ *and any* $\varepsilon > 0$, *there exists* $\sigma = \sigma(\varepsilon) > 0$ *such that* $\|F(t, t_0, x_0)\| < \varepsilon$ *for all* $\|x_0\| < \eta$, $t_0 \in R_+$, *and* $t \geq t_0 + \sigma$. *In this case, we say that the domain* B_η *is contained in the domain of attraction of the trivial solution of system (1).*

Definition 3. *The trivial solution of system (1) is called uniformly asymptotically stable if it is uniformly stable and uniformly attracting.*

Definition 4. *We say that a function* $v : R_+ \times B_H \to R$ *belongs to the class* V_0 *if the function* v *is continuous on the set* $G = E \times B_H$ *and satisfies the Lipschitz condition with respect to* x *on* G, $v(t, 0) \equiv 0$ *for* $t \in R_+$, *and, for any* $k \in N$, *the following finite limits exist:*

$$\lim_{t \to \tau_k - 0} v(t, x) = v(\tau_k, x), \qquad \lim_{t \to \tau_k + 0} v(t, x) = v(\tau_k + 0, x).$$

Let K denote the class of the Hahn functions [14, p. 21], i.e., the class of functions $g : R_+ \to R_+$ continuous, strictly increasing, and satisfying the condition $g(0) = 0$.

The theorem below is a modification of a theorem proved in [8, 10].

Theorem 1. *Suppose that conditions* (H_1)–(H_3) *are satisfied and there exist functions* $v \in V_0$, $g \in K$, $b \in K$, *and* $c \in K$ *such that*

$$v(t, x) \geq g(\|x\|) \quad \text{for} \quad t \in R_+, \quad x \in B_H, \tag{2}$$

$$v(t, x) \leq b(\|x\|) \quad \text{for} \quad t \in R_+, \quad x \in B_H, \tag{3}$$

$$D^+ v(t, x) \leq -c(\|x\|) \quad \text{for} \quad (t, x) \in G, \tag{4}$$

where $D^+ v(t, x)$ *denotes the upper right-hand Dini derivative along the solution* $x(t)$, *and*

$$v(\tau_k + 0, x + J_k(x)) \leq v(\tau_k, x) \quad \text{for} \quad k \in N, \quad x \in B_H. \tag{5}$$

Then the solution $x = 0$ *of system (1) is uniformly asymptotically stable and there exists* $h > 0$, $h < H$, *such that* B_h *is contained in the domain of its attraction.*

Let us show that Theorem 1 is invertible. For this purpose, we first prove the following auxiliary statements:

Lemma 1. *Let* $\psi(\tau): R_+ \to R_+$ *be a nonnegative bounded piecewise-continuous function that tends to zero as* $\tau \to +\infty$, *has discontinuity points of the first kind* $\tau_1, \ldots, \tau_n, \ldots$, *where* $\tau_i - \tau_{i-1} \geq \theta > 0$ *and* $\tau_0 = 0$, *and is left-continuous at the discontinuity points, i.e.,* $\psi(\tau_i) = \psi(\tau_i - 0)$, $i \in N$, *and assume that, on the set*

$$E = \bigcup_{i=1}^{\infty} (\tau_{i-1}, \tau_i),$$

the derivative $\psi'(\tau)$ *of the function* $\psi(\tau)$ *exists and satisfies the condition* $|\psi'(\tau)| \leq P$. *Then, for any* $t \in R_+$, *the function* $f(t) = \sup_{t \leq \tau < \infty} \psi(\tau)$ *has one-sided derivatives satisfying the conditions*

$$-P \leq f'(t-0) \leq 0, \quad -P \leq f'(t+0) \leq 0. \tag{6}$$

Proof. Note that, for $t \in R_+$, the curve $y = f(t)$ is formed of alternating segments of the curve $y = \psi(t)$ where $\psi(t)$ does not decrease and segments where the function $f(t)$ is constant, i.e., $f(t)$ is a piecewise-continuous monotonically nondecreasing function that tends to zero as $t \to +\infty$. The discontinuity points can only be the points $t = \tau_i$, $i \in N$. For $t \in R_+$, this function has one-sided derivatives $f'(t \pm 0)$ satisfying conditions (6), which was to be proved.

Lemma 2. *Let* $f_1: R_+ \to R_+$ *and* $f_2: R_+ \to R_+$ *be nonnegative bounded piecewise-continuous functions such that*

$$\lim_{t \to \infty} f_1(t) = 0, \qquad \lim_{t \to \infty} f_2(t) = 0.$$

Then the following inequality is true:

$$\left| \sup_{t \in R_+} f_1(t) - \sup_{t \in R_+} f_2(t) \right| \leq \sup_{t \in R_+} |f_1(t) - f_2(t)|.$$

Proof. In the case where $\sup_{t \in R_+} f_1(t) = \sup_{t \in R_+} f_2(t)$, Lemma 2 is obvious. Let $\sup_{t \in R_+} f_1(t) \neq \sup_{t \in R_+} f_2(t)$. Without loss of generality, we can assume that $\sup_{t \in R_+} f_1(t) > \sup_{t \in R_+} f_2(t)$. By virtue of the fact that the nonnegative functions $f_1(t)$ and $f_2(t)$ tend to zero as $t \to \infty$, there exist finite values t_1, t_2, and t_3 such that $\sup_{t \in R_+} f_1(t) = f_1(t_1)$, $\sup_{t \in R_+} f_2(t) = f_2(t_2)$, and $\sup_{t \in R_+} |f_1(t) - f_2(t)| = |f_1(t_3) - f_2(t_3)|$. Consequently, $f_1(t_1) - f_2(t_2) \leq f_1(t_1) - f_2(t_1) \leq |f_1(t_3) - f_2(t_3)|$, which proves Lemma 2.

Theorem 2. *Suppose that conditions* (H_1)–(H_3) *are satisfied, the solution* $x = 0$ *of system (1) is uniformly asymptotically stable, and the set* B_h, $0 < h < H$, *is contained in the domain of its attraction. Then there exist constants* $P > 0$ *and* $L_1 > 0$ *and functions* $g \in K$, $b \in K$, $c \in K$, *and* $v: R_+ \times B_h \to R_+$ *such* *that* $v \in V_0$, $|v(t, x_1) - v(t, x_2)| \le L_1 \|x_1 - x_2\|$ *for* $t \in R_+$, $x_1 \in B_h$, $x_2 \in B_h$, *and conditions (2)–(5) and the inequality* $D^+ v(t, x) \ge -P$ *are satisfied.*

If system (1) is periodic with period ω, *then the function* v *can also be chosen to be* ω*-periodic.*

Proof. Let $\varphi(t)$ be a scalar monotonically decreasing continuous function that satisfies the inequality

$$\|F(t, t_0, x_0)\| \le \varphi(t - t_0), \qquad t \ge t_0 \tag{7}$$

for any $x_0 \in B_h$ and is such that $\lim_{t \to \infty} \varphi(t) = 0$. The existence of such a function $\varphi(t)$ follows from the property of uniform asymptotic stability in the sense of Definition 3. (It is sufficient to choose any continuous positive function monotonically decreasing to zero and satisfying the inequality $\varphi(t) > \varepsilon$ for $t \in [\sigma(\varepsilon), \sigma(\varepsilon/2)]$ as the function $\varphi(t)$.)

Let $M(t)\colon R_+ \to R_+$ denote a monotonically increasing continuous function such that $\lim_{t \to \infty} M(t) = +\infty$. It was shown in [15, pp. 310–315] that there exists a continuously differentiable function $g = g(\varphi)\colon R_+ \to R_+$ such that

$$g \in K, \qquad g' \in K, \tag{8}$$

$$\int_0^\infty g(\varphi(\tau))\, d\tau = N_1 < +\infty, \tag{9}$$

$$\int_0^\infty g'(\varphi(\tau)) M(\tau)\, d\tau = N_1 < +\infty, \tag{10}$$

$$g'(\varphi(\tau)) M(\tau) < N_3 \qquad \text{for all} \quad \tau \ge 0, \tag{11}$$

where N_1, N_2, and N_3 are positive constants.

Let us show that the function

$$v(t, x) = \int_t^\infty g(\|F(\tau, t, x)\|)\, d\tau + \sup_{t \le \tau < \infty} g(\|F(\tau, t, x)\|) \tag{12}$$

satisfies all conditions of the theorem.

Integral (9) converges. Therefore, by virtue of estimate (7), the integral on the right-hand side of (12) also converges. Consequently, the function v is defined in the domain

$$R_+ \times B_h. \tag{13}$$

Note that $\sup_{t \le \tau < \infty} (\|F(\tau, t, x)\|) \ge \|x\|$. By virtue of (8), we have

$$\int_t^\infty g(\|F(\tau,t,x)\|)d\tau \ \ge\ 0, \qquad \sup_{t\le\tau<\infty} g(\|F(\tau,t,x)\|) \ \ge\ g(\|x\|),$$

i.e., the function v satisfies inequality (2).

Using estimate (7), we obtain $\|F(t,t_0,x_0)\| \le \varphi(0)$ for all t_0, x_0 belonging to domain (13). Therefore,

$$v(t,x) \ \le\ \int_0^\infty g(\varphi(\tau))d\tau \ +\ g(\varphi(0)) \ =\ N_4 \ =\ \text{const.}$$

Consequently, the function v is uniformly bounded in domain (13). Let us show that the function v satisfies the Lipschitz condition with respect to x. Using Lemma 2, we obtain

$$\begin{aligned} |v(t,x_1)-v(t,x_2)| &= \Bigg|\int_t^\infty [g(\|F(\tau,t,x_1)\|)-g(\|F(\tau,t,x_2)\|)]d\tau \\ &\quad + \Bigg[\sup_{t\le\tau<\infty} g(\|F(\tau,t,x_1)\|)-\sup_{t\le\tau<\infty} g(\|F(\tau,t,x_2)\|)\Bigg]\Bigg| \\ &\le \int_t^\infty g'_\varphi(\sup(\|F(\tau,t,x_1)\|,\|F(\tau,t,x_2)\|))\|F(\tau,t,x_1)-F(\tau,t,x_2)\|d\tau \\ &\quad + \sup_{t\le\tau<\infty} \big|(g(\|F(\tau,t,x_1)\|)-g(\|F(\tau,t,x_2)\|))\big|. \end{aligned} \tag{14}$$

According to estimate (2.24) from [1], we have

$$\|F(\tau,t,x_1)-F(\tau,t,x_2)\| \ <\ M(\tau-t)\|x_1-x_2\|, \tag{15}$$

where $M\colon R_+\to R_+$ is a monotonically increasing positive continuous function such that

$$M(\tau-t) \ >\ (1+L)^p e^{L(\tau-t)},$$

where p is the number of points τ_i on the interval $[t;\tau]$. By virtue of property (H_3), such a function exists. Taking into account inequality (15), applying the mean-value theorem to the second term on the right-hand side of (14), and using estimates (10) and (11), we obtain

$$\begin{aligned} |v(t,x_1)-v(t,x_2)| &\le \|x_1-x_2\|\Bigg[\int_t^\infty g'_\varphi(\sup(\|F(\tau,t,x_1)\|,\|F(\tau,t,x_2)\|))M(\tau-t)d\tau \\ &\quad + \sup_{t\le\tau<\infty} \big(g'_\varphi(\varphi(\tau-t))\big)M(\tau-t)\Bigg] \le (N_2+N_3)\|x_1-x_2\|, \end{aligned} \tag{16}$$

which proves that v satisfies the Lipschitz condition with respect to x uniformly in t. The Lipschitz condition implies the existence of a function $b \in K$ for which inequality (3) is satisfied. As the function $b(\|x\|)$, one can choose the function $(N_2 + N_3)\|x\|$.

We now verify the continuity of the function $v(t,x)$ for $(t,x) \in G$. By virtue of property (16), it suffices to verify the continuity of the function v with respect to t for $t \in (\tau_i, \tau_{i+1})$. The first term on the right-hand side of (12) is a function continuous in t for $t \in R_+$ and differentiable with respect to t for $t \neq \tau_i$. The second term is continuous in t for $t \neq \tau_i$ and has bounded nonpositive left and right derivatives with respect to t for $t \in R_+$ by virtue of Lemma 1. Consider $D^+ v(t,x)$ along the solutions $F(t, t_0, x_0)$ of system (1). We have $D^+ v = D^+ \bar{v}$, where $\bar{v}$ is the result of the substitution of an arbitrary solution $F(t, t_0, x_0)$ of Eqs. (1) into the function v. On the other hand, we have

$$\bar{v} = \int_t^\infty g(\|F(\tau, t, F(t, t_0, x_0))\|)d\tau + \sup_{t \le \tau < \infty} g(\|F(\tau, t, F(t, t_0, x_0))\|)$$

$$= \int_t^\infty g(\|F(\tau, t_0, x_0)\|)d\tau + \sup_{t \le \tau < \infty} g(\|F(\tau, t, F(t, t_0, x_0))\|)$$

because $F(\tau, t, F(t, t_0, x_0)) \equiv F(\tau, t_0, x_0)$. Hence, for $t = t_0$, we get

$$D^+ v(t, F(t, t_0, x_0))\Big|_{t=t_0} = \frac{d}{dt}\left(\int_t^\infty g(\|F(\tau, t_0, x_0)\|)\right)d\tau\Big|_{t=t_0}$$

$$+ \lim_{\Delta t \to 0+0} \sup\left(\sup_{t_0 + \Delta t \le \tau < \infty} g(\|F(\tau, t_0, x_0)\|) - \sup_{t_0 \le \tau < \infty} g(\|F(\tau, t_0, x_0)\|)\right).$$

The second term on the right-hand side of the last equality is nonpositive. Therefore,

$$D^+ v(t, F(t, t_0, x_0))\Big|_{t=t_0} \le -g(\|F(t_0, t_0, x_0)\|) = -g(\|x_0\|),$$

i.e., $D^+ v$ satisfies relation (4).

For $x = F(\tau_k, t_0, x_0)$, we have $x + J_k(x) = F(\tau_k + 0, t_0, x_0)$. Hence, taking into account that the second term in inequality (12) is a nonincreasing function, we conclude that inequality (5) is satisfied along the trajectory $x = F(t, t_0, x_0)$ of system (1).

Now assume that system (1) is periodic with period ω. This means that $f(t+\omega, x) \equiv f(t,x)$ and there exists $q \in N$ such that $J_k(x) \equiv J_{k+q}(x)$ and $\tau_{k+q} = \tau_k + \omega$ for any natural k. We show that, in this case, the function $v(t,x)$ defined by equality (12) possesses the property $v(t+\omega, x) \equiv v(t,x)$. Indeed,

$$v(t+\omega, x) = \int_{t+\omega}^\infty g(\|F(\tau, t+\omega, x)\|)d\tau + \sup_{t+\omega \le \tau < \infty} g(\|F(\tau, t+\omega, x)\|).$$

Introducing a new variable s according to the formula $\tau = s + \omega$, we get

$$v(t+\omega, x) = \int_t^\infty g(\|F(s+\omega, t+\omega, x)\|)ds + \sup_{t \le s < \infty} g(\|F(s+\omega, t+\omega, x)\|). \tag{17}$$

Using the obvious property of solutions of periodic systems

$$F(t+\omega, t_0+\omega, x_0) = F(t, t_0, x_0), \tag{18}$$

we obtain $v(t+\omega, x) \equiv v(t, x)$ by virtue of equalities (17) and (18), which was to be proved. The theorem is proved.

We demonstrate one of possible applications of the theorem proved. Assume that system (1) has the uniformly asymptotically stable solution

$$x = 0. \tag{19}$$

Along with system (1), we consider the system

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) + f_*(t, x), \qquad t \neq \tau_i, \\ \Delta x\big|_{t=\tau_i} &= J_i(x) + J_i^*(x) = I_i(x), \qquad i \in N, \end{aligned} \tag{20}$$

where $f_*(t, x)$ and $J_i^*(x)$ are continuous functions satisfying the Lipschitz condition with respect to x uniformly in $i \in N$ and such that $f_*(t, 0) \equiv 0$ and $J_i^*(0) = 0$ for $i \in N$. Under the assumptions imposed on the right-hand sides of systems (1) and (20), the following theorem is true:

Theorem 3. *If the trivial solution of system (1) is uniformly asymptotically stable and the limit relations*

$$\lim_{t\to\infty} f_*(t, x) = 0, \qquad \lim_{i\to\infty} J_i^*(x) = 0 \tag{21}$$

are satisfied uniformly in $x \in B_H$, $0 < H < \infty$, *then solution (19) of system (20) is also uniformly asymptotically stable.*

Proof. Since the trivial solution of system (1) is uniformly asymptotically stable, there exist a function $v(t, x)$ and Hahn functions g, b, and c that satisfy the conditions of the previous theorem. Using the Yoshizawa theorem [16], we first estimate $D^+v(t, x)\big|_{(20)}$ along the solution $x(t)$ of system (20) for $x \in B_h$:

$$\begin{aligned} D^{+}v(t,x)\Big|_{(20)} &= \lim_{\xi\to 0+0}\sup\frac{v(t+\xi, x+\xi f(t,x)+\xi f_{*}(t,x)) - v(t,x)}{\xi} \\ &\le \lim_{\xi\to 0+0}\sup\frac{v(t+\xi, x+\xi f(t,x)+\xi f_{*}(t,x)) - v(t+\xi, x+\xi f(t,x))}{\xi} \\ &\qquad + \lim_{\xi\to 0+0}\sup\frac{v(t+\xi, x+\xi f(t,x)) - v(t,x)}{\xi} \\ &\le L_1\left\|f_{*}(t,x)\right\| + D^{+}v(t,x)\Big|_{(1)}. \end{aligned} \tag{22}$$

By analogy, we estimate the value Δv_i of the jump of the function v along the trajectory $x(t)$ of system (20) at time τ_i:

$$\begin{aligned} \Delta v_i &= v\big(\tau_i+0, x+J_i(x)+J_i^{*}(x)\big) - v(\tau_i, x) \\ &= \big[v\big(\tau_i+0, x+J_i(x)+J_i^{*}(x)\big) - v(\tau_i+0, x+J_i(x))\big] + \big[v(\tau_i+0, x+J_i(x)) - v(\tau_i, x)\big] \\ &\le v\big(\tau_i+0, x+J_i(x)+J_i^{*}(x)\big) - v(\tau_i+0, x+J_i(x)) \le L_1\left\|J_i^{*}(x)\right\|. \end{aligned} \tag{23}$$

Recall that L_1 in inequalities (22) and (23) denotes the Lipschitz constant for the function v.

Let us show that solution (19) of system (20) is uniformly stable. We choose arbitrary $\varepsilon_1 > 0$, $\varepsilon_1 < h < H$. Let $t_1 \in R_{+}$ be a sufficiently large initial moment of time. We show that there exists $\delta_1 = \delta_1(\varepsilon_1) > 0$ such that the solution $x(t) = x(t, t_1, x_1)$ of system (20) satisfies the condition $\|x(t)\| < \varepsilon_1$ for $t > t_1$ if $x_1 \in B_{\delta_1}$. Denote $\delta_1 = b^{-1}\big(g\big(b^{-1}(g(\varepsilon_1))/(2L)\big)\big)$, where b^{-1} is the function inverse to b, and L is the Lipschitz constant for the functions $I_i(x)$, which is assumed to be greater than unity ($L > 1$). Assume that the value t_1 satisfies the inequality $t_1 \ge T_1$, where T_1 is so large that $L_1\|f_{*}(t,x)\| < \gamma_1$ for $t \ge T_1$ and $x \in B_{\varepsilon_1}$, and $L_1\|J_i^{*}(x)\| < \theta\gamma_1$ for $\tau_i \ge T_1$, $x \in B_{\varepsilon_1}$, and $\gamma_1 = c(\delta_1)/2$. Since the limit relations (21) are satisfied uniformly in $x \in B_H$, and γ_1 depends only on ε_1, we can choose the value of T_1 dependent only on ε_1. Let us show that if $\|x_1\| < \delta_1$, then $\|x(t)\| < \|x(t, t_1, x_1)\| < \varepsilon_1$ for $t > t_1$. The trajectory $x(t)$ can leave B_{δ_1} in one of the following two ways: either there exists a number $k \in N$ such that $x(\tau_k) \in B_{\delta_1}$ and $x(\tau_k) + I_k(x(\tau_k)) \overline{\in} B_{\delta_1}$, or there exists a moment of time $t_{*} \in (\tau_{k-1}, \tau_k)$ such that $\|x(t_{*})\| = \delta_1$ and $\|x(\tau_k)\| > \delta_1$. In the first case ($x(\tau_k) \in B_{\delta_1}$, $\tau_k > t_1$, $x(\tau_k) + I_k(x(\tau_k)) \overline{\in} B_{\delta_1}$), we obtain

$$\left\|x(\tau_k) + I_k(x(\tau_k))\right\| < b^{-1}(g(\varepsilon_1)) \tag{24}$$

because $\|I_k(x(\tau_k))\| \le L\|x(\tau_k)\|$ and

$$\left\|x(\tau_k)\right\| \le \delta_1 = b^{-1}\left(g\left(\frac{b^{-1}(g(\varepsilon_1))}{2L}\right)\right) \le \frac{b^{-1}(g(\varepsilon_1))}{2L} < \frac{b^{-1}(g(\varepsilon_1))}{2}.$$

In the second case (there exists a moment of time $t_* \in (\tau_{k-1}, \tau_k)$ such that $\|x(t_*)\| = \delta_1$, $\|x(\tau_k)\| > \delta_1$), using the condition $\|x(t_*)\| = \delta_1$ we get

$$g(\|x(\tau_k)\|) \leq v(\tau_k, x(\tau_k)) < v(t_*, x(t_*)) \leq b(\delta_1) = g\left(\frac{b^{-1}(g(\varepsilon_1))}{2L}\right),$$

whence $\|x(\tau_k)\| < b^{-1}(g(\varepsilon_1))/(2L)$, and, consequently, inequality (24) is satisfied.

Now consider the function $\bar{v}(t) = v(t, x(t))$ for $t \in (\tau_k, \tau_{k+1}]$. Then

$$g(\|x(t)\|) \leq \bar{v}(t) \leq \bar{v}(\tau_k + 0) \leq b\left(b^{-1}(g(\varepsilon_1))\right) = g(\varepsilon_1),$$

whence

$$\|x(t)\| \leq \varepsilon_1 \tag{25}$$

by virtue of the condition $g \in K$. In the case where the solution $x(t)$ has points belonging to B_{δ_1} on the interval (τ_k, τ_{k+1}), one can prove by analogy that $\|x(\tau_{k+1}) + I_{k+1}(x(\tau_{k+1}))\| < b^{-1}(g(\varepsilon_1))$, and inequality (25) holds for all $t \in (\tau_{k+1}, \tau_{k+2})$. If the solution $x(t)$ satisfies the condition $\delta_1 \leq \|x(t)\| \geq \varepsilon_1$ on the interval (τ_k, τ_{k+1}), then

$$\bar{v}(\tau_{k+1}) \leq \bar{v}(\tau_k) + \Delta v_k + \int_{\tau_k}^{\tau_{k+1}} D^+ \bar{v}(t) dt \leq \bar{v}(\tau_k) + \gamma_1 \theta - 2\gamma_1 \theta = \bar{v}(\tau_k) - \gamma_1 \theta. \tag{26}$$

Using estimates (23) and (26), we obtain

$$\bar{v}(\tau_{k+1} + 0) = \bar{v}(\tau_{k+1}) + \Delta v_{k+1} \leq \bar{v}(\tau_k),$$

which implies that inequality (25) is also satisfied on $(\tau_{k+1}, \tau_{k+2}]$. By analogy, we prove that, for $t \in (\tau_{k+2}, \tau_{k+3}], \ldots, t \in (\tau_{k+s}, \tau_{k+s+1}]$, the solution $x(t)$ satisfies inequality (25). Thus, we have proved that, for any $\varepsilon_1 > 0$, there exists $T_1 = T_1(\varepsilon_1) > 0$ such that, for any $t_1 \geq T_1$, one can find $\delta_1 = \delta_1(\varepsilon_1) > 0$ such that the inequality $\|x_1\| < \delta_1$ yields $\|x(t)\| = \|x(t, t_1, x_1)\| < \varepsilon_1$ for $t > t_1$. It follows from inequality (2.24) in [1] and hypothesis (H_3) that there exists $\delta > 0$ such that, for any $t_0 \in [0, T_1]$ and $x_0 \in B_\delta$, the solution $x(t, t_0, x_0)$ satisfies the inequality $\|x(T_1, t_0, x_0)\| < \delta_1$ and, consequently, the inequality $\|x(t, t_1, x_1)\| < \varepsilon_1$ for $t > t_0$. Since δ_1 and T_1 depend only on ε_1, the quantity δ also depends only on ε_1, which proves the uniform stability of solution (19) of system (20).

Let us show that solution (19) of system (20) is uniformly attracting. For this purpose, we choose an arbitrary $\varepsilon_1 > 0$, $\varepsilon_1 < h$, and the corresponding value $\delta = \delta(\varepsilon_1) > 0$ from the definition of uniform stability. We show that, for any $\varepsilon_2 > 0$, $\varepsilon_2 < \varepsilon_1$, there exists $\sigma = \sigma(\varepsilon_2) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon_2$ for all $\|x_0\| < \delta$, $t_0 \in R_+$, and $t \geq t_0 + \sigma$. For this purpose, we choose $\delta_2 = \delta_2(\varepsilon_2) > 0$ so that the solution $x(t, t_0, x_0)$ of system (20) that hits B_{δ_2} at a certain moment of time does not leave the domain B_{ε_2} afterwards. Such a choice of δ_2 is possible by virtue of the proved fact that solution (19) of system (20) is uniformly stable. As proved above, the inequality $\|x(t, t_0, x_0)\| < \varepsilon_1$ holds for $t > t_0$. Let us estimate the period of time during which the solution $x(t, t_0, x_0)$ can belong to the set

$$\delta_2 \le \|x\| < \varepsilon_1. \tag{27}$$

Let t_2 denote a moment of time such that $L_1 \|f_*(t, x)\| < \gamma_2$ for $t \ge t_2$ and $x \in B_{\varepsilon_1}$, and $L_1 \| J_m^*(x) \| < \theta\gamma_2$ for $\tau_m \ge t_2$, $x \in B_{\varepsilon_1}$, and $\gamma_2 = c(\delta_2)/2$. Since the limit relations (21) hold uniformly in $x \in B_H$, and γ_2 depends only on ε_2, we can choose the value of t_2 dependent only on ε_2. Let us show that solution (19) of system (20) can belong to set (27) during a period of time not larger than $t_3 = b(\varepsilon_1)/\gamma_2(\varepsilon_2)$. For this purpose, we estimate the value $\bar{v}(\tau_m + s\theta)$, where s is a natural number, taking into account assumption (H_3):

$$\bar{v}(\tau_m + s\theta) \le \bar{v}(\tau_m) + s\gamma_2 + \int_{\tau_m}^{\tau_m + s\theta} D^+ \bar{v}(t)\,dt$$

$$\le \bar{v}(\tau_m) + s\gamma_2\theta - 2s\gamma_2\theta = \bar{v}(\tau_m) - s\gamma_2\theta \le b(\varepsilon_1) - s\gamma_2\theta. \tag{28}$$

Indeed, by virtue of estimate (28), for $s\theta > t_3$ we obtain $\bar{v}(\tau_m + s\theta) < 0$, which is impossible because $v(t, x)$ is a strictly positive function. Thus, σ can be chosen in the form $\sigma = t_2 + t_3$, where t_2 and t_3 depend only on ε_2. This proves that the trivial solution of system (20) is uniformly attracting and the set B_δ belongs to the domain of its attraction. The theorem is proved.

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