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CONVERGENCE OF EIGENVALUES AND EIGENFUNCTIONS OF NONLINEAR DIRICHLET PROBLEMS IN DOMAINS WITH FINE-GRAIN BOUNDARY

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We study the behavior of eigenvalues and eigenfunctions of the Dirichlet problem for nonlinear elliptic second-order equations in domains with fine-grain boundary.

In the present work, we study the convergence of the eigenvalues of nonlinear Dirichlet problems in a sequence of domains with fine-grain boundary. The averaging of linear problems in such domains was studied by many authors beginning with Marchenko and Khruslov [1]. Linear problems on eigenvalues in perforated domains are also well investigated (see, e.g., [2, 3]). For nonlinear equations, the Dirichlet problem in perforated domains was considered in [4–6]. One should also note the works [7–10], where nonlinear problems on eigenvalues were studied in a fixed domain.

1. Formulation of Conditions and Results

Let $\Omega_s = \Omega \setminus \bigcup_{i=1}^{I(s)} F_i^{(s)}$, where $\Omega \subset R^n$ is an arbitrary domain and let, for any natural value of s, a finite number of disjoint closed domains $F_i^{(s)}$, $i = \overline{1, I(s)}$, contained in Ω be defined. Let $d_i^{(s)}$ denote the lower bound of the radii of balls containing $F_i^{(s)}$ and let $x_i^{(s)}$ be the center of a ball of radius $d_i^{(s)}$ such that $F_i^{(s)} \subset \overline{B\big(x_i^{(s)}, d_i^{(s)}\big)}$. Denote $r_i^{(s)} = \operatorname{dist}\Big\{ B\big(x_i^{(s)}, d_i^{(s)}\big), \bigcup_{j \neq i} B\big(x_j^{(s)}, d_j^{(s)}\big) \cup \partial\Omega \Big\}$. Assume that the following conditions are satisfied:

B_1) $d_i^{(s)} \le C_0 r_i^{(s)}$, where C_0 is a constant independent of i and s, and

$$\lim_{s \to \infty} \max_{1 \le i \le I(s)} r_i^{(s)} = 0;$$

B_2) for a certain continuous nondecreasing function $\alpha : [0, +\infty) \to [0, +\infty)$ satisfying the conditions $\alpha(0) = 0$ and $\alpha(t)/t \xrightarrow[t \to \infty]{} \infty$, the following inequality is true:

$$\sum_{i=1}^{I(s)} C_m\big(F_i^{(s)}\big) \left\{ \frac{C_m\big(F_i^{(s)}\big) + \alpha^{n-1}\big(d_i^{(s)}\big)}{\big(r_i^{(s)}\big)^n} \right\}^{1/(m-1)} \le C_1.$$

Here and below, all constants C_j, $j = 0, 1, \ldots$, are positive and independent of s, and $C_m(E)$ is the m-capacity of the set $E \subset B(x_0, 1/2)$, i.e.,

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$$C_m(E) = \inf_{\varphi \in V(E)} \int_{B(x_0,1)} |\nabla\varphi|^m dx,$$

where $V(E) = \{\varphi(x) \in C_0^\infty(B(x_0,1)) : \varphi(x) = 1,\ x \in E\}$.

Consider the functionals

$$\Phi_s : \overset{0}{W}{}^1_m(\Omega_s) \times \overset{0}{W}{}^1_m(\Omega_s) \to R^1, \qquad G_s : \overset{0}{W}{}^1_m(\Omega_s) \to R^1,$$

$$\Phi_s(u, v) = \int_{\Omega_s} f(x, v(x), \nabla u(x))dx, \qquad G_s(u) = \int_{\Omega_s} g(x, u(x))dx.$$

Denote

$$f_i(x, u, p) = \frac{\partial f(x, u, p)}{\partial p_i}, \qquad f_0(x, u, p) = \frac{\partial f(x, u, p)}{\partial u},$$

$$g_0(x, u) = \frac{\partial g(x, u)}{\partial u}.$$

Assume that the functions $f(x, u, p)$ and $g(x, u)$ satisfy the following conditions:

$A_1)$ the functions $f(x, u, p)$ and $g(x, u)$ are measurable in x for any $u \in R^1$ and $p \in R^n$ and belong to the class C^1 in u and p, and $x \in \Omega$ a.e.;

$A_2)$ there exist positive constants C_2, C_3, and C_4 such that, for $2 \le m < n$, one has $m \le m_1 < nm/(n-m)$, $i = 1, \dots, n$, and, for all values of $x \in \Omega$, $p, q \in R^n$, and $u, v \in R^1$, the following inequalities are true:

$$|f_i(x, u, p) - f_i(x, v, q)| \le C_2\left(1 + |u|^{m_1} + |v|^{m_1} + |p|^m + |q|^m\right)^{(m-2)/m}(|p-q| + |u-v|),$$

$$\left|f_0(x, u, p)\right| \le C_2\left(|u|^{m_1} + |p|^m\right)^{(m_1-1)/m_1} + \varphi(x),$$

$$f_0(x, u, p)u \ge -(C_3 - C_4)\left|p\right|^m - \varphi(x)(1 + |u|),$$

$$\left|g_0(x, u)\right| \le C_2|u|^{m_1-1},$$

where $\varphi(x) \in L_{r_1}(\Omega)$ and $r_1 > n/m$. Moreover, $f_i(x, u, 0) = 0$ for $x \in \Omega$ and $u \in R^1$, $i = 1, \dots, n$;

$A_3)$ $\forall\, x \in \overline{\Omega}$ $\forall\, p, q \in R^n$ $\forall\, u \in R^1$:

$$\sum_{i=1}^{n} (f_i(x, u, p) - f_i(x, u, q))(p_i - q_i) \ge C_3(1 + |p| + |q|)^{m-2}|p-q|^2;$$

$A_4)$ $\forall\, x \in \overline{\Omega}$ $\forall\, u \in R^1$:

$$g(x, 0) = 0, \quad f_0(x, 0, 0) = g_0(x, 0) = 0, \quad g_0(x, u)u > 0, \quad u \neq 0,$$

In the domain Ω_s, we consider the problem on eigenvalues

$$\sum_{j=1}^{n} \frac{d}{dx_j} f_j(x, u_s(x), \nabla u_s(x)) - f_0(x, u_s(x), \nabla u_s(x)) = \lambda_s g_0(x, u_s(x)), \quad x \in \Omega_s, \tag{1}$$

$$u_s(x) = 0, \quad x \in \partial\Omega_s. \tag{2}$$

To formulate one more condition for sets $F_i^{(s)}$ that guarantees the possibility of constructing an averaged problem, we need auxiliary functions $v_i^{(s)}(x, k)$. Let $\psi_0(x) \in C_0^\infty(B(0, 1))$ and $\psi_0(x) = 1$, $x \in B(0, 1/2)$. For an arbitrary real k with $d_i^{(s)} < 1/2$, we denote by $v_i^{(s)}(x, k)$ a function that belongs to the space $k\psi_0\big(x - x_i^{(s)}\big) + \overset{0}{W}{}_m^1\big(\Omega_i^{(s)}\big)$ and satisfies the integral identity

$$\begin{gathered} \sum_{j=1}^{n} \int_{\Omega_i^{(s)}} f_j\big(x, 0, \nabla v_i^{(s)}(x)\big) \frac{\partial \psi(x)}{\partial x_j} dx = 0 \quad \forall\, \psi(x) \in \overset{0}{W}{}_m^1\big(\Omega_i^{(s)}\big), \\ \Omega_i^{(s)} = B\big(x_i^{(s)}, 1\big) \setminus F_i^{(s)} = B_i^{(s)} \setminus F_i^{(s)}. \end{gathered} \tag{3}$$

The existence and uniqueness of the function $v_i^{(s)}(x, k)$ were proved in [4]. Outside $\Omega_i^{(s)}$, we assume that $v_i^{(s)}(x, k) = k\psi_0\big(x - x_i^{(s)}\big)$.

We also assume that the following condition is satisfied:

$C)$ there exists a function $c(x, u)$ continuous in x for any $u \in R^1$ and belonging to the class C^1 in u for almost all $x \in \Omega$. Moreover, for any ball $B \subset \Omega$,

$$\lim_{s \to \infty} \sum_{i \in I_s(B)} \sum_{j=1}^{n} \int_0^u \frac{1}{\mu} \int_\Omega f_j\big(x, 0, \nabla v_i^{(s)}(x, \mu)\big) \frac{\partial v_i^{(s)}(x, \mu)}{\partial x_j} dx\, d\mu = \int_B c(x, u)\, dx,$$

and the convergence to the limit is uniform in u on any bounded interval of u.

We introduce the functionals $\overset{0}{W}{}_m^1(\Omega) \times \overset{0}{W}{}_m^1(\Omega) \to R^1$, $G: \overset{0}{W}{}_m^1(\Omega) \to R^1$, and $C: \overset{0}{W}{}_m^1(\Omega) \to R^1$, where

$$\Phi(u, v) = \int_\Omega f(x, v(x), \nabla u(x))\, dx, \qquad G(u) = \int_\Omega g(x, u(x))\, dx,$$

$$C(u) = \int_\Omega c(x, u(x))\, dx.$$

In addition, denote

$$F_c(u) = \Phi(u, u) + C(-u), \qquad c_0(x, u) = \frac{\partial c(x, u)}{\partial u},$$

$$M(G) = \left\{ u(x) \in \overset{0}{W}{}_m^1(\Omega) \colon G(u) = 1 \right\},$$

$$M_s(G_s) = \left\{ u_s(x) \in \overset{0}{W}{}_m^1(\Omega_s) \colon G_s(u_s) = 1 \right\}.$$

Consider the limit problem on eigenvalues

$$\sum_{j=1}^{n} \frac{d}{dx_j} f_j(x, u(x), \nabla u(x)) - f_0(x, u(x), \nabla u(x)) + c_0(x, -u(x)) = \lambda g_0(x, u(x)), \qquad x \in \Omega. \tag{4}$$

$$u(x) = 0, \qquad x \in \partial\Omega.. \tag{5}$$

The main result of the present work is the following theorem:

Theorem 1. *Let* $\lambda_s = \min\limits_{v_s \in M_s(G_s)} F(v_s) = F(u_s)$ *be the eigenvalue of problem (1), (2), let* $\lambda = \min\limits_{u \in M(G)} F_c(u) = F_c(\tilde{u})$ *be the eigenvalue of problem (4), (5), and let* $u_s(x)$ *and* $\tilde{u}(x)$ *be the corresponding eigenfunctions. Suppose that conditions* B_1), B_2), A_1) – A_4), *and* C) *are satisfied. Then* $\lim_{s \to \infty} \lambda_s = \lambda$ *and the sequence* $\{u_s(x)\}_{s=1}^{\infty}$ *converges as* $s \to \infty$ *to* $\tilde{u}(x)$ *strongly in* $W_r^1(\Omega)$ *for any* $r < m$ *and weakly in* $W_m^1(\Omega)$.

2. Estimates for Solutions of the Model Problem

In [4, 6], the following lemmas were proved for the function $v_i^{(s)}(x, k)$:

Lemma 1 (Theorem 2.2 [4]). *Let conditions* A_1) – A_3) *be satisfied and let* $|k| \leq N$. *Then there exists a constant* C_5 *that depends only on* n, m, *and* N *and is such that, for* $1 \leq i \leq I(s)$, $s = 1, 2, \ldots$, *the following inequalities are true:*

$$\left\| v_i^{(s)} \right\|_{W_2^1\left(B_i^{(s)}\right)}^2 + \left\| v_i^{(s)} \right\|_{W_m^1\left(B_i^{(s)}\right)}^m \leq C_5 \left\{ |k|^m C_m\left(F_i^{(s)}\right) \right\}^{2/m} \left\{ |k|^m C_m\left(F_i^{(s)}\right) + \left(d_i^{(s)}\right)^n \right\}^{(m-2)/m},$$

$$\left\| \nabla \bar{v}_i^{(s)} - \nabla \tilde{v}_i^{(s)} \right\|_{L_2\left(B_i^{(s)}\right)}^2 + \left\| \nabla \bar{v}_i^{(s)} - \nabla \tilde{v}_i^{(s)} \right\|_{L_m\left(B_i^{(s)}\right)}^2$$

$$\leq C_5 \left| \bar{k} - \tilde{k} \right|^2 \left\{ C_m\left(F_i^{(s)}\right) \right\}^{2/m} \left\{ C_m\left(F_i^{(s)}\right) + \left(d_i^{(s)}\right)^n \right\}^{(m-2)/m},$$

where

$$\bar{v}_i^{(s)} = v_i^{(s)}\left(x, \bar{k}\right), \ \ \tilde{v}_i^{(s)} = v_i^{(s)}\left(x, \tilde{k}\right), \ \ \left|\bar{k}\right| \le N, \ \ \left|\tilde{k}\right| \le N,$$

$$\left|v_i^{(s)}(x,k)\right| \le C_5 |k| \left(\frac{C_m\left(F_i^{(s)}\right)}{\left|x - x_i^{(s)}\right|^{n-m}} \right)^{1/(m-1)}, \qquad x \in B_i^{(s)} \setminus B\left(x_i^{(s)}, d_i^{(s)}\right).$$

Lemma 2. *Let conditions* $A_1) - A_3)$ *be satisfied. Then there exists a constant* C_6 *that depends only on* n *and* m *and is such that*

$$\left|v_i^{(s)}(x,k)\right| \ge C_6 |k| \left(\frac{C_m\left(F_i^{(s)}\right)}{\left|x - x_i^{(s)}\right|^{n-m}} \right)^{1/(m-1)}, \qquad x \in B_i^{(s)} \setminus B\left(x_i^{(s)}, d_i^{(s)}\right).$$

Proof. Denote $d_i^{(s)} = d$, $F_i^{(s)} = F$, $v_i^{(s)}(x,k) = v(x,k)$, and $x_i^{(s)} = x_0$. To solve the model problem, we introduce the test function $\psi(x) = [k - v(x,k)]\eta^m(x)$ in the integral identity (3). Here,

$$\eta(x) = \begin{cases} 1, & x \in B(x_0, \rho), \\ 0, & x \notin B(x_0, 2\rho), \end{cases} \qquad \text{for} \quad 2d \le \rho < 1.$$

We get

$$\sum_{j=1}^{n} \int_{B(x_0,1)\setminus F} f_j(x, 0, \nabla v(x,k)) \frac{\partial v(x,k)}{\partial x_j} \eta^m(x)\, dx$$

$$\le m \sum_{j=1}^{n} \int_{B(x_0,1)\setminus F} [k - v(x,k)] f_j(x, 0, \nabla v(x,k)) \frac{\partial \eta(x)}{\partial x_j} \eta^{m-1}(x)\, dx.$$

We now use conditions $A_2)$ and $A_3)$ and the Hölder inequality. Assuming that $\sigma < (m-1)/m$, we obtain

$$\int_{B(x_0,1)\setminus F} (1 + |\nabla v(x,k)|)^{m-2} |\nabla v(x,k)|^2 \eta^m(x)\, dx$$

$$\le \frac{C_7 k}{\rho} \sum_{j=1}^{n} \int_{B(x_0,2\rho)\setminus B(x_0,\rho)} \left| f_j(x, 0, \nabla v(x,k)) \right| dx$$

$$\le C_7 C_2 \frac{kn}{\rho} \int_{B(x_0,2\rho)\setminus B(x_0,\rho)} \left(1 + |\nabla v(x,k)|^m\right)^{(m-2)/m} |\nabla v(x,k)|\, dx$$

$$\le \frac{C_8 k}{\rho}\left(\int\limits_{B(x_0,2\rho)\setminus B(x_0,\rho)} (1+|\nabla v(x,k)|)^m |v(x,k)|^{-\sigma m/(m-1)}\,dx\right)^{(m-1)/m}$$

$$\times\left(\int\limits_{B(x_0,2\rho)\setminus B(x_0,\rho)} |v(x,k)|^{\sigma m}\,dx\right)^{1/m}.$$

Denote $M_1(\rho) = \max_{x\in\partial B(x_0,\rho)} v(x,k)$. Then

$$\int\limits_{B(x_0,1)\setminus F} (1+|\nabla v(x,k)|)^{m-2}|\nabla v(x,k)|^2\eta^m(x)\,dx \le kC_9 M_1^{\sigma}(\rho)\rho^{n/m}\frac{1}{\rho}$$

$$\times\left(\int\limits_{B(x_0,2\rho)\setminus B(x_0,\rho)} (1+|\nabla v(x,k)|)^m |v(x,k)|^{-\sigma m/(m-1)}\,dx\right)^{(m-1)/m}.$$

We define one more cut-off function

$$\chi(x) = \begin{cases} 1, & x\in B(x_0,2\rho),\\ 0, & x\notin B(x_0,4\rho)\setminus B(x_0,\rho/2).\end{cases}$$

Then

$$\int\limits_{B(x_0,2\rho)} (1+|\nabla v(x,k)|)^{m-2}|\nabla v(x,k)|^2\eta^m(x)\,dx \le kC_9 M_1(\rho)\rho^{n/m}\frac{1}{\rho}$$

$$\times\left(\int\limits_{B(x_0,4\rho)} (1+|\nabla v(x,k)|)^m |v(x,k)|^{-\sigma m/(m-1)}\chi^m(x)\,dx\right)^{(m-1)/m}. \qquad (6)$$

For the estimation of the integral on the right-hand side of inequality (6), we introduce the test function $\psi(x) = [v(x,k)]^{1-\sigma m/(m-1)}\chi^m(x)$ in the integral identity (3). As a result, we obtain

$$\sum_{j=1}^{n}\int\limits_{B(x_0,4\rho)} f_j(x,0,\nabla v(x,k))\left(1-\sigma\frac{m}{m-1}\right)(v(x,k))^{-\sigma m/(m-1)}\frac{\partial v(x,k)}{\partial x_j}\chi^m(x)\,dx$$

$$= -m\sum_{j=1}^{n}\int\limits_{B(x_0,4\rho)} f_j(x,0,\nabla v(x,k))(v(x,k))^{1-\sigma m/(m-1)}\frac{\partial\chi(x)}{\partial x_j}\chi^{m-1}(x)\,dx.$$

Using conditions A_2) and A_3), we get

$$\left(1-\sigma\frac{m}{m-1}\right)\int\limits_{B(x_0,4\rho)}(1+|\nabla v(x,k)|)^{m-2}|\nabla v(x,k)|^2(v(x,k))^{-\sigma m/(m-1)}\chi^m(x)\,dx$$

$$\leq\frac{C_{10}}{\rho}\int\limits_{B(x_0,4\rho)\setminus B(x_0,\rho/2)}(1+|\nabla v(x,k)|)^{m-1}(v(x,k))^{1-\sigma m/(m-1)}\chi^{m-1}(x)\,dx.$$

Applying the Young inequality with exponents m and $m/(m-1)$ to the factors

$$a=(1+|\nabla v(x,k)|)^{m-1}(v(x,k))^{-\sigma}\chi^{m-1}(x)\quad\text{and}\quad b=\frac{1}{\rho}(v(x,k))^{1-\sigma/(m-1)},$$

we get

$$\int\limits_{B(x_0,4\rho)}(1+|\nabla v(x,k)|)^{m}(v(x,k))^{-\sigma m/(m-1)}\chi^m(x)\,dx$$

$$\leq C_{11}\varepsilon^{m/(m-1)}\int\limits_{B(x_0,4\rho)\setminus B(x_0,\rho/2)}(1+|\nabla v(x,k)|)^{m}(v(x,k))^{-\sigma m/(m-1)}\chi^m(x)\,dx$$

$$+\frac{C_{11}}{\rho^m\varepsilon^m}\int\limits_{B(x_0,4\rho)\setminus B(x_0,\rho/2)}(v(x,k))^{m-\sigma m/(m-1)}\,dx.$$

We now choose ε from the condition $C_{11}\varepsilon^{m/(m-1)}=1/2$. Then

$$\int\limits_{B(x_0,4\rho)}(1+|\nabla v(x,k)|)^{m}(v(x,k))^{-\sigma m/(m-1)}\chi^m(x)\,dx$$

$$\leq\frac{C_{12}}{\rho^m}\int\limits_{B(x_0,4\rho)\setminus B(x_0,\rho/2)}(v(x,k))^{m-\sigma m/(m-1)}\,dx\leq C_{13}\left(M\left(\frac{\rho}{2}\right)\right)^{m-\sigma m/(m-1)}\rho^{n-m}.$$

Returning to estimate (6) and using the inequality proved above, we obtain

$$\int\limits_{B(x_0,1)\setminus F}(1+|\nabla v(x,k)|)^{m-2}|\nabla v(x,k)|^2\eta^m(x)\,dx$$

$$\leq kC_{14}M_1^{\sigma}(\rho)\rho^{n/m}\frac{1}{\rho}\left[\left(M_1\left(\frac{\rho}{2}\right)\right)^{m-\sigma m/(m-1)}\rho^{n-m}\right]^{(m-1)/m}$$

$$=kC_{14}\rho^{n-m}M_1^{\sigma}(\rho)M_1^{m-1-\sigma}\left(\frac{\rho}{2}\right)\leq kC_{14}\rho^{n-m}M_1^{m-1}\left(\frac{\rho}{2}\right).$$

This yields

$$\int\limits_{B(x_0,\,2\rho)} |\nabla(v(x,k)\eta(x))|^m\,dx \;\le\; kC_{15}\,\rho^{n-m}M_1^{m-1}\left(\frac{\rho}{2}\right)$$

$$+\; C_{15}\int\limits_{B(x_0,\,2\rho)} |\nabla\eta(x)|^m|v(x,k)|^m\,dx \;\le\; kC_{15}\,\rho^{n-m}M_1^{m-1}\left(\frac{\rho}{2}\right)$$

$$+\; \frac{C_{16}}{\rho^m}k\int\limits_{B(x_0,\,2\rho)} |v(x,k)|^{m-1}\,dx \;\le\; kC_{17}\,\rho^{n-m}M_1^{m-1}\left(\frac{\rho}{2}\right).$$

The left-hand side of the last inequality is estimated using the definition of capacity as follows:

$$\int\limits_{B\left(x_i^{(s)},\,2\rho\right)} \left|\nabla\left(v_i^{(s)}(x,k)\eta(x)\right)\right|^m dx \;\ge\; k^m C_m\left(F_i^{(s)}\right).$$

Finally, we get

$$k^m C_m\left(F_i^{(s)}\right) \;\le\; kC_{17}\,\rho^{n-m}M_1^{m-1}\left(\frac{\rho}{2}\right).$$

It is easy to see that

$$M_1(\rho) \;\ge\; C_{18}k\left\{\frac{C_m\left(F_i^{(s)}\right)}{\rho^{n-m}}\right\}^{1/(m-1)}.$$

Using the Harnack inequality, we get

$$\min_{|x|=\rho} v_i^{(s)}(x,k) \;\ge\; C_{19}k\left\{\frac{C_m\left(F_i^{(s)}\right)}{\rho^{n-m}}\right\}^{1/(m-1)}$$

which was to be proved.

Using the scheme of the proof of an analogous inequality in [5, 6], we get the following pointwise estimate:

Lemma 3. *Let conditions* $A_1) - A_3)$ *be satisfied and let* $|k| \le N$. *Then there exists a constant* C_{20} *that depends only on* n, m, *and* N *and is such that, for* $1 \le i \le I(s)$, $s = 1, 2, \ldots$, *the following inequality is true:*

$$\left|\nabla v_i^{(s)}(x,k)\right| \;\le\; C_{18}\frac{|k|}{\left|x - x_i^{(s)}\right|}\left\{\frac{C_m\left(F_i^{(s)}\right)}{\left|x - x_i^{(s)}\right|^{n-m}}\right\}^{1/(m-1)},$$

$$x \in B\left(x_i^{(s)}, \frac{3}{4}\right)\setminus B\left(x_i^{(s)}, 2d_i^{(s)}\right).$$

Proof. We use the same notation as in the proof of Lemma 2. Assume for definiteness that $k > 0$. Since $m < n$, there exists a constant C_{21} that depends only on s and is such that

$$\inf\left\{ \int\limits_{B(x_0,\rho)} |\nabla\varphi|^m dx\colon \varphi \in C_0^\infty(B(x_0,\rho)),\ \varphi(x) = 1,\ x \in F \right\} \le C_{21} C_m(F).$$

Hence, for any $\varepsilon > 0$, there exists a function $\varphi(x) \in C_0^\infty(B(x_0,\rho))$, $\varphi(x) = 1$ for $x \in F$, such that

$$\int\limits_{B(x_0,\rho)} |\nabla\varphi|^m dx \le C_{21}(C_m(F) + \varepsilon).$$

Let $z(x) = \max\{2\varphi(x) - 1, 0\}$ and $G = \{x \in B(x_0,\rho)\colon z(x) > 0\} = \{x \in B(x_0,\rho)\colon \varphi(x) > 1/2\}$. Using the Poincaré inequalities, we get

$$\operatorname{meas} G \le 2^m \int\limits_{B(x_0,\rho)} |\varphi(x)|^m dx \le C_{22}\rho^m \int\limits_{B(x_0,\rho)} |\nabla\varphi(x)|^m dx \le C_{23}\rho^m(C_m(F) + \varepsilon). \tag{7}$$

Let M be an arbitrary number from the interval $(0, k)$. To solve the model problem for $v(x,k)$, we introduce the test function $\psi(x) = v_M(x) - Mz(x)$ in the integral identity (3). Here,

$$v_M(x) = \begin{cases} v(x,k), & x \in E_M = \{x \in B(x_0,1)\setminus F\colon 0 \le v(x,k) \le M\}, \\ M, & x \in B(x_0,1)\setminus E_M. \end{cases}$$

Using conditions A_2) and A_3), the Young inequality, and relation (7), we obtain

$$\int\limits_{E_{M'}} \left(|\nabla v(x,k)|^2 + |\nabla v(x,k)|^m\right) dx \le C_{24} M (C_m(F) + \varepsilon)$$

and then pass to the limit as $\varepsilon \to 0$. Choosing

$$M = M' = \max_{B(x_0,1)\setminus B(x_0,2\rho/3)} |v(x,k)| \le C_5 |k| \left\{ \frac{C_m(F)}{\rho^{n-m}} \right\}^{1/(m-1)},$$

we get

$$\int\limits_{E_{M'}} \left(|\nabla v(x,k)|^2 + |\nabla v(x,k)|^m\right) dx \le C_{25} |k| \left\{ \frac{C_m(F)}{\rho^{n-m}} \right\}^{1/(m-1)} C_m(F). \tag{8}$$

We choose ρ so that $2d \le \rho \le 3/4$. Proceeding by analogy with the proof of Theorem 5 in [5] and using the Moser method, we obtain

$$\operatorname{vrai\,max}\left\{|\nabla v(x,k)|^m : \frac{3}{4}\rho \le |x-x_0| \le \frac{5}{4}\rho\right\} \le C_{25}\rho^{-n}\int\limits_{E_{M'}}(1+|\nabla v(x,k)|)^{m-2}|\nabla v(x,k)|^2\psi^2(x)\,dx,$$

where $\psi(x) \in C_0^\infty(B(x_0,1))$ is the function equal to 1 for $3\rho/4 \le |x-x_0| \le 5\rho/4$ and to 0 outside the set $2\rho/3 \le |x-x_0| \le 4\rho/3$ and such that $0 \le \psi(x) \le 1$, $|\nabla\psi| \le 20/\rho$. This and (8) yield

$$|\nabla v(x,k)|^m \le C_{26}\rho^{-n}\int\limits_{E_{M'}}(1+|\nabla v(x,k)|)^{m-2}|\nabla v(x,k)|^2\,dx \le C_{27}|k|\rho^{-n}\left\{\frac{C_m(F)}{\rho^{n-m}}\right\}^{1/(m-1)}C_m(F).$$

Finally, we obtain

$$|\nabla v(x,k)|^m \le C_{28}|k|\left\{\frac{C_m(F)}{\rho^{n-m}}\right\}^{m/(m-1)}\frac{1}{\rho^m}$$

for $x : 3\rho/4 \le |x-x_0| \le 5\rho/4$, which proves the required estimate.

3. Proof of the Main Theorem

Condition A_2) implies that the functionals $\Phi(v_1, v_2)$ and $G(v)$ are differentiable, i.e., $\Phi'(v_1, v_2) = \Phi_1'(v_1, v_2) + \Phi_2'(v_1, v_2)$, where

$$(\Phi_1'(v_1,v_2),u) = \int\limits_\Omega\sum_{i=1}^n f_i(x,v_2,\nabla v_1)\frac{\partial u}{\partial x_i}\,dx,$$

$$(\Phi_2'(v_1,v_2),u) = \int\limits_\Omega f_0(x,v_2,\nabla v_1)u\,dx \quad \forall\, u \in \overset{0}{W}{}_m^1(\Omega)$$

and

$$(G'(v),u) = \int\limits_\Omega g_0(x,v)u\,dx \quad \forall\, u \in \overset{0}{W}{}_m^1(\Omega).$$

By analogy, we can define the derivatives of the functionals $\Phi_s(v_1, v_2)$ and $G_s(v)$.

Conditions A_2) and A_3) also imply that the functional $G(u)$ is weakly continuous, i.e., the weak convergence of the sequence $\{u_n(x)\}_{n=1}^\infty$ to $u_0(x)$ in $\overset{0}{W}{}_m^1(\Omega)$ yields the convergence of $G(u_n)$ to $G(u_0)$ as $n \to \infty$. In this case, the functional $\Phi(u, v)$ has the following properties:

(a) for any function $v(x) \in \overset{0}{W}{}_m^1(\Omega)$ and $C \in R^1$, the set $\Phi_{C,v} = \{u \in \overset{0}{W}{}_m^1(\Omega) : \Phi(u,v) \le C\}$ is convex;

(b) for any bounded set $D \subset \overset{0}{W}{}^1_m(\Omega)$ and any sequence $v_n(x) \in \overset{0}{W}{}^1_m(\Omega)$, the convergence $v_n(x) \rightharpoonup v_0(x)$ implies that $\Phi(u, v_n) \to \Phi(u, v_0)$ uniformly in $u \in D$; moreover, for any $v(x) \in \overset{0}{W}{}^1_m(\Omega)$, the functional $\Phi(\cdot, v)\colon \overset{0}{W}{}^1_m(\Omega) \to R^1$ is continuous. [Here, $\rightharpoonup$ denotes weak convergence in $\overset{0}{W}{}^1_m(\Omega)$].

Condition A_4) implies that the sets $M_s(G_s)$ and $M(G)$ are manifolds of the class C^1 such that $G_s'(u) \neq 0$ for $u \in M_s(G_s)$ and $G'(u) \neq 0$ for $u \in M(G)$. In [7, Theorem 5], the following statement was proved:

Theorem 2. *Let* V *be a reflexive Banach space, let* $\Phi(u, v)\colon V \times V \to R^1$ *be a real functional convex with respect to* u, *let* $G(v)$ *be a weakly continuous real functional on* V, *let* $F(v) = \Phi(v, v)$, *and let* $v \in V$. *Assume that* F *and* G *are differentiable in* V, *and, for a given* $c \in R^1$, *the set* $M = \{v \in V\colon G(v) = c\}$ *is not empty and* $G'(v) \neq 0$ *for* $v \in M$. *Let* $F(v) \to \infty$ *as* $\|v\| \to \infty$ *on* M. *Then there exist* $v_0 \in M$ *and* $\lambda \in R^1$ *such that* $F'(v_0) = \lambda G'(v_0)$ *and* $v_0 = \min_{v \in M} F(v)$.

In other words, if conditions $A_1) - A_4)$ are satisfied, then there exists the eigenvalue $\lambda_s = \min_{v_s \in M_s(G_s)} F(v_s) = F(u_s)$ of problem (1), (2), and $u_s(x)$ is the corresponding eigenfunction. Using Lemmas 1–3, we can prove that the function $c(x, u)$ is differentiable with respect to u and

$$|c(x,u)| \leq C_{29}|u|^m, \qquad |c_0(x,u)| \leq C_{29}|u|^{m-1},$$

$$c(x,u) \geq 0, \qquad c_0(x,0) = 0.$$

Then there exists the eigenvalue $\lambda = \min_{u \in M(G)} F_c(u) = F_c(\tilde{u})$ of problem (4), (5), and $\tilde{u}(x)$ is the corresponding eigenfunction. One can prove that $\operatorname{vrai\,max}_{x \in \Omega}|\tilde{u}(x)|$ is bounded. Let $\{\tilde{u}_k(x)\}_{k=1}^{\infty}$ be a uniformly bounded sequence of functions from $C^{\infty}(\Omega)$ that converges to $\tilde{u}(x)$ in $W^1_m(\Omega)$.

We construct a sequence $\left\{\tilde{\tilde{u}}_{s,k}(x)\right\}_{s=1}^{\infty}$ that weakly converges to $\tilde{u}_k(x)$ in $\overset{0}{W}{}^1_m(\Omega)$ as $s \to \infty$, $\tilde{\tilde{u}}_{s,k}(x) \in M_s(G)$. To this end, by using the function $\alpha(t)$ and condition B_2), we define a nondecreasing function $\omega\colon [0, +\infty) \to [0, +\infty)$ such that $\omega(t) \leq C_0 t/(1 + C_0)$, where the constant C_0 is taken from condition B_1), and

$$\omega(t) \leq t^{n/(n-1)}, \qquad \frac{\omega(t)}{t} \xrightarrow[t \to 0]{} 0,$$

$$\frac{\alpha^{n-1}(\omega(t))}{t^n} \xrightarrow[t \to 0]{} \infty, \qquad \frac{t^n}{\omega^{n-m}(t)} \xrightarrow[t \to \infty]{} 0.$$

We also introduce the sequence $\rho_i^{(s)} = \max\left\{d_i^{(s)}, \omega\left(r_i^{(s)}\right)\right\}$ and the subsets of indices $I_s' = \{i\colon i = 1, \ldots, I(s),\ d_i^{(s)} \geq \omega\left(r_i^{(s)}\right)\}$, and $I_s'' = \{i\colon i = 1, \ldots, I(s),\ d_i^{(s)} < \omega\left(r_i^{(s)}\right)\}$. In [4], the following statement was proved:

Lemma 4 (Lemma 3.1 in [4])**.** *If conditions* B_1) *and* B_2) *are satisfied, then*

$$\lim_{s\to\infty}\sum_{i\in I_s'} C_m\left(F_i^{(s)}\right) = 0, \qquad \lim_{s\to\infty}\sum_{i\in I_s''}\left(\rho_i^{(s)}\right)^n = 0, \qquad \sum_{i=1}^{I(s)} C_m\left(F_i^{(s)}\right) \le C_{30}.$$

We define cut-off functions:

(i)
$$\psi_i^{(s)}(x) \in \overset{0}{W}{}_m^1\left(B\left(x_i^{(s)}, \left(1+\frac{1}{2C_0}\right)d_i^{(s)}\right)\right), \qquad \psi_i^{(s)}(x) = 1, \quad x\in F_i^{(s)},$$

$$0 \le \psi_i^{(s)}(x) \le 1, \qquad \int_\Omega \left|\nabla\psi_i^{(s)}(x)\right|^m dx \le C_{31}\left\{C_m\left(F_i^{(s)}\right) + 2^{-i-s}\right\},$$

(ii) we choose numbers τ_1 and τ_2 and functions $\varphi_i^{(s)}(x) \in C_0^\infty(\Omega)$ so that $1 < \tau_1 < \tau_2 < 1 + (1/2C_0)$ and

$$0 \le \varphi_i^{(s)}(x) \le 1, \qquad \left|\nabla\varphi_i^{(s)}\right| \le \frac{C_{32}}{\rho_i^{(s)}},$$

$$\varphi_i^{(s)}(x) = \begin{cases} 1, & x\in B\left(x_i^{(s)}, \tau_1\rho_i^{(s)}\right), \\ 0, & x\notin B\left(x_i^{(s)}, \tau_2\rho_i^{(s)}\right). \end{cases}$$

The supports of the functions $\varphi_i^{(s)}(x)$ $(\psi_i^{(s)}(x))$ do not intersect for a given s and different i. By $\tilde{u}_{k,i}^s$, we denote the average of the function $\tilde{u}_k(x)$ over the ball:

$$\tilde{u}_{k,i}^s = \frac{1}{\operatorname{meas} D_i^{(s)}}\int_{D_i^{(s)}} \tilde{u}_k(x)\,dx.$$

For any fixed $\mu\in(0,1)$, we define a function $\tilde{u}_{s,k}(x) \in \overset{0}{W}{}_m^1(\Omega_s)$:

$$\tilde{u}_{s,k}(x) = \tilde{u}_k(x) + \frac{1}{\mu}\sum_{i=1}^{3}\tilde{q}_{s,k}^{(l)}(x), \tag{9}$$

$$\tilde{q}_{s,k}^{(1)}(x) = \mu\sum_{i\in I_s'}\left(\tilde{u}_{k,i}^{(s)} - \tilde{u}_k(x)\right)\psi_i^{(s)}(x) + \mu\sum_{i\in I_s''}\left(\tilde{u}_{k,i}^{(s)} - \tilde{u}_k(x)\right)\varphi_i^{(s)}(x),$$

$$\tilde{q}_{s,k}^{(2)}(x) = \sum_{i\in I_s'} v_i^{(s)}\left(x, -\mu\tilde{u}_{k,i}^{(s)}\right)\varphi_i^{(s)}(x), \tag{10}$$

$$\tilde{q}_{s,k}^{(3)}(x) = \sum_{i\in I_s''} v_i^{(s)}\left(x, -\mu\tilde{u}_{k,i}^{(s)}\right)\varphi_i^{(s)}(x).$$

By using the properties of a solution of the model problem and Lemma 4, we prove the following statements:

Lemma 5. *If conditions* $A_1) - A_3), B_1)$, *and* $B_2)$ *are satisfied, then the sequences* $\tilde{q}^{(1)}_{s,k}(x)$ *and* $\tilde{q}^{(2)}_{s,k}(x)$ *strongly converge to zero in* $W^1_m(\Omega)$ *as* $s \to \infty$.

Lemma 6. *If conditions* $A_1) - A_3), B_1)$, *and* $B_2)$ *are satisfied, then the sequence* $\tilde{q}^{(3)}_{s,k}(x)$ *strongly converges to zero as* $s \to \infty$ *in* $W^1_r(\Omega)$ *for any* $r < m$ *and weakly converges in* $W^1_m(\Omega)$.

Conditions $A_2)$ and $A_4)$ imply that, for any fixed s, there is a constant $l(s)$ such that $G_s\big(l(s)\tilde{u}_{s,k}(x)\big) = 1$. One can prove that

$$\lim_{s\to\infty} l(s) = 1. \tag{11}$$

Indeed, by construction, $\tilde{u}_{s,k}(x) \rightharpoonup \tilde{u}_k(x)$ in $W^1_m(\Omega)$ as $s \to \infty$. Let $\lim_{s\to\infty} l(s) = l \neq 1$. The weak convergence of the functional G yields

$$1 = \lim_{k\to\infty}\lim_{s\to\infty} G_s\big(l(s)\tilde{u}_{s,k}(x)\big) = \lim_{k\to\infty}\lim_{s\to\infty} G\big(l(s)\tilde{u}_{s,k}(x)\big) = G(l\tilde{u}(x)),$$

i.e., $G(l\tilde{u}(x)) = G(\tilde{u}(x)) = 1$. It is obvious that this yields $l = 1$, and (11) is proved.

We choose $\tilde{\tilde{u}}_{s,k}(x) = l(s)\tilde{u}_{s,k}(x)$. For this choice of the function $\tilde{\tilde{u}}_{s,k}(x)$, we obtain the following statement by using Lemmas 5 and 6 and condition C).

Lemma 7. *Let conditions* $A_1) - A_3), B_1), B_2)$, *and* $C)$ *be satisfied. Then*

$$\lim_{k\to\infty}\lim_{s\to\infty}\int_0^1\int_\Omega f_0\Big(x, \tilde{u}_k(x) + \theta\big(\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x)\big), \nabla\tilde{\tilde{u}}_{s,k}(x)\Big)\Big[\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x)\Big]dx\,d\theta = 0, \tag{12}$$

$$\lim_{k\to\infty}\lim_{s\to\infty}\int_0^1\int_\Omega\sum_{j=1}^n f_j\Big(x, \tilde{u}_k(x), \nabla\tilde{u}_k(x) + \mu\big(\nabla\tilde{\tilde{u}}_{s,k}(x) - \nabla\tilde{u}_k(x)\big)\Big)$$

$$\times\frac{\partial\Big[\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x)\Big]}{\partial x_j}dx\,d\mu = \int_\Omega c(x, -\tilde{u}(x))\,dx. \tag{13}$$

Proof. Indeed, using condition A_2), we get

$$\left|\int_0^1\int_\Omega f_0\Big(x, \tilde{u}_k(x) + \theta\big(\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x)\big), \nabla\tilde{\tilde{u}}_{s,k}(x)\Big)\Big[\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x)\Big]dx\,d\theta\right|$$

$$\leq C_{32}\int_0^1\int_\Omega\left[\left(1 + \left|\tilde{u}_k(x) + \theta\big(\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x)\big)\right|^{m_1} + \left|\nabla\tilde{\tilde{u}}_{s,k}(x)\right|^m\right)^{(m_1-1)/m_1} + \varphi(x)\right]\left|\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x)\right|dx\,d\theta.$$

The integral on the right-hand side of the last inequality tends to zero as $s \to \infty$ because the sequence $\Big\{\big(\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x)\big)\Big\}_{s=1}^{\infty}$ converges weakly to zero in $\overset{0}{W}{}_m^1(\Omega)$, i.e., (12) is proved. Then

$$\int_0^1 \int_\Omega \sum_{j=1}^n f_j\Big(x, \tilde{u}_k(x), \nabla\tilde{u}_k(x) + \mu\big(\nabla\tilde{\tilde{u}}_{s,k}(x) - \nabla\tilde{u}_k(x)\big)\Big)\frac{\partial\big[\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x)\big]}{\partial x_j}\,dx\,d\mu$$

$$= I_1^{(s,k)} + I_2^{(s,k)} + I_3^{(s,k)}, \tag{14}$$

where

$$I_1^{(s,k)} = (l(s)-1)\int_0^1 \int_\Omega \sum_{j=1}^n f_j\Big(x, \tilde{u}_k(x), \nabla\tilde{u}_k(x) + \mu\big(\nabla\tilde{\tilde{u}}_{s,k}(x) - \nabla\tilde{u}_k(x)\big)\Big)\frac{\partial\tilde{u}_k(x)}{\partial x_j}\,dx\,d\mu,$$

$$I_2^{(s,k)} = l(s)\int_0^1 \frac{1}{\mu}\int_\Omega \sum_{j=1}^n f_j\Big(x, \tilde{u}_k(x), \nabla\tilde{u}_k(x) + \mu\big(\nabla\tilde{\tilde{u}}_{s,k}(x) - \nabla\tilde{u}_k(x)\big)\Big)\,\frac{\partial\big(\tilde{q}_{s,k}^{(1)}(x) + \tilde{q}_{s,k}^{(2)}(x)\big)}{\partial x_j}\,dx\,d\mu,$$

$$I_3^{(s,k)} = l(s)\int_0^1 \frac{1}{\mu}\int_\Omega \sum_{j=1}^n f_j\Big(x, \tilde{u}_k(x), \nabla\tilde{u}_k(x) + \mu\big(\nabla\tilde{\tilde{u}}_{s,k}(x) - \nabla\tilde{u}_k(x)\big)\Big)\frac{\partial\tilde{q}_{s,k}^{(3)}(x)}{\partial x_j}\,dx\,d\mu.$$

Since $\lim_{s\to\infty}(l(s)-1) = 0$, we have $\lim_{s\to\infty} I_1^{(s,k)} = 0$. Lemma 5 yields $\lim_{s\to\infty} I_2^{(s,k)} = 0$. We separately consider

$$I_3^{(s,k)} = \int_0^1 \frac{1}{\mu}\int_\Omega \sum_{j=1}^n f_j\big(x, 0, \nabla\tilde{q}_{s,k}^{(3)}(x)\big)\frac{\partial\tilde{q}_{s,k}^{(3)}}{\partial x_j}\,dx\,d\mu + I_{3,1}^{(s,k)} + I_{3,2}^{(s,k)} + I_{3,3}^{(s,k)} + I_{3,4}^{(s,k)}, \tag{15}$$

where

$$I_{3,1}^{(s,k)} = l(s)\int_0^1 \int_\Omega \sum_{j=1}^n \Big[f_j\Big(x, \tilde{u}_k(x), \nabla\tilde{u}_k(x) + \mu\big(\nabla\tilde{\tilde{u}}_{s,k}(x) - \nabla\tilde{u}_k(x)\big)\Big)$$

$$- f_j\big(x, \tilde{u}_k(x), l(s)\nabla\tilde{q}_{s,k}^{(3)}(x)\big)\Big]\frac{\partial\tilde{q}_{s,k}^{(3)}}{\partial x_j}\,dx\,d\mu,$$

$$I_{3,2}^{(s,k)} = l(s)\int_0^1 \frac{1}{\mu}\int_\Omega \sum_{j=1}^n \Big[f_j\big(x, \tilde{u}_k(x), l(s)\nabla\tilde{q}_{s,k}^{(3)}(x)\big) - f_j\big(x, \tilde{u}_k(x), \nabla\tilde{q}_{s,k}^{(3)}(x)\big)\Big]\frac{\partial\tilde{q}_{s,k}^{(3)}}{\partial x_j}\,dx\,d\mu,$$

$$I_{3,3}^{(s,k)} = l(s)\int_0^1 \frac{1}{\mu}\int_\Omega \sum_{j=1}^n \Big[f_j\big(x, \tilde{u}_k(x), \nabla\tilde{q}_{s,k}^{(3)}(x)\big) - f_j\big(x, 0, \nabla\tilde{q}_{s,k}^{(3)}(x)\big)\Big]\frac{\partial\tilde{q}_{s,k}^{(3)}}{\partial x_j}\,dx\,d\mu,$$

$$I_{3,4}^{(s,k)} = [l(s)-1]\int_0^1 \frac{1}{\mu}\int_\Omega \sum_{j=1}^n f_j\big(x, 0, \nabla\tilde{q}_{s,k}^{(3)}(x)\big)\frac{\partial \tilde{q}_{s,k}^{(3)}}{\partial x_j}\,dx\,d\mu.$$

For the estimation of $I_{3,1}^{(s,k)}$, we use condition A_2) and the Hölder inequality:

$$\big|I_{3,1}^{(s,k)}\big| \le C_{33}\int_0^1 \frac{1}{\mu}\left(\int_\Omega \Big(1+|\tilde{u}_k|^{m_1}+|\nabla\tilde{u}_k|^m+\big|\nabla\tilde{q}_{s,k}^{(1)}\big|^m+\big|\nabla\tilde{q}_{s,k}^{(2)}\big|^m+\big|\nabla\tilde{q}_{s,k}^{(3)}\big|^m\Big)dx\right)$$

$$\times\left\{\left[\int_\Omega \Big(\big|\nabla\tilde{q}_{s,k}^{(1)}\big|+\big|\nabla\tilde{q}_{s,k}^{(2)}\big|\Big)^{m/2}\big|\nabla\tilde{q}_{s,k}^{(3)}\big|^{m/2}dx\right]^{2/m}+\left[\int_\Omega \Big(|\nabla\tilde{u}_k|\big|\nabla\tilde{q}_{s,k}^{(3)}\big|\Big)^{m/2}dx\right]^{2/m}\right\}d\mu.$$

The first term in braces tends to zero as $s\to\infty$ by virtue of Lemmas 5 and 6. Consider the second term:

$$\int_\Omega \Big(|\nabla\tilde{u}_k|\big|\nabla\tilde{q}_{s,k}^{(3)}\big|\Big)^{m/2}dx \le C_{34}\sum_{i\in I_s''}\left(\int_{D_i^{(s)}}|\nabla\tilde{u}_k|^m\,dx\right)^{1/2}\left(\int_{B_i^{(s)}}\Big|\nabla\Big(v_i^{(s)}\big(x,-\mu\tilde{u}_{k,i}^{(s)}\big)\varphi_i^{(s)}(x)\Big)\Big|^m\,dx\right)^{1/2}.$$

Its convergence to zero follows from the absolute continuity of the integral because

$$\lim_{s\to\infty}\int_{\bigcup_{i\in I_s''}D_i^{(s)}}|\nabla\tilde{u}_k|^m\,dx = 0.$$

Moreover,

$$\int_\Omega \Big(1+|\tilde{u}_k|^{m_1}+|\nabla\tilde{u}_k|^m+\big|\nabla\tilde{q}_{s,k}^{(1)}\big|^m+\big|\nabla\tilde{q}_{s,k}^{(2)}\big|^m+\big|\nabla\tilde{q}_{s,k}^{(3)}\big|^m\Big)dx \le C_{35},$$

where the constant C_{35} is independent of s. This yields $\lim_{s\to\infty} I_{3,1}^{(s,k)} = 0$, $\lim_{s\to\infty} I_{3,2}^{(s,k)} = 0$, and $\lim_{s\to\infty} I_{3,4}^{(s,k)} = 0$ by virtue of (10), and $\lim_{s\to\infty} I_{3,3}^{(s,k)} = 0$ by virtue of Lemmas 5 and 6. We represent the remaining integral on the right-hand side of (15) as follows:

$$\int_0^1 \frac{1}{\mu}\int_\Omega \sum_{j=1}^n f_j\big(x, 0, \nabla\tilde{q}_{s,k}^{(3)}(x)\big)\frac{\partial\tilde{q}_{s,k}^{(3)}}{\partial x_j}\,dx\,d\mu = I_4^{(s,k)} + I_5^{(s,k)}, \tag{16}$$

where

$$I_4^{(s,k)} = \int_0^1 \frac{1}{\mu}\int_\Omega \sum_{i\in I_s''}\sum_{j=1}^n f_j\Big(x, 0, \nabla\tilde{v}_i^{(s)}\big(x, -\mu\,\tilde{u}_{k,i}^{(s)}\big)\Big)\frac{\partial\tilde{v}_i^{(s)}\big(x,-\mu\,\tilde{u}_{k,i}^{(s)}\big)}{\partial x_j}\,dx\,d\mu,$$

$$I_5^{(s,k)} = \int_0^1 \frac{1}{\mu}\int_\Omega \sum_{i\in I_s''}\sum_{j=1}^{n}\bigg[f_j\Big(x, 0, \nabla\Big(\tilde v_i^{(s)}\big(x, -\mu\,\tilde u_{k,i}^{(s)}\big)\varphi_i^{(s)}(x)\Big)\Big)\frac{\partial\Big(\tilde v_i^{(s)}\big(x, -\mu\,\tilde u_{k,i}^{(s)}\big)\varphi_i^{(s)}(x)\Big)}{\partial x_j}$$

$$- f_j\Big(x, 0, \nabla\tilde v_i^{(s)}\big(x, -\mu\,\tilde u_{k,i}^{(s)}\big)\Big)\frac{\partial\tilde v_i^{(s)}\big(x, -\mu\,\tilde u_{k,i}^{(s)}\big)}{\partial x_j}\bigg]dx\,d\mu.$$

Using the methods presented in Chap. 9 in [4], we prove that $\lim_{s\to\infty} I_5^{(s,k)} = 0$.

Given the number k, we determine $d = d(k) > 0$ so that the oscillation of the function $\tilde u_k(x)$ does not exceed $(1/k)^{m/2}$ on any set $E \subset \Omega$ whose diameter is less than $2d$. The possibility of such a choice for $d(k)$ follows from the continuity of the indicated functions in $\overline{\Omega}$. We represent the set $\overline{\Omega}$ as the union of disjoint sets $\overline{\Omega}_l$, $l = 1, \dots, L(k)$, with piecewise smooth boundaries so that the diameter of every set Ω_l is less than d.

We choose a number $s_1 = s_1(k)$ such that, for $s \ge s_1$, the inequality $r_i^{(s)} + d_i^{(s)} < d$ is satisfied for $i = \overline{1, I(s)}$. By $I_s(\Omega_l)$, we denote the set of indices $i \in I_s''$ such that $x_i^{(s)} \in \Omega_l$. We define the following average:

$$\tilde u_l^{(k)} = \frac{1}{\operatorname{meas}\Omega_l}\int_{\Omega_l}\tilde u_k(x)\,dx.$$

Then

$$I_4^{(s,k)} = \int_0^1 \frac{1}{\mu}\int_\Omega \sum_{l=1}^{L}\sum_{i\in I_s(\Omega_l)}\sum_{j=1}^{n} f_j\Big(x, 0, \nabla\tilde v_i^{(s)}\big(x, -\mu\,\tilde u_l^{(k)}\big)\Big)\frac{\partial\tilde v_i^{(s)}\big(x, -\mu\,\tilde u_l^{(k)}\big)}{\partial x_j}dx\,d\mu$$

$$+ \int_0^1 \frac{1}{\mu}\int_\Omega \sum_{l=1}^{L}\sum_{i\in I_s(\Omega_l)}\sum_{j=1}^{n}\bigg[f_j\Big(x, 0, \nabla\tilde v_i^{(s)}\big(x, -\mu\,\tilde u_{k,i}^{(s)}\big)\Big)\frac{\partial\tilde v_i^{(s)}\big(x, -\mu\,\tilde u_{k,i}^{(s)}\big)}{\partial x_j}$$

$$- f_j\Big(x, 0, \nabla\tilde v_i^{(s)}\big(x, -\mu\,\tilde u_l^{(k)}\big)\Big)\frac{\partial\tilde v_i^{(s)}\big(x, -\mu\,\tilde u_l^{(k)}\big)}{\partial x_j}\bigg]dx\,d\mu = I_6^{(s,k)} + I_7^{(s,k)}.$$

The choice of the partition of the domain Ω yields $\lim_{s\to\infty} I_7^{(s,k)} = 0$. In view of the given partition $\{\Omega_l\}_{l=1}^{L}$ of the domain Ω, we choose a number $s_2 = s_2(k)$ so that, for $|t| \le M$, $s \ge s_2$, $l = 1, \dots, L$, we have

$$\left|\int_0^1 \frac{1}{\mu}\int_\Omega \sum_{i\in I_s''}\sum_{j=1}^{n} f_j\big(x, 0, \nabla\tilde v_i^{(s)}(x, -\mu)\big)\frac{\partial\tilde v_i^{(s)}(x, -\mu)}{\partial x_j}dx\,d\mu - \int_\Omega c(x, t)\,dx\right| \le \frac{1}{Lk}. \tag{17}$$

We represent $I_6^{(s,k)}$ as follows:

$$I_6^{(s,k)} = \sum_{l=1}^{L}\sum_{i\in I_s(\Omega_l)}\sum_{j=1}^{n}\int_0^{-\tilde u_l^{(k)}} \frac{1}{\mu}\int_\Omega f_j\big(x, 0, \nabla\tilde v_i^{(s)}(x, \mu)\big)\frac{\partial\tilde v_i^{(s)}(x, \mu)}{\partial x_j}dx\,d\mu$$

$$= \int_\Omega c(x, -\tilde u(x))\,dx + I_{6,1}^{(s,k)} + I_{6,2}^{(k)} + I_{6,3}^{(k)}, \tag{18}$$

where

$$I_{6,1}^{(s,k)} = \sum_{l=1}^{L} \left[\sum_{i \in I_s(\Omega_l)} \sum_{j=1}^{n} \int_{0}^{-\tilde{u}_l^{(k)}} \frac{1}{\mu} \int_{\Omega} f_j\left(x, 0, \nabla \tilde{v}_i^{(s)}(x,\mu)\right) \frac{\partial \tilde{v}_i^{(s)}(x,\mu)}{\partial x_j} dx\, d\mu - \int_{\Omega_l} c\left(x, -\tilde{u}_l^{(k)}\right) dx \right],$$

$$I_{6,2}^{(k)} = \sum_{l=1}^{L} \int_{\Omega_l} c\left(x, -\tilde{u}_l^{(k)}\right) dx - \int_{\Omega} c(x, -\tilde{u}_k(x))\, dx,$$

$$I_{6,3}^{(k)} = \int_{\Omega} \left[c(x, -\tilde{u}_k(x)) - c(x, -\tilde{u}(x)) \right] dx.$$

Inequality (17) yields $\lim_{k\to\infty} I_{6,1}^{(s,k)} = 0$, and the continuity of the function $c(x, t)$ leads to $\lim_{k\to\infty} I_{6,m}^{(k)} = 0$, $m = 2, 3$. Relations (14)–(16) and (18) yield (13), and the proof of Lemma 7 is completed.

Then, using the definition of an eigenvalue of problem (1), (2) and the Lagrange theorem, we get

$$\begin{aligned}
\lambda_s \le F\left(\tilde{\tilde{u}}_{s,k}\right) &= \int_{\Omega} f(x, \tilde{u}_k(x), \nabla \tilde{u}_k(x))\, dx \\
&+ \int_{\Omega} \left[f\left(x, \tilde{\tilde{u}}_{s,k}(x), \nabla \tilde{\tilde{u}}_{s,k}(x)\right) - f\left(x, \tilde{u}_k(x), \nabla \tilde{\tilde{u}}_{s,k}(x)\right) \right] dx \\
&+ \int_{\Omega} \left[f\left(x, \tilde{u}_k(x), \nabla \tilde{\tilde{u}}_{s,k}(x)\right) - f(x, \tilde{u}_k(x), \nabla \tilde{u}_k(x)) \right] dx = \int_{\Omega} f(x, \tilde{u}_k(x), \nabla \tilde{u}_k(x))\, dx \\
&+ \int_{0}^{1} \int_{\Omega} f_0\left(x, \tilde{u}_k(x) + \theta\left(\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x)\right), \nabla \tilde{\tilde{u}}_{s,k}(x)\right) \left[\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x) \right] dx\, d\theta \\
&+ \int_{0}^{1} \int_{\Omega} \sum_{j=1}^{n} f_j\left(x, \tilde{u}_k(x), \nabla \tilde{u}_k(x) + \mu\left(\nabla \tilde{\tilde{u}}_{s,k}(x) - \nabla \tilde{u}_k(x)\right)\right) \frac{\partial\left[\tilde{\tilde{u}}_{s,k}(x) - \tilde{u}_k(x) \right]}{\partial x_j} dx\, d\mu.
\end{aligned}$$

Using relations (12) and (13) and the strong convergence of $\tilde{u}_k(x)$ to $\tilde{u}(x)$, we obtain

$$\limsup_{s\to\infty} \lambda_s \le \int_{\Omega} f(x, \tilde{u}(x), \nabla \tilde{u}(x))\, dx + \int_{\Omega} c(x, -\tilde{u}(x))\, dx = \lambda.$$

Thus, we get

$$\limsup_{s\to\infty} \lambda_s \le \lambda. \tag{19}$$

Below, we prove the inequality

$$\liminf_{s\to\infty} \lambda_s \ \geq\ \lambda. \tag{20}$$

Relation (19) yields the estimate $F(u_s) = \lambda_s \leq C_{36}$ with some constant C_{36}. By using the conditions imposed on the function $f(x, u, p)$, we get

$$C_{37}\int\limits_{\Omega} |\nabla u_s|^m\, dx \ \leq\ \int\limits_{\Omega} f(x, u_s(x), \nabla u_s(x))\,dx \ \leq\ C_{36}.$$

This yields $\|u_s\|_{\overset{0}{W}{}^1_m(\Omega)} \leq C_{38}$. Hence, there exists a subsequence of the sequence $u_s(x)$ that converges weakly in $\overset{0}{W}{}^1_m(\Omega)$ as $s \to \infty$. Denote its limit by $\bar{u}(x) \in \overset{0}{W}{}^1_m(\Omega)$.

Fixing a constant $\mu \in (0, 1)$, we represent the asymptotic expansion of the function $u_s(x)$ as follows:

$$u_s(x) \ =\ \bar{u}_k(x) \ +\ \frac{1}{\mu}\sum_{l=1}^{3} \bar{q}_{s,k}^{(l)}(x) \ +\ \omega_{s,k}(x),$$

where $\bar{q}_{s,k}^{(l)}(x)$ have the same form as in (9) and are constructed by analogy using solutions of auxiliary model problems, cut-off functions, and the function $\bar{u}(x)$, and $\omega_{s,k}(x) \in \overset{0}{W}{}^1_m(\Omega_s)$ is the remainder of the expansion. By analogy with the proof of Theorem 3.1 in [4], we conclude that $\omega_{s,k}(x)$ converges to zero strongly in $\overset{0}{W}{}^1_m(\Omega)$ as $s \to \infty$, and $G_s(u_s) = 1$ by the definition of eigenfunction. Since G is a weakly continuous functional, we have $G(\bar{u}) = 1$. Hence,

$$\lambda_s \ =\ F(u_s) \ =\ \int\limits_{\Omega} f(x, u_s(x), \nabla u_s(x))\,dx \ =\ \int\limits_{\Omega} f(x, \bar{u}_k(x), \nabla \bar{u}_k(x))\,dx$$

$$+\ \int\limits_0^1 \int\limits_{\Omega} f_0(x, \bar{u}_k(x) + \theta(u_s(x) - \bar{u}_k(x)), \nabla u_s(x))[\,u_s(x) - \bar{u}_k(x)\,]\,dx\,d\theta$$

$$+\ \int\limits_0^1 \int\limits_{\Omega} \sum_{j=1}^{n} f_j(x, \bar{u}_k(x), \nabla \bar{u}_k(x) + \mu(\nabla u_s(x) - \nabla \bar{u}_k(x)))\frac{\partial[\,u_s(x) - \bar{u}_k(x)\,]}{\partial x_j}\,dx\,d\mu.$$

As in the proof of Lemma 7, we prove that

$$\liminf_{s\to\infty} \lambda_s \ =\ \liminf_{s\to\infty} F(u_s) \ =\ F_c(\bar{u}) \ \geq\ \inf_{u\in M(G)} F_c(u) \ =\ \lambda.$$

Thus, we have proved inequality (20). Relations (19) and (20) prove the statement of Theorem 1.

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REFERENCES

1. V. A. Marchenko and E. Ya. Khruslov, *Boundary-Value Problems in Domains with Fine-Grain Boundary* [in Russian], Naukova Dumka, Kiev (1974).
2. G. A. Iosif'yan, O. A. Oleinik, and A. S. Shamaev, "Averaging of eigenvalues and eigenfunctions of a boundary-value problem in elasticity theory for a perforated domain," *Vestn. Mosk. Univ., Ser. 1. Mat. Mekh.,* No. 4, 53–63 (1983).
3. O. A. Oleinik and T. A. Shaposhnikova, "On averaging of solutions of Dirichlet problems for a partially perforated domain of the general form with aperiodic structure," *Vestn. Mosk. Univ, Ser. 1. Mat. Mekh.,* No. 2, 49–55 (1995).
4. I. V. Skrypnik, *Methods for Investigation of Nonlinear Elliptic Boundary-Value Problems* [in Russian], Fizmatlit, Moscow (1990).
5. I. V. Skrypnik, "Asymptotics of solutions of nonlinear elliptic problems for perforated domains," *Mat. Sb.,* **184**, No. 10, 67–90 (1993).
6. I. V. Skrypnik, "New conditions for averaging of nonlinear Dirichlet problems in perforated domains," *Ukr. Mat. Zh.,* **48**, No. 5, 675–694 (1996).
7. F. Browder, "Variation methods for nonlinear elliptic eigenvalue problems," *Bull. Amer. Math. Soc.,* **71**, No. 1, 176–183 (1965).
8. F. Browder, "Infinite-dimensional manifolds and nonlinear elliptic eigenvalue problems," *Ann. Math.,* **82**, No. 3, 459–477 (1965).
9. F. Fucik, J. Necas, and V. Soucek, *Spectral Analysis of Nonlinear Operators,* Springer, Berlin (1973).
10. S. I. Pokhozhaev, "On eigenfunctions of quasilinear elliptic problems," *Mat. Sb.,* **82**, No. 2, 192–212 (1970).