Convergence of variational eigenvalues and eigenfunctions to the Dirichlet problem for the p-Laplacian in domains with fine-grained boundary

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Abstract

We study the problem of homogenization to the Dirichlet eigenvalue problems for the p-Laplace operator in sequence of perforated domains with fine-grained boundary. Using asymptotic expansion method, we derive the homogenized problem for the new equation with additional term of capacity type. Moreover, we show that a sequence of eigenvalues to the problem in perforated domains converges to the corresponding critical levels of the homogenized problem.

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1. Introduction and the problem statement

This paper is concerned with convergence of solutions to the Dirichlet eigenvalue problem for the p-Laplacian in a sequence of domains with a complex geometry. Neither numerical nor analytical methods are useful in solving problems in perforated domains. In the same time the complex microstructure of perforated domains brings about no additional complications into solvability of such problems. Under special conditions on perforations it can be shown that solutions of problems in perforated domains are closed in some sense to solution of homogenized problem for certain new differential equation in a simple domain.

Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be a bounded domain. For every fixed $s \in \mathbb{N}$ we consider a finite number $I(s)$ of disjoint closed domains $\mathcal{F}_i^{(s)} \subset \Omega,\ i = 1, \dots, I(s)$, with

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nonempty interior (i.e. none of $\mathcal{F}_i^{(s)}$ degenerate to a point). While the number $I(s)$ of small sets tends to infinity as $s \to \infty$, diameters of these small sets $\mathcal{F}_i^{(s)}$, $i = 1, \dots, I(s)$, tend to zero. Let us introduce the following notations for perforations

$$\mathcal{F}_s := \bigcup_{i=1}^{I(s)} \mathcal{F}_i^{(s)}, \quad \Omega_s := \Omega \setminus \mathcal{F}_s.$$

A sequence of perforated domains Ω_s in the case when the perforations are small disjoint components, are called domains with fine-grained boundary ([9]) or strongly perforated domains ([11]).

V.A. Marchenko and E.Ya. Khruslov were the first who investigated the Dirichlet boundary value problems and eigenvalue problems for linear operators in strongly perforated domains (e.g. [11] and references therein). Subsequently, homogenization of the eigenvalue problems for linear operators with rapidly oscillating coefficients and in perforated domains was studied by many authors; we only mention here [13, 14, 23, 24, 16, 17].

To the best to our knowledge there are only few results about the homogenization of nonlinear eigenvalue problems. For example, asymptotic behavior of the eigenvalues of a class of family of nonlinear monotone elliptic operators in a periodic heterogeneous medium was studied in [2], [4]. In comparison with these works, we consider a non–periodical structure of perforation. We also use a different approach which gives us a possibility to obtain more detailed information about asymptotic behavior of solutions to nonlinear eigenvalue problems in perforated domains.

We study asymptotic behavior of solutions to the following Dirichlet eigenvalue problems:

$$-\Delta_p u_s = \lambda_s |u_s|^{p-2} u_s, \quad x \in \Omega_s, \tag{1.1}$$

$$u_s = 0, \; x \in \partial\Omega_s, \tag{1.2}$$

where $\Delta_p u := \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_j}\right)$ is the standard p-Laplacian, $2 \le p < n$.

Definition 1.1. *We say that a number λ_s is an eigenvalue of problem (1.1), (1.2) if there exists a function $u_s \in W_0^{1,p}(\Omega_s)$, $u_s \not\equiv 0$, called an eigenfunction of (1.1), (1.2), such that*

$$\int_{\Omega_s} \sum_{j=1}^n |\nabla u_s|^{p-2} \frac{\partial u_s}{\partial x_j} \frac{\partial \varphi_s}{\partial x_j} dx = \lambda_s \int_{\Omega_s} |u_s|^{p-2} u_s \varphi_s dx,$$

for every $\varphi_s \in W_0^{1,p}(\Omega_s)$.

Extending an eigenfunction $u_s \in W_0^{1,p}(\Omega_s)$ into $\mathcal{F}_s$ by zero and keeping the same notation, we obtain a function $u_s \in W_0^{1,p}(\Omega)$ (see [11], [19]).

There are several possibilities how to characterize sequences of variational eigenvalues and eigenfunctions for the p-Laplacian as critical levels and critical points of the corresponding energy functional on a symmetric manifold. Some of them can be found in [1, 3, 7, 15, 22, 25] and it is not possible to discuss them in detail here. In our article we apply a definition which was introduced by Drábek and Robinson (see [5], [6]). We use this definition since it perfectly fits with the technique of the proof of our main result. On the other hand, one could use the other definitions as well. For example, the definitions based on the notions of the Krasnoselskij genus or Ljusternik-Schnirelmann category (see e.g. Struwe [22] and Zeidler [25]) are certainly possible to be employed. However, the proofs would be much more technical.

The main result of this paper is the homogenization of problem (1.1), (1.2). We show that eigenvalues to problem (1.1), (1.2) in perforated domains Ω_s converge to the corresponding critical levels of homogenized problem in the simple domain Ω. The differential equation for the constructed homogenized problem has an additional term of capacity character. The behavior of the first eigenvalue to the Dirichlet problem for nonlinear second order elliptic equation in domains with fine-grained boundary was studied in [21]. Our aim is to investigate higher eigenvalues of problem (1.1), (1.2) using the approach introduced in [21]. The proof is based on the method of asymptotic expansion developed by I.V. Skrypnik [19] for quasilinear elliptic equations. In the framework of this method, we construct the asymptotic expansion of the solutions to problems (1.1), (1.2). The asymptotic expansion is built in the terms of auxiliary functions which are solutions of the appropriate model problems. Knowing the behavior of solutions of the model problems, we construct the corresponding homogenized problem.

The paper is organized as follows. In Section 2 we formulate the assumptions on the sequence of perforated domains Ω_s, $s = 1, 2, \ldots$, and the main result of the paper. Section 3 contains the preliminary information about behaviour of the solutions of the model problems and the construction of asymptotic expansion. The Main Theorem is proved in Section 4. Proofs of the auxiliary Lemmas are in Appendix (Section 5).

2. Formulation of the conditions and the main result

To formulate conditions on Ω_s we introduce a qualitative characterization of massiveness of the set $\mathcal{F}_i^{(s)}$, $\forall\, s = 1, 2, \ldots, \forall\, i = 1, \ldots, I(s)$.

Definition 2.1. *By $\mathcal{C}_p(E)$ we denote the p-capacity of a set $E \subset B(x_0, \frac{1}{2})$, i.e.*

$$\mathcal{C}_p(E) = \inf_{\varphi \in V(E)} \int\limits_{B(x_0,1)} |\nabla \varphi|^p \, dx,$$

where $V(E) = \{\varphi \in C_0^\infty(B(x_0, 1)) : \varphi(x) = 1,\ x \in E\}$, (see [12]).

Note that in the linear case $p = 2$ and in three dimensions, $n = 3$, the value $\mathcal{C}_p(E)$ is the Newton capacity.

It the sequel, by $C_j, \forall j = 0, 1, 2, \ldots$, we denote positive constants not depending on s but depending on n and p only.

Let $B(x_i^{(s)}, r_i^{(s)})$ be a minimal ball with radius $r_i^{(s)}$ and center $x_i^{(s)}$ such that $\mathcal{F}_i^{(s)} \subset \overline{B(x_i^{(s)}, r_i^{(s)})}$. By $d_i^{(s)}$ we denote the distance between $B(x_i^{(s)}, r_i^{(s)})$ and $\bigcup_{j \neq i} B(x_j^{(s)}, r_j^{(s)}) \cup \partial\Omega$. We assume that $d_i^{(s)} > 0$ for any choice of s and i (see Figure 1).

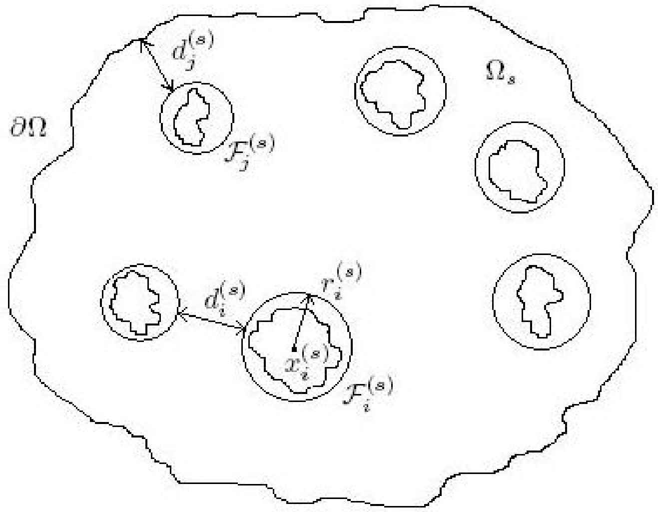

Figure 1. Domain with fine-grained boundary.

Assume that the following conditions are satisfied for all $s = 1, 2, \dots,\ i = 1, \dots, I(s)$:

$(\mathbf{B_1})$ $r_i^{(s)} \leq C_0 d_i^{(s)}$, *where* C_0 *is a constant not depending on* i *and* s, *and*

$$\lim_{s\to\infty} \max_{1\leq i\leq I(s)} d_i^{(s)} = 0;$$

$(\mathbf{B_2})$ *there exist a positive constant* C_1 *and a continuous nondecreasing function* $\alpha : [0, +\infty) \to [0, +\infty)$ *satisfying* $\alpha(0) = 0$ *and* $\frac{\alpha(t)}{t} \underset{t\to\infty}{\to} \infty$, *such that*

$$\sum_{i=1}^{I(s)} \mathcal{C}_p(\mathcal{F}_i^{(s)}) \Big\{ \frac{\mathcal{C}_p(\mathcal{F}_i^{(s)}) + \alpha^{n-1}(r_i^{(s)})}{(d_i^{(s)})^n} \Big\}^{\frac{1}{p-1}} \leq C_1$$

for every s, i.

Example 2.1. *Let* $p = 2, n = 3$ *and the sets* $\mathcal{F}_i^{(s)}$ *are such that the diameters of* $\mathcal{F}_i^{(s)}$ *are congruous with its inradiuses. In this case capacities* $\mathcal{C}_p(\mathcal{F}_i^{(s)})$ *of the sets* $\mathcal{F}_i^{(s)}$ *are equal to its diameters and condition* $(\mathbf{B_2})$ *has the following form*

$$\sum_{i=1}^{I(s)} \frac{(r_i^{(s)})^2}{(d_i^{(s)})^3} \leq C_1.$$

Then conditions $(\mathbf{B_1}), (\mathbf{B_2})$ *mean that diameters of sets* $\mathcal{F}_i^{(s)}$ *tend to zero faster than distances between these sets as* $s \to \infty$.

To formulate an additional condition for the sets $\mathcal{F}_i^{(s)}$ that guarantees the possibility of construction of an averaged problem, we define auxiliary functions $v_i^{(s)}(x,t)$ which are solutions of the appropriate model problems near small sets $\mathcal{F}_i^{(s)}$.

Let $x_i^{(s)}$ be as above. We use the following notations

$$B_i^{(s)} := B(x_i^{(s)}, 1), \ \Omega_i^{(s)} := B_i^{(s)} \setminus \mathcal{F}_i^{(s)}.$$

For every fixed $t \in \mathbb{R}$ we consider the following model problem

$$\Delta_p v_i^{(s)} = 0, \ x \in \Omega_i^{(s)}, \tag{2.1}$$

$$v_i^{(s)} = 0, \ x \in \partial B_i^{(s)}, \ \ v_i^{(s)} = t, \ x \in \partial \mathcal{F}_i^{(s)}. \tag{2.2}$$

Definition 2.2. *A function $v_i^{(s)} = v_i^{(s)}(x,t)$ is said to be a weak solution of problem (2.1), (2.2) if*

(i) the function $v_i^{(s)} - t\psi_0(x - x_i^{(s)})$ belongs to the space $W_0^{1,p}(\Omega_i^{(s)})$, where $\psi_0 \in C_0^\infty(B(0,1))$ and $\psi_0(x) = 1$, $x \in B(0, \frac{1}{2})$;

(ii) for every function $\psi \in W_0^{1,p}(\Omega_i^{(s)})$ the following integral identity holds:

$$\sum_{j=1}^{n} \int\limits_{\Omega_i^{(s)}} |\nabla v_i^{(s)}|^{p-2} \frac{\partial v_i^{(s)}}{\partial x_j} \frac{\partial \psi}{\partial x_j} dx = 0. \tag{2.3}$$

Outside of $\Omega_i^{(s)}$ we set $v_i^{(s)}(x,t) = t\psi_0(x - x_i^{(s)})$ and keep for the redefined function the same notation.

Remark 2.1. *The existence and uniqueness of a solution $v_i^{(s)}$ to problem (2.1), (2.2) were proved in [19] for every fixed $s \in \mathbb{N}$ and $i = 1, \ldots, I(s)$.*

We assume that the following condition is satisfied:

(**C**) *There exists a function $c(x,t)$ which is continuous in x for any $t \in \mathbb{R}$ and of the class C^1 in t for almost all $x \in \Omega$, such that for every ball $B \subset \Omega$ we have*

$$\lim_{s\to\infty} \sum_{i \in I_s(B)} \int\limits_{\Omega} |\nabla v_i^{(s)}(x,t)|^p dx = \int\limits_{B} c(x,t) dx$$

and convergence to the limit is uniform in t on any bounded interval for t. Here $I_s(B)$ is the set of indices $i, 1 \le i \le I(s)$ such that $x_i^{(s)} \in B$.

Some examples of the sets $\mathcal{F}_s$ for which the condition (**C**) is satisfied can be found in [9] and [10] for the case $p = 2$.

Example 2.2. *Let the set $\mathcal{F}_s$ consists from a finite number of congruous domains such that some of their points constitute a space cube grating in a bounded domain $\Omega \subset \mathbb{R}^3$. The period of this grating equals to $l = l^{(s)}$. Let us use the following notations*

$$\varrho := \varrho_i^{(s)} = diam\, \mathcal{F}_i^{(s)}, \ \ \mathcal{C} := \mathcal{C}_i^{(s)} = \mathcal{C}_2(\mathcal{F}_i^{(s)}).$$

Let a sequence of problems be such that $l \to 0$, $\varrho \le \frac{l}{3}$ as $s \to \infty$, and the following limit exists

$$\lim_{l\to 0} \frac{\mathcal{C}}{l^3} = q.$$

*Then it was shown in [10] that condition (**C**) is satisfied with a function $c(x) \equiv q$.*

We consider the following homogenized problem in Ω:

$$-\Delta_p u - c_0(x,u(x)) = \lambda |u|^{p-2}u, \; x \in \Omega, \tag{2.4}$$
$$u = 0, \; x \in \partial\Omega, \tag{2.5}$$

where $c_0(x,t) = \frac{\partial c(x,t)}{\partial t}$, see (**C**) for $c(x,t)$.

Definition 2.3. *We say that a number λ is an eigenvalue of problem (2.4), (2.5), if there exists a function $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, called an eigenfunction of (2.4), (2.5), such that*

$$\int_\Omega \sum_{j=1}^n |\nabla u|^{p-2} \frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_j}\, dx - \int_\Omega c_0(x,u(x))\varphi\, dx = \lambda \int_\Omega |u|^{p-2}u\,\varphi\, dx,$$

for every $\varphi \in W_0^{1,p}(\Omega)$.

Let us define sequences of critical levels and critical points following the approach introduced in [5],[6].

Definition 2.4. *Define the symmetric manifolds:*

$$\mathcal{L} := \Big\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^p\, dx = 1 \Big\}, \quad \mathcal{L}_s := \Big\{ u_s \in W_0^{1,p}(\Omega_s) : \int_{\Omega_s} |u_s|^p\, dx = 1 \Big\}.$$

For every $k, s \in \mathbb{N}$ by $\Theta_k, \Theta_{k,s}$ we denote the following sets

$\Theta_k := \{\mathcal{A} \subset \mathcal{L} : \mathcal{A}$ *is the image of a continuous odd function* $h : \mathcal{L}^{k-1} \to \mathcal{L}\}$,
$\Theta_{k,s} := \{\mathcal{A}_s \subset \mathcal{L}_s : \mathcal{A}_s$ *is the image of a continuous odd function* $h_s : \mathcal{L}^{k-1} \to \mathcal{L}_s\}$,

where $\mathcal{L}^{k-1}$ represents the unit sphere in $\mathbb{R}^k$.

We define now critical levels:

$$\lambda_k := \inf_{\mathcal{A}\in\Theta_k} \sup_{u\in\mathcal{A}} \frac{\int_\Omega |\nabla u|^p\, dx - \int_\Omega c(x,u(x))\, dx}{\int_\Omega |u|^p\, dx}, \quad k \in \mathbb{N}, \tag{2.6}$$

$$\lambda_k(\Omega_s) := \inf_{\mathcal{A}_s\in\Theta_{k,s}} \sup_{u_s\in\mathcal{A}_s} \frac{\int_\Omega |\nabla u_s|^p\, dx}{\int_\Omega |u_s|^p\, dx}, \quad k, s \in \mathbb{N}. \tag{2.7}$$

Corresponding critical points will be denoted by $u_k, u_{k,s}$ respectively. On these functions the minimaxes in (2.6), (2.7) are achieved. Notice that both u_k and $u_{k,s}$ are not determined uniquely.

It was shown in [5],[6] that $\{\lambda_k(\Omega_s)\}_{k=1}^\infty$ is a sequence of eigenvalues for (1.1), (1.2) and $u_{k,s}$ are the corresponding eigenfunctions. However, it is an open problem for the p-Laplacian in higher dimensions, whether $\{\lambda_k(\Omega_s)\}_{k=1}^\infty$ and $\{u_{k,s}\}_{k=1}^\infty$ exhaust the sequence of all eigenvalues and eigenfunctions, respectively. Similarly as in [5], compactness argument yields that every λ_k, $k = 1, 2, \dots$, is a critical level for

$$u \longmapsto \int\limits_\Omega |\nabla u|^p \, dx - \int\limits_\Omega c(x, u(x)) \, dx$$

subject to

$$\int\limits_\Omega |u|^p \, dx = 1.$$

The existence of critical levels λ_k, $k \in \mathbb{N}$, follows from properties of solutions to model problem (2.1), (2.2) which will be mentioned later in Section 3.1 (cf. Remark 3.1).

Let $u_k(x)$ be a critical point associated with λ_k. The Lagrange multiplier method implies the existence of a multiplier μ_k such that

$$p \int\limits_\Omega \sum_{j=1}^n |\nabla u_k|^{p-1} \frac{\partial u_k}{\partial x_j} \frac{\partial \varphi}{\partial x_j} \, dx - \int\limits_\Omega c_0(x, u_k(x)) \varphi \, dx = \mu_k p \int\limits_\Omega |u_k|^{p-2} u_k \varphi \, dx$$

holds for every $\varphi \in W_0^{1,p}(\Omega)$. One can see immediately that $\mu_k = \lambda_k$ if and only if

$$\frac{1}{p} \, c_0(x, u_k(x)) u_k(x) = c(x, u_k(x)).$$

Notice that this is the case when c is homogeneous with respect to the second variable. On the other hand, if c is not homogeneous, λ_k is not an eigenvalue of (2.4), (2.5), in general. To show homogeneity of function c is the subject of further investigations. However, the critical point u_k is an eigenfunction of (2.4), (2.5) associated with the eigenvalue μ_k.

Finally, we also note that methods from [19] imply that there exists a positive constant $\tilde{C}$, depending on n, k only, such that the following inequality is valid

$$ess \sup_{x \in \Omega} |u_k(x)| \le \tilde{C}, \; k \in \mathbb{N}.$$

Now we are in a position to formulate the main result of the paper.

Main Theorem. *Let conditions* $(\mathbf{B_1}), (\mathbf{B_2}), (\mathbf{C})$ *be satisfied. Then for every* $k \in \mathbb{N}$ *and for every sequence of critical points* $\{u_{k,s_m}\}_{m=1}^\infty$ *(here* $\{s_m\}_{m=1}^\infty \subset \mathbb{N}$*) there exist a subsequence* $\{u_{k,s_{m_l}}\}_{l=1}^\infty$ *and an eigenfunction* u_k *of (2.4), (2.5) such that* $\{u_{k,s_{m_l}}\}_{l=1}^\infty$ *converges strongly to* u_k *in* $W_0^{1,p'}(\Omega)$ *for every* $p' < p$ *and weakly in* $W_0^{1,p}(\Omega)$ *as* $l \to \infty$*, and the following statement holds:*

$$\lim_{l \to \infty} \lambda_k(\Omega_{s_{m_l}}) = \lambda_k.$$

3. Preliminaries

3.1. Estimates of solutions of the model problems (2.1), (2.2)

The study of asymptotic behavior of solutions to problem (1.1), (1.2) is based on integral and pointwise estimates of the auxiliary functions $v_i^{(s)}$ which are solutions of problems of the type (2.1), (2.2). We recall the properties of these solutions which were proved in [19],[21] applying the technique from [18],[20].

Lemma 3.1. *(Theorem 2.2, [19]). Let N be a fixed positive number and $|t| \le N$. Then there exists a constant C_2 depending only on n, p, N such that for solution $v_i^{(s)}$ of problem (2.1), (2.2) for every $s = 1, 2, \dots,\ 1 \le i \le I(s)$, the following inequalities hold*

$$(i) \quad |v_i^{(s)}(x,t)| \le C_2 |t| \Big(\frac{\mathcal{C}_p(\mathcal{F}_i^{(s)})}{|x - x_i^{(s)}|^{n-p}} \Big)^{\frac{1}{p-1}}, \quad x \in B_i^{(s)} \setminus B(x_i^{(s)}, r_i^{(s)}), \tag{3.1}$$

$$(ii) \quad \|v_i^{(s)}\|^2_{W^{1,2}(B_i^{(s)})} + \|v_i^{(s)}\|^p_{W^{1,p}(B_i^{(s)})}$$
$$\le C_2 \{|t|^p \mathcal{C}_p(\mathcal{F}_i^{(s)})\}^{\frac{2}{p}} \{|t|^p \mathcal{C}_p(\mathcal{F}_i^{(s)}) + (r_i^{(s)})^n\}^{\frac{p-2}{p}}, \tag{3.2}$$

$$(iii) \quad \|\nabla \bar{v}_i^{(s)} - \nabla \hat{v}_i^{(s)}\|^2_{L_2(B_i^{(s)})} + \|\nabla \bar{v}_i^{(s)} - \nabla \hat{v}_i^{(s)}\|^p_{L_p(B_i^{(s)})}$$
$$\le C_2 |\bar{t} - \hat{t}|^2 \{\mathcal{C}_p(\mathcal{F}_i^{(s)})\}^{\frac{2}{p}} \{\mathcal{C}_p(\mathcal{F}_i^{(s)}) + (r_i^{(s)})^n\}^{\frac{p-2}{p}}, \tag{3.3}$$

where $\bar{v}_i^{(s)} := v_i^{(s)}(x, \bar{t}),\ \hat{v}_i^{(s)} := v_i^{(s)}(x, \hat{t}), |\bar{t}| \le N,\ |\hat{t}| \le N,$

$$(iv) \quad \|\nabla v_i^{(s)}\|^p_{L_p(E_\theta)} \le C_2 \frac{\theta}{t} \{|t|^p \mathcal{C}_p(\mathcal{F}_i^{(s)})\}^{\frac{2}{p}} \{|t|^p \mathcal{C}_p(\mathcal{F}_i^{(s)}) + (r_i^{(s)})^n\}^{\frac{p-2}{p}}, \tag{3.4}$$

for every $\theta : \quad 0 < \theta < |t|,$ *here* $E_\theta := \{x \in B_i^{(s)} : 0 \le v_i^{(s)}(x,t) \le \theta\}.$

Lemma 3.2. *(Lemma 2, [21]). There exists a positive constant C_3, depending on n and p only, such that the following inequality is valid*

$$|v_i^{(s)}(x,t)| \ge C_3 |t| \Big(\frac{\mathcal{C}_p(\mathcal{F}_i^{(s)})}{|x - x_i^{(s)}|^{n-p}} \Big)^{\frac{1}{p-1}}, \quad x \in B_i^{(s)} \setminus B(x_i^{(s)}, r_i^{(s)}),$$

for $s = 1, 2, \dots,\ 1 \le i \le I(s)$.

Lemma 3.3. *(Lemma 3, [21]). Let N be a fixed positive number and $|t| \le N$. Then there exists a positive constant C_4, depending on n, p, N only, such that the following inequality is valid*

$$|\nabla v_i^{(s)}(x,t)| \le C_4 \frac{|t|}{|x - x_i^{(s)}|} \Big\{ \frac{\mathcal{C}_p(\mathcal{F}_i^{(s)})}{|x - x_i^{(s)}|^{n-p}} \Big\}^{\frac{1}{p-1}},\ x \in B\Big(x_i^{(s)}, \frac{3}{4}\Big) \setminus B(x_i^{(s)}, 2d_i^{(s)}),$$

for $s = 1, 2, \dots,\ 1 \le i \le I(s)$.

Remark 3.1. *It follows from Lemma 3.3 that the function $c(x,u)$ from condition* (**C**) *satisfies*

$$c(x,u) \le C_5 |u|^p,\ |c_0(x,u)| \le C_6 |u|^{p-1},\ c_0(x,0) = 0 \tag{3.5}$$

for all $u \in \mathbb{R}$ and almost all $x \in \Omega$. Inequalities (3.5) guarantee that for every $k \in \mathbb{N}$ a critical level λ_k and a corresponding critical point $u_k(x)$ associated with (2.4), (2.5) exist and are finite.

3.2. Cut-off functions and construction of asymptotic expansion

In the framework of approach developed by I.V. Skrypnik in [19], we construct the asymptotic expansion of solutions to the non–linear eigenvalue problem in perforated domains in terms of solutions of the model problems (2.1), (2.2). In this Section we build for a function from $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ the special functions which are near the critical points of the corresponding functionals from (2.7).

We define a nondecreasing function $\omega : [0, +\infty) \to [0, +\infty)$ such that $\omega(t) \le \frac{C_0 t}{1+C_0}$ and ω satisfies the following properties

$$\omega(t) \le t^{\frac{n}{n-1}}, \ \frac{\omega(t)}{t} \underset{t\to 0}{\to} 0, \ \frac{\alpha^{n-1}(\omega(t))}{t^n} \underset{t\to\infty}{\to} \infty, \ \frac{t^n}{\omega^{n-m}(t)} \underset{t\to\infty}{\to} 0.$$

(We used the constant C_0 from ($\mathbf{B_1}$) and the function $\alpha(t)$ from condition ($\mathbf{B_2}$).) The construction of this function can be found in [19]. We also introduce a sequence $\rho_i^{(s)} = \max\{r_i^{(s)}, \omega(d_i^{(s)})\}$ and subsets of indices

$$I'_s = \{i : i = 1, \dots, I(s), \ r_i^{(s)} \ge \omega(d_i^{(s)})\},$$
$$I''_s = \{i : i = 1, \dots, I(s), r_i^{(s)} < \omega(d_i^{(s)})\}.$$

In [19] the following statement was proved.

Lemma 3.4. *(Lemma 3.1, [19]). Let conditions* ($\mathbf{B_1}$), ($\mathbf{B_2}$) *be satisfied. Then*

$$\lim_{s\to\infty} \sum_{i\in I'_s} \mathcal{C}_p(\mathcal{F}_i^{(s)}) = 0, \quad \lim_{s\to\infty} \sum_{i\in I''_s} (\rho_i^{(s)})^n = 0, \quad \sum_{i=1}^{I(s)} \mathcal{C}_p(\mathcal{F}_i^{(s)}) \le C_7. \tag{3.6}$$

We define cut-off functions $\psi_i^{(s)}$, $\varphi_i^{(s)}$ with the following properties:

$$(i) \quad \psi_i^{(s)} \in W_0^{1,p}\Big(B\Big(x_i^{(s)}, \Big(1 + \frac{1}{2C_0}\Big) r_i^{(s)}\Big)\Big), \quad \psi_i^{(s)} = 1, \ x \in \mathcal{F}_i^{(s)},$$
$$0 \le \psi_i^{(s)} \le 1, \quad \int_\Omega |\nabla \psi_i^{(s)}|^p dx \le C_8\{\mathcal{C}_p(\mathcal{F}_i^{(s)}) + 2^{-i-s}\}; \tag{3.7}$$

(ii) we choose numbers $\tau_1, \tau_2: \quad 1 < \tau_1 < \tau_2 < 1 + \dfrac{1}{2C_0}$,

and functions $\varphi_i^{(s)} \in C_0^\infty(\Omega)$ such that

$$0 \ \le \varphi_i^{(s)}(x) \le 1, \ |\nabla \varphi_i^{(s)}| \le \frac{C_9}{\rho_i^{(s)}}, \ \varphi_i^{(s)} = \begin{cases} 1, \ x \in B(x_i^{(s)}, \tau_1 \rho_i^{(s)}), \\ 0, \ x \notin B(x_i^{(s)}, \tau_2 \rho_i^{(s)}). \end{cases} \tag{3.8}$$

Let us note that the supports of functions $\varphi_i^{(s)}$ ($\psi_i^{(s)}$) do not mutually intersect.

Let $\vartheta \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ be an arbitrary function. We denote by $\vartheta_{s,i}$ the average of the function ϑ over the ball $\mathcal{D}_i^{(s)} = B(x_i^{(s)}, \tau_2\rho_i^{(s)})$, namely

$$\vartheta_{s,i} = \frac{1}{\operatorname{meas}\mathcal{D}_i^{(s)}} \int\limits_{\mathcal{D}_i^{(s)}} \vartheta\, dx. \tag{3.9}$$

We use the following "ansatz" for construction of asymptotic expansion of solution to nonlinear eigenvalue problems in perforated domains. We define a function $\vartheta_s \in W_0^{1,p}(\Omega_s)$ such that

$$\vartheta_s(x) = \vartheta(x) + \sum_{m=1}^{3} q_s^{(m)}(x), \tag{3.10}$$

where

$$\begin{aligned} q_s^{(1)}(x) &= \sum_{i\in I_s'}(\vartheta_{s,i} - \vartheta(x))\psi_i^{(s)}(x) + \sum_{i\in I_s''}(\vartheta_{s,i} - \vartheta(x))\varphi_i^{(s)}(x), \\ q_s^{(2)}(x) &= -\sum_{i\in I_s'} v_i^{(s)}(x, \vartheta_{s,i})\varphi_i^{(s)}(x), \\ q_s^{(3)}(x) &= -\sum_{i\in I_s''} v_i^{(s)}(x, \vartheta_{s,i})\varphi_i^{(s)}(x). \end{aligned} \tag{3.11}$$

This construction will be used in proof of the Main Theorem. The aim of this ansatz is to represent functions which converge weakly as a sum of the weak limit and functions constructed with the help of solutions to the appropriate model problems. The asymptotic behaviour of these special functions is possible to study. Namely, using the properties of the auxiliary functions $v_i^{(s)}(x,t)$, Lemma 3.4, and methods from [19], one can prove the following properties of $q_s^{(m)}(x),\ m = 1, 2, 3.$

Lemma 3.5. *Let conditions* $(\mathbf{B_1}), (\mathbf{B_2})$ *be satisfied, then the sequences* $\{q_s^{(1)}\}_{s=1}^\infty$ *and* $\{q_s^{(2)}\}_{s=1}^\infty$ *converge to zero strongly in* $W^{1,p}(\Omega)$ *as* $s \to \infty$.

Lemma 3.6. *Let conditions* $(\mathbf{B_1}), (\mathbf{B_2})$ *be satisfied, then the sequence* $\{q_s^{(3)}\}_{s=1}^\infty$ *converges to zero strongly in* $W^{1,p'}(\Omega)$ *for* $p' < p$ *and converges to zero weakly in* $W^{1,p}(\Omega)$ *as* $s \to \infty$.

Lemma 3.7. *Let conditions* $(\mathbf{B_1}), (\mathbf{B_2}), (\mathbf{C})$ *be satisfied, then*

$$\lim_{s\to\infty} \int\limits_\Omega |\nabla q_s^{(3)}|^p\, dx = -\int\limits_\Omega c(x, \vartheta(x))\, dx. \tag{3.12}$$

The detailed proof of Lemma 3.7 is postponed to Appendix 5.1.

4. Proof of the Main Theorem

To prove the Main Theorem, we apply the induction with respect to $k \in \mathbb{N}$:

– For $k = 1$ the Main Theorem was proved in paper [21].

– Suppose that the statement of the Main Theorem is true for the natural numbers $1, \ldots, k-1$. That is, for every $\rho = 1, \ldots, k-1$, a subsequence of sequence

$\{u_{\rho,s}\}_{s=1}^{\infty}$ converges to u_ρ strongly in $W_0^{1,p'}(\Omega)$ for every $p' < p$, and weakly in $W_0^{1,p}(\Omega)$ as $s \to \infty$, and for the corresponding subsequence of sequence $\{\lambda_\rho(\Omega_s)\}_{s=1}^{\infty}$ the following equalities are valid

$$\lim_{s\to\infty} \lambda_\rho(\Omega_s) = \lambda_\rho.$$

Hereafter we use for subsequences the same notations as for sequences.

– We show now that the same statement is true for $\rho = k$.

To prove this convergence we need to show the boundedness of the sequence $\{\lambda_k(\Omega_s)\}_{s=1}^{\infty}$ by a constant not depending on s, and the existence of a subsequence of eigenvalues satisfying the following inequalities

$$\lambda_k \le \liminf_{s\to\infty} \lambda_k(\Omega_s) \le \lim_{s\to\infty} \lambda_k(\Omega_s) \le \limsup_{s\to\infty} \lambda_k(\Omega_s) \le \lambda_k. \tag{4.1}$$

First, let us show the boundedness of the sequence $\{\lambda_k(\Omega_s)\}_{s=1}^{\infty}$. For every fixed $k \in \mathbb{N}$ and $\varepsilon > 0$ let us denote by $\mathcal{A}_k(\Omega_s)$ the following sets

$$\begin{aligned} \mathcal{A}_k(\Omega_s) := \Big\{ \tfrac{u^{(s)}}{\|u^{(s)}\|_{L_p(\Omega)}} : u^{(s)}(x) = \sum_{\rho=1}^{k} \alpha_\rho(\varepsilon) u_\rho^{(s)}(x),\ \sum_{\rho=1}^{k} |\alpha_\rho(\varepsilon)| = 1, \\ u_\rho^{(s)} = u_\rho + \sum_{m=1}^{3} q_{s,\rho}^{(m)} \in W_0^{1,p}(\Omega_s)\backslash\{0\},\ \rho = 1, ..., k \Big\}. \end{aligned} \tag{4.2}$$

Here we used the induction assumptions and the results of Section 3.2 for the construction of the functions $u_\rho^{(s)},\ \rho = 1, \dots, k$. The functions $q_{s,\rho}^{(m)}$, $m = 1, 2, 3$, are defined analogously to (3.11). The set $\mathcal{A}_k(\Omega_s)$ is the symmetric and compact subset of $\mathcal{L}$. Let us define map $f_{k,s} : \mathcal{A}_k(\Omega_s) \to \mathbb{R}^k \setminus \{0\}$ satisfying the following relation

$$f_{k,s}\left(\pm \frac{u_\rho^{(s)}}{\|u_\rho^{(s)}\|_{L_p(\Omega_s)}}\right) = \pm e_\rho^k,$$

where $e_\rho^k, \rho = 1, ..., k$, is the standard unit basis of $\mathbb{R}^k$. For the function

$$u^{(s)} = \sum_{\rho=1}^{k} \alpha_\rho(\varepsilon) u_\rho^{(s)} \frac{1}{\|\sum\limits_{\rho=1}^{k} \alpha_\rho(\varepsilon) u_\rho^{(s)}\|_{L_p(\Omega_s)}}$$

we set

$$f_{k,s}(u^{(s)}) = \sum_{\rho=1}^{k} \alpha_\rho(\varepsilon) f_{k,s}\left(\frac{u_\rho^{(s)}}{\|u_\rho^{(s)}\|_{L_p(\Omega_s)}}\right).$$

The function $f_{k,s}$ is an odd homeomorphism between $\mathcal{A}_k(\Omega_s)$ and an unit sphere in $\mathbb{R}^k$. Such maps and their properties were considered in [8]. In particular, the set $\mathcal{A}_k(\Omega_s)$ belongs to the system of sets $\Theta_{k,s}$ from Definition 2.4 of the critical levels. Then

$$\lambda_k(\Omega_s) \le \sup_{u_s \in \mathcal{A}_k(\Omega_s)} \frac{\int\limits_\Omega |\nabla u_s|^p\, dx}{\int\limits_\Omega |u_s|^p\, dx} = \frac{\int\limits_\Omega |\nabla u^{(s,\varepsilon)}|^p\, dx}{\int\limits_\Omega |u^{(s,\varepsilon)}|^p\, dx}, \tag{4.3}$$

where $u^{(s,\varepsilon)}$ is a function from the compact set $\mathcal{A}_k(\Omega_s)$ on which the supremum in (4.3) is achieved, namely,

$$u^{(s,\varepsilon)} := \sum_{\rho=1}^{k} \tilde{\alpha}_\rho(\varepsilon) u_\rho^{(s)} = \sum_{\rho=1}^{k} \tilde{\alpha}_\rho(\varepsilon) \left(u_\rho + \sum_{m=1}^{3} q_{s,\rho}^{(m)} \right).$$

We consider the right-hand side of (4.3). Without loss of generality, $\tilde{\alpha}_\rho(\varepsilon)$, $\rho = 1, \dots, k$, can be chosen such that

$$\lim_{\varepsilon \to 0} \tilde{\alpha}_k(\varepsilon) = 1, \quad \lim_{\varepsilon \to 0} \tilde{\alpha}_\rho(\varepsilon) = 0, \quad \rho = 1, \dots, k-1.$$

Then

$$\int\limits_{\Omega_s} |\nabla u^{(s,\varepsilon)}|^p dx = \int\limits_{\Omega_s} \left| \nabla \left(\sum_{\rho=1}^{k} \tilde{\alpha}_\rho(\varepsilon) \left(u_\rho + \sum_{m=1}^{3} q_{s,\rho}^{(m)} \right) \right) \right|^p dx = \int\limits_{\Omega_s} |\nabla u_k^{(s)}|^p dx + \gamma_1(\varepsilon, s), \tag{4.4}$$

$$\int\limits_{\Omega_s} |u^{(s,\varepsilon)}|^p dx = \int\limits_{\Omega_s} \left| \sum_{\rho=1}^{k} \tilde{\alpha}_\rho(\varepsilon) \left(u_\rho + \sum_{m=1}^{3} q_{s,\rho}^{(m)} \right) \right|^p dx = \int\limits_{\Omega} |u_k^{(s)}|^p dx + \gamma_2(\varepsilon, s), \tag{4.5}$$

where $\lim\limits_{\varepsilon \to 0} \gamma_1(\varepsilon, s) = 0$, $\lim\limits_{\varepsilon \to 0} \gamma_2(\varepsilon, s) = 0$ uniformly with respect to $s \in \mathbb{N}$.
We use also the following auxiliary statement.

Lemma 4.1. *Let conditions* $(\mathbf{B_1})$, $(\mathbf{B_2})$, $(\mathbf{C})$ *be satisfied. Then the following equalities are valid*

$$\int\limits_{\Omega} |\nabla u_k^{(s)}|^p dx = \int\limits_{\Omega} |\nabla u_k|^p dx - \int\limits_{\Omega} c(x, u_k(x)) dx + \gamma_3(s), \tag{4.6}$$

$$\int\limits_{\Omega_s} |u_k^{(s)}|^p dx = \int\limits_{\Omega} |u_k|^p dx + \gamma_4(s), \tag{4.7}$$

where $\lim\limits_{s \to \infty} \gamma_3(s) = 0$, $\lim\limits_{s \to \infty} \gamma_4(s) = 0$.

The proof of Lemma 4.1 is rather technical and therefore we postpone it to Appendix 5.2.

Using (4.3)-(4.7), we derive

$$\lambda_k(\Omega_s) \leq \frac{\left(\int\limits_{\Omega} |\nabla u_k|^p dx - \int\limits_{\Omega} c(x, u_k(x)) dx \right) + \gamma_1(\varepsilon, s) + \gamma_3(s)}{\int\limits_{\Omega} |u_k|^p dx + \gamma_2(\varepsilon, s) + \gamma_4(s)}. \tag{4.8}$$

The sequences $\gamma_1(\varepsilon, s)$, $\gamma_3(\varepsilon, s)$ tend to zero as $\varepsilon \to 0$ uniformly with respect to $s \in \mathbb{N}$, $\gamma_2(s)$, $\gamma_4(s)$ tend to zero as $s \to \infty$. Then we have $|\gamma_i| \leq 1/6$ for ε small enough and s big enough. This yields that there exists a constant C_{10} not depending on ε, s such that the following inequality holds

$$|\lambda_k(\Omega_s)| \leq C_{10}. \tag{4.9}$$

It follows from (4.9), that we can pass to the limit in (4.8) for $s \to \infty$. As result we obtain

$$\limsup_{s\to\infty} \lambda_k(\Omega_s) \le \limsup_{s\to\infty} \lim_{\varepsilon\to 0} \frac{\int\limits_\Omega |\nabla u_k|^p dx - \int\limits_\Omega c(x, u_k(x))dx + \gamma_1(\varepsilon, s) + \gamma_3(s)}{\int\limits_\Omega |u_k|^p dx + \gamma_2(\varepsilon, s) + \gamma_4(s)} = \lambda_k. \tag{4.10}$$

We derived the inequality on the right-hand side of (4.1). It remains to prove the second inequality from (4.1), namely:

$$\liminf_{s\to\infty} \lambda_k(\Omega_s) \ge \lambda_k. \tag{4.11}$$

We normalize a k-th eigenfunction $u_{s,k}$ to problem (1.1), (1.2) as

$$\|u_{k,s}\|_{L_p(\Omega)} = 1.$$

Then, the definition of eigenvalues and inequality (4.9) imply that

$$\int\limits_\Omega |\nabla u_{k,s}|^p dx = \lambda_k(\Omega_s) \le C_{10}.$$

Hence, there exists a subsequence of the sequence $\{u_{k,s}\}_{s=1}^\infty$ that converges weakly in $W_0^{1,p}(\Omega)$. We use for this subsequence the same notation and denote its weak limit by $\bar{u}_k \in W_0^{1,p}(\Omega)$. Applying methods from [19] and (4.9), it can be checked the existence of a constant M, not depending on s, such that the following inequality is valid:

$$ess \sup_{x\in\Omega} |u_{k,s}(x)| \le M.$$

We employ *the asymptotic expansion* of the function $u_{k,s}$, that is

$$u_{k,s}(x) = \bar{u}_k(x) + \sum_{p=1}^{3} \bar{q}_{s,k}^{(p)}(x) + \bar{\omega}_{s,k}(x), \tag{4.12}$$

where $\bar{q}_{s,k}^{(p)}$ have the same form as in (3.11) and are constructed analogously using solutions of model problems (2.1), (2.2), cut-off functions (3.7), (3.8), and the weak limit $\bar{u}_k$. Therefore, for the functions $\bar{q}_{s,k}^{(p)}$ we obtain the same statements as in Lemmas 3.5, 3.6, 3.7. The function $\bar{\omega}_{s,k} \in W_0^{1,p}(\Omega_s)$ is a remainder term of the expansion. Similarly as in the proof of Theorem 3.1 in [19], we prove that the sequence $\{\bar{\omega}_{s,k}\}_{s=1}^\infty$ converges strongly to zero in $W_0^{1,p}(\Omega)$ as $s \to \infty$. The proof is omitted here.

Then, the subsequence $\{u_{k,s}\}_{s=1}^\infty$ converges to $\bar{u}_k$ strongly in $W^{1,p'}(\Omega)$, $p' < p$, and weakly in $W^{1,p}(\Omega)$ as $s \to \infty$. Using the same argument as in the proof of equality (4.6), we obtain

$$\lambda_k(\Omega_s) = \int\limits_\Omega |\nabla u_{k,s}|^p dx = \int\limits_\Omega |\nabla \bar{u}_k|^p dx - \int\limits_\Omega c(x, \bar{u}_k(x))dx + \gamma_5(s), \tag{4.13}$$

where $\lim_{s\to\infty} \gamma_5(s) = 0$.

Assume that inequality (4.11) is not valid, that is

$$\liminf_{s\to\infty} \lambda_k(\Omega_s) < \lambda_k. \tag{4.14}$$

From the weak convergence of $u_{k,s}$ to $\bar{u}_k$ in $W_0^{1,p}(\Omega)$ it follows that $\bar{u}_k \in \mathcal{L}$. Analogously to (4.2), we define a set $\bar{\mathcal{A}}_k \in \Theta_k$ in the following way

$$\bar{\mathcal{A}}_k := \left\{ \frac{\tilde{u}}{\|\tilde{u}\|_{L_p(\Omega)}} : \tilde{u} = \sum_{\rho=1}^{k-1} \alpha_\rho(\varepsilon) u_\rho + \alpha_k(\varepsilon)\bar{u}_k,\ \sum_{\rho=1}^{k} |\alpha_\rho(\varepsilon)| = 1 \right\},$$

where u_ρ, $\rho = 1, \dots, k-1$, are eigenfunctions to problem (1.1), (1.2).

Using (4.13) on the left-hand side of inequality (4.14) and passing to the limit as $s \to \infty$, we derive

$$\frac{\int\limits_\Omega |\nabla \bar{u}_k|^p\,dx - \int\limits_\Omega c(x, \bar{u}_k(x))\,dx}{\int\limits_\Omega |\bar{u}_k|^p\,dx} = \liminf_{s\to\infty} \frac{\int\limits_\Omega |\nabla u_{k,s}|^p\,dx}{\int\limits_\Omega |u_{k,s}|^p\,dx} = \liminf_{s\to\infty} \lambda_k(\Omega_s)$$

$$< \lambda_k = \inf_{\mathcal{A}\in\Theta_k} \sup_{u\in\mathcal{A}} \frac{\int\limits_\Omega |\nabla u|^p\,dx - \int\limits_\Omega c(x, u(x))dx}{\int\limits_\Omega |u|^p\,dx}$$

$$\le \sup_{u\in\bar{\mathcal{A}}_k} \frac{\int\limits_\Omega |\nabla u|^p\,dx - \int\limits_\Omega c(x, u(x))\,dx}{\int\limits_\Omega |u|^p\,dx} = \frac{\int\limits_\Omega |\nabla \tilde{u}^{(\varepsilon)}|^p\,dx - \int\limits_\Omega c(x, \tilde{u}^{(\varepsilon)}(x))\,dx}{\int\limits_\Omega |\tilde{u}^{(\varepsilon)}|^p\,dx}, \tag{4.15}$$

where $\tilde{u}^{(\varepsilon)} := \sum\limits_{\rho=1}^{k-1} \bar{\alpha}_\rho(\varepsilon) u_\rho + \bar{\alpha}_k(\varepsilon)\bar{u}_k$ is the function belonging to the set $\bar{\mathcal{A}}_k$ on which the supremum of the functional on the right-hand side of (4.15) is achieved. Let us choose $\bar{\alpha}_\rho(\varepsilon)$, $\rho = 1, \dots, k$, such that

$$\lim_{\varepsilon\to 0} \bar{\alpha}_k(\varepsilon) = 1, \quad \lim_{\varepsilon\to 0} \bar{\alpha}_\rho(\varepsilon) = 0, \quad \rho = 1, \dots, k-1.$$

Then, using the continuity of the function $c(x, u)$, we can pass to the limit in (4.15) as $\varepsilon \to 0$ and get

$$\frac{\int\limits_\Omega |\nabla \bar{u}_k|^p\,dx - \int\limits_\Omega c(x, \bar{u}_k(x))\,dx}{\int\limits_\Omega |\bar{u}_k|^p\,dx} < \lambda_k \le \frac{\int\limits_\Omega |\nabla \bar{u}_k|^p\,dx - \int\limits_\Omega c(x, \bar{u}_k(x))\,dx}{\int\limits_\Omega |\bar{u}_k|^p\,dx}.$$

This contradiction means that (4.11) is true. Relations (4.10), (4.11) prove (4.1). That is, we proved the existence of the subsequence of eigenvalues $\{\lambda_k(\Omega_s)\}_{s=1}^\infty$ such that

$$\lim_{s\to\infty} \lambda_k(\Omega_s) = \lambda_k = \int\limits_\Omega |\nabla \bar{u}_k|^p dx - \int\limits_\Omega c(x, \bar{u}_k(x))dx. \tag{4.16}$$

The last equality follows from (4.13) after passing to the limit as $s \to \infty$. Therefore the function $\bar{u}_k$ is the k-th eigenfunction to problem (2.4), (2.5). Moreover, it was shown that the subsequence $\{u_{k,s}\}_{s=1}^{\infty}$ converges to $\bar{u}_k$ strongly in $W^{1,p'}(\Omega)$, $p' < p$, and weakly in $W^{1,p}(\Omega)$ as $s \to \infty$. Consequently, the statement of the Main Theorem follows from (4.1) and (4.16).

5. Appendix

Remark 5.1. *By $K_j, j = 1, 2, \dots,$ we denote different positive constants depending on n, p, C_i, $i = 0, 1, 2, \dots,$ only.*

5.1. Proof of Lemma 3.7

We investigate the following integral

$$\int\limits_{\Omega} \left|\nabla q_s^{(3)}\right|^p \, dx = J_1^{(s)} + J_2^{(s)}, \tag{5.1}$$

where

$$J_1^{(s)} := -\sum_{i\in I_s''} \int\limits_{\Omega} |\nabla v_i^{(s)}(x, \vartheta_{s,i})|^p \, dx,$$

$$J_2^{(s)} := \int\limits_{\Omega} \left|\nabla q_s^{(3)}\right|^p \, dx + \sum_{i\in I_s''} \int\limits_{\Omega} |\nabla v_i^{(s)}(x, \vartheta_{s,i})|^p \, dx.$$

First, let us show that

$$\lim_{s\to\infty} J_2^{(s)} = 0. \tag{5.2}$$

Using the definition of $q_s^{(3)}$ we derive

$$J_2^{(s)} = -\sum_{i\in I_s''} \int\limits_{\Omega} |\nabla(v_i^{(s)}(x, \vartheta_{s,i})\varphi_i^{(s)})|^p dx + \sum_{i\in I_s''} \int\limits_{\Omega} |\nabla v_i^{(s)}(x, \vartheta_{s,i})|^p \, dx$$

$$= \sum_{i\in I_s''} \int\limits_{\Omega} |\nabla v_i^{(s)}|^p dx - \sum_{i\in I_s''} \int\limits_{\Omega} \Big(|\nabla v_i^{(s)}|^2 (\varphi_i^{(s)})^2 + 2(\nabla\varphi_i^{(s)}, \nabla v_i^{(s)})\varphi_i^{(s)} v_i^{(s)}$$

$$+ (v_i^{(s)})^2 |\nabla\varphi_i^{(s)}|^2\Big)^{\frac{p}{2}} dx = \sum_{i\in I_s''} \int\limits_{\Omega} (|\nabla v_i^{(s)}|^2)^{\frac{p}{2}} dx - \sum_{i\in I_s''} \int\limits_{\Omega} \Big(|\nabla v_i^{(s)}|^2$$

$$+ ((\varphi_i^{(s)})^2 - 1)|\nabla v_i^{(s)}|^2 + 2(\nabla\varphi_i^{(s)}, \nabla v_i^{(s)})\varphi_i^{(s)} v_i^{(s)} + (v_i^{(s)})^2 |\nabla\varphi_i^{(s)}|^2\Big)^{\frac{p}{2}} dx$$

$$= -\sum_{i\in I_s''} \int\limits_{\Omega} \int\limits_0^1 \frac{d}{d\xi} \Big(|\nabla v_i^{(s)}|^2 + \xi\Big(((\varphi_i^{(s)})^2 - 1)|\nabla v_i^{(s)}|^2 + 2(\nabla v_i^{(s)}, \nabla\varphi_i^{(s)}) v_i^{(s)} \varphi_i^{(s)}$$

$$+ (v_i^{(s)})^2 |\nabla\varphi_i^{(s)}|^2\Big)\Big)^{\frac{p}{2}} d\xi dx = -\frac{p}{2} \sum_{i\in I_s''} \int\limits_{\Omega} \int\limits_0^1 \Big(|\nabla v_i^{(s)}|^2 + \xi\Big(((\varphi_i^{(s)})^2 - 1)|\nabla v_i^{(s)}|^2$$

$$+2(\nabla\varphi_i^{(s)},\nabla v_i^{(s)})\varphi_i^{(s)}v_i^{(s)}+\left(v_i^{(s)}\right)^2|\nabla\varphi_i^{(s)}|^2\Big)\Big)^{\frac{p}{2}-1}\Big(((\varphi_i^{(s)})^2-1)|\nabla v_i^{(s)}|^2$$

$$+2(\nabla\varphi_i^{(s)},\nabla v_i^{(s)})\varphi_i^{(s)}v_i^{(s)}+(v_i^{(s)})^2|\nabla\varphi_i^{(s)}|^2\Big)d\xi dx.$$

Since $0\le\xi\le 1$ it means that

$$J_2^{(s)}\le K_1\sum_{i\in I_s''}\int_\Omega|\nabla v_i^{(s)}|^{p-2}\Big(((\varphi_i^{(s)})^2-1)|\nabla v_i^{(s)}|^2$$

$$+2(\nabla\varphi_i^{(s)},\nabla v_i^{(s)})\varphi_i^{(s)}v_i^{(s)}+|v_i^{(s)}|^2|\nabla\varphi_i^{(s)}|^2\Big)dx$$

$$+K_1\sum_{i\in I_s''}\int_\Omega\Big(|(\varphi_i^{(s)})^2-1||\nabla v_i^{(s)}|^2+2(\nabla\varphi_i^{(s)}\nabla v_i^{(s)})\varphi_i^{(s)}v_i^{(s)}+|v_i^{(s)}|^2|\nabla\varphi_i^{(s)}|^2\Big)^{\frac{p}{2}}dx.$$

We can estimate the right-hand side of the last inequality using (3.8). Denote by $\mathcal{K}_i^{(s)}$ the following ring

$$\mathcal{K}_i^{(s)}:=\{x\in\mathbb{R}^n:\tau_1\rho_i^{(s)}\le|x-x_i^{(s)}|\le\tau_2\rho_i^{(s)}\}.$$

Then

$$J_2^{(s)}\le K_2\sum_{i\in I_s''}\int_{\mathcal{K}_i^{(s)}}|\nabla v_i^{(s)}|^p(1-(\varphi_i^{(s)})^2)dx+K_2\sum_{i\in I_s''}\int_{\mathcal{K}_i^{(s)}}|\nabla v_i^{(s)}|^{p-2}|v_i^{(s)}|^2|\nabla\varphi_i^{(s)}|^2dx$$

$$+K_2\sum_{i\in I_s''}\int_{\mathcal{K}_i^{(s)}}|\nabla v_i^{(s)}|^{p-2}(\nabla\varphi_i^{(s)},\nabla v_i^{(s)})\varphi_i^{(s)}v_i^{(s)}dx+K_2\sum_{i\in I_s''}\int_{\mathcal{K}_i^{(s)}}(1-(\varphi_i^{(s)})^2)^{\frac{p}{2}}|\nabla v_i^{(s)}|^pdx$$

$$+K_2\sum_{i\in I_s''}\int_{\mathcal{K}_i^{(s)}}(\nabla\varphi_i^{(s)},\nabla v_i^{(s)})^{\frac{p}{2}}(\varphi_i^{(s)})^{\frac{p}{2}}(v_i^{(s)})^{\frac{p}{2}}+K_2\sum_{i\in I_s''}\int_{\mathcal{K}_i^{(s)}}|v_i^{(s)}|^p|\nabla\varphi_i^{(s)}|^pdx$$

$$\le K_3\sum_{i\in I_s''}\int_{\mathcal{K}_i^{(s)}}|\nabla v_i^{(s)}|^p\,dx+K_3\sum_{i\in I_s''}\int_{\mathcal{K}_i^{(s)}}|\nabla v_i^{(s)}|^{p-2}(\nabla\varphi_i^{(s)},\nabla v_i^{(s)})v_i^{(s)}dx$$

$$+K_3\sum_{i\in I_s''}\frac{1}{(\rho_i^{(s)})^2}\int_{\mathcal{K}_i^{(s)}}|\nabla v_i^{(s)}|^{p-2}|v_i^{(s)}|^2dx$$

$$+K_3\sum_{i\in I_s''}\int_{\mathcal{K}_i^{(s)}}(\nabla\varphi_i^{(s)},\nabla v_i^{(s)})^{\frac{p}{2}}(v_i^{(s)})^{\frac{p}{2}}dx+K_3\sum_{i\in I_s''}\frac{1}{(\rho_i^{(s)})^p}\int_{\mathcal{K}_i^{(s)}}|v_i^{(s)}|^p\,dx. \tag{5.3}$$

Let us consider the integrals on the right-hand side of (5.3). Using (3.1) we obtain

$$\sum_{i\in I_s''}\frac{1}{(\rho_i^{(s)})^p}\int_{\mathcal{K}_i^{(s)}}|v_i^{(s)}(x,\vartheta_{s,i})|^p\,dx\le K_4\sum_{i\in I_s''}(\rho_i^{(s)})^{n-p}\left(\frac{\mathcal{C}_p(\mathcal{F}_i^{(s)})}{(\rho_i^{(s)})^{n-p}}\right)^{\frac{p}{p-1}}$$

$$\leq K_5 \sum_{i\in I''_s} \left(\frac{\mathcal{C}^p_p(\mathcal{F}^{(s)}_i)}{(\rho^{(s)}_i)^{n-p}} \right)^{\frac{1}{p-1}} \leq K_6 \max_{1\leq i\leq I(s)} \left(\frac{(d^{(s)}_i)^n}{\omega^{n-p}(d^{(s)}_i)} \right)^{\frac{1}{p-1}} \sum_{i=1}^{I(s)} \left(\frac{\mathcal{C}^p_p(\mathcal{F}^{(s)}_i)}{(d^{(s)}_i)^n} \right)^{\frac{1}{p-1}}.$$

The right-hand side of the last inequality tends to zero as $s \to \infty$ due to the properties of function $\omega(t)$. This means that

$$\lim_{s\to\infty} \sum_{i\in I''_s} \frac{1}{(\rho^{(s)}_i)^p} \int_{\mathcal{K}^{(s)}_i} |v^{(s)}_i(x, \vartheta_{s,i})|^p dx = 0. \tag{5.4}$$

We denote by $\theta^{(s)}_i := \max_{x\in\mathcal{K}^{(s)}_i} |v^{(s)}_i(x, \vartheta_{s,i})|$. Then from (3.1) we have

$$|v^{(s)}_i(x, \vartheta_{s,i})| \leq \theta^{(s)}_i \leq C_2 |\vartheta_{s,i}| \left(\frac{\mathcal{C}_p(\mathcal{F}^{(s)}_i)}{(\rho^{(s)}_i)^{n-p}} \right)^{\frac{1}{p-1}}, \quad x \in \mathcal{K}^{(s)}_i. \tag{5.5}$$

Using (3.4), (5.5) and the definition of I''_s we derive

$$\sum_{i\in I''_s} \int_{\mathcal{K}^{(s)}_i} |\nabla v^{(s)}_i(x, \vartheta_{s,i})|^p \, dx \leq K_7 \sum_{i\in I''_s} \theta^{(s)}_i (\mathcal{C}_p(\mathcal{F}^{(s)}_i) + (r^{(s)}_i)^n)$$

$$\leq K_8 \sum_{i\in I''_s} \left(\frac{\mathcal{C}_p(\mathcal{F}^{(s)}_i)}{(\rho^{(s)}_i)^{n-p}} \right)^{\frac{1}{p-1}} (\mathcal{C}_p(\mathcal{F}^{(s)}_i) + (r^{(s)}_i)^n)$$

$$= K_9 \sum_{i\in I''_s} \left(\frac{\mathcal{C}^p_p(\mathcal{F}^{(s)}_i)}{(\rho^{(s)}_i)^{n-p}} \right)^{\frac{1}{p-1}} + K_9 \sum_{i\in I''_s} (\rho^{(s)}_i)^n.$$

Analogously to the proof of (5.4), we deduce from the properties of the function $\omega(t)$ and (3.6) that

$$\lim_{s\to\infty} \sum_{i\in I''_s} \int_{\mathcal{K}^{(s)}_i} |\nabla v^{(s)}_i|^p \, dx = 0. \tag{5.6}$$

After applying Hölder's inequality and (5.6), (5.4) to other terms in (5.3) we derive (5.2). It remains to prove that

$$\lim_{s\to\infty} J^{(s)}_1 = -\int_\Omega c(x, \vartheta(x)) dx.$$

Let $\{\vartheta_\ell\}_{\ell=1}^\infty$ be a sequence of functions from $C^\infty_0(\Omega)$ that converges to $\vartheta(x)$ in $W^{1,p}_0(\Omega)$ as $\ell \to \infty$. The sequence $\{\vartheta_\ell\}_{\ell=1}^\infty$ has a subsequence which converges to $\vartheta(x)$ almost everywhere. Let us keep the same notation for this subsequence. We define

$$\vartheta_{\ell,\nu} = \frac{1}{meas\,\Omega_\nu} \int_{\Omega_\nu} \vartheta_\ell \, dx.$$

For a fixed number $\ell \in \mathbb{N}$ we define a small number $d = d(\ell) > 0$ such that the following inequality is valid

$$|\vartheta_{\ell,\nu} - \vartheta(x)| \leq \left(\frac{1}{\ell}\right)^{\frac{p}{2}} \tag{5.7}$$

almost everywhere on an arbitrary set $E \subset \Omega$ whose diameter is less then $2d$.

We represent the set $\overline{\Omega}$ as a union of disjoint subsets $\Omega_\nu,\ \nu = 1, \dots, \Upsilon(\ell)$, with piecewise smooth boundaries so that the diameter of every set Ω_ν is less than d. We choose a number $s_1 = s_1(\ell)$ such that for $s \geq s_1$ the inequality $r_i^{(s)} + d_i^{(s)} < d$ is satisfied for every $i = 1, \dots, I(s)$. By $I_s(\Omega_\nu)$ we denote a set of indices $i \in I_s''$ such that $x_i^{(s)} \in \Omega_\nu$. Then we have the following representation for the integral $J_1^{(s)}$:

$$J_1^{(s)} = J_3^{(s,\ell)} + J_4^{(s,\ell)}, \tag{5.8}$$

where

$$J_3^{(s,\ell)} := -\sum_{\nu=1}^{\Upsilon(\ell)} \sum_{i\in I_s(\Omega_\nu)} \int_\Omega |\nabla v_i^{(s)}(x, \vartheta_{\ell,\nu})|^p \, dx,$$

$$J_4^{(s,\ell)} \sum_{\nu=1}^{\Upsilon(\ell)} \sum_{i\in I_s(\Omega_\nu)} \int_\Omega (-|\nabla v_i^{(s)}(x, \vartheta_{s,i})|^p + |\nabla v_i^{(s)}(x, \vartheta_{\ell,\nu})|^p) \, dx.$$

We use for solutions of model problems (2.1), (2.2) with $\bar{t} \sim \vartheta_{\ell,\nu}$ and $\hat{t} \sim \vartheta_{s,i}$ the following notations

$$\bar{v}_i^{(s)} := v_i^{(s)}(x, \vartheta_{\ell,\nu}), \quad \hat{v}_i^{(s)} := v_i^{(s)}(x, \vartheta_{s,i}).$$

Then using Hölder's inequality and estimates (3.2), (3.3) from Lemma 3.1, we obtain

$$J_4^{(s,\ell)} = \sum_{\nu=1}^{\Upsilon(\ell)} \sum_{i\in I_s(\Omega_\nu)} \int_\Omega (|\nabla \bar{v}_i^{(s)}|^p - |\nabla \hat{v}_i^{(s)}|^p) \, dx \leq \sum_{i\in I_s''} \int_\Omega (|\nabla \bar{v}_i^{(s)}|^p - |\nabla \hat{v}_i^{(s)}|^p) \, dx$$

$$= \sum_{i\in I_s''} \int_\Omega \int_0^1 \frac{d}{d\zeta} (|\nabla(\hat{v}_i^{(s)} + \zeta(\bar{v}_i^{(s)} - \hat{v}_i^{(s)}))|^2)^{\frac{p}{2}} \, d\zeta dx$$

$$= \sum_{i\in I_s''} \int_\Omega \int_0^1 p|\nabla(\hat{v}_i^{(s)} + \zeta(\bar{v}_i^{(s)} - \hat{v}_i^{(s)}))|^{p-2} \big(\nabla(\hat{v}_i^{(s)} + \zeta(\bar{v}_i^{(s)} - \hat{v}_i^{(s)})), \nabla(\bar{v}_i^{(s)} - \hat{v}_i^{(s)})\big) \, d\zeta dx$$

$$\leq K_{10} \sum_{i\in I_s''} \int_0^1 \int_\Omega \left(|\nabla \hat{v}_i^{(s)}|^{p-2} + \zeta^{p-2} |\nabla(\bar{v}_i^{(s)} - \hat{v}_i^{(s)})|^{p-2}\right)$$

$$\times \left((\nabla \tilde{v}_i^{(s)}, \nabla(\bar{v}_i^{(s)} - \tilde{v}_i^{(s)})) + \zeta |\nabla(\bar{v}_i^{(s)} - \tilde{v}_i^{(s)})|^2\right) dx d\zeta$$

$$\le K_{11}\sum_{i\in I_s''}\Big(\left(\int\limits_\Omega|\nabla\hat v_i^{(s)}|^p\,dx\right)^{\frac{p-1}{p}}\left(\int\limits_\Omega|\nabla(\bar v_i^{(s)}-\hat v_i^{(s)})|^p\,dx\right)^{\frac1p}$$

$$+\left(\int\limits_\Omega|\nabla\hat v_i^{(s)}|^p\,dx\right)^{\frac1p}\left(\int\limits_\Omega|\nabla(\bar v_i^{(s)}-\hat v_i^{(s)})|^p\,dx\right)^{\frac{p-1}{p}}$$

$$+\left(\int\limits_\Omega|\nabla\hat v_i^{(s)}|^p\,dx\right)^{\frac{p-2}{p}}\left(\int\limits_\Omega|\nabla(\bar v_i^{(s)}-\hat v_i^{(s)})|^p\,dx\right)^{\frac2p}+\int\limits_\Omega|\nabla(\bar v_i^{(s)}-\hat v_i^{(s)})|^p\,dx\Big)$$

$$\le K_{12}\sum_{i\in I_s''}(\mathcal C_p(\mathcal F_i^{(s)}))^{\frac2p}(\mathcal C_p(\mathcal F_i^{(s)})+(r_i^{(s)})^n)^{\frac{p-2}{p}}$$

$$\times\left(|\vartheta_{\ell,\nu}-\vartheta_{s,i}|^{\frac2p}+|\vartheta_{\ell,\nu}-\vartheta_{s,i}|^{\frac{2(p-1)}{p}}+|\vartheta_{\ell,\nu}-\vartheta_{s,i}|^{\frac4p}+|\vartheta_{\ell,\nu}-\vartheta_{s,i}|^2\right).$$

Finally from (5.7) and Lemma 3.4 we obtain

$$J_4^{(s,\ell)}\le K_{13}\sum_{i\in I_s''}|\vartheta_{\ell,\nu}-\vartheta_{s,i}|^{\frac2p}(\mathcal C_p(\mathcal F_i^{(s)})+(r_i^{(s)})^n)$$

$$\le\frac{K_{14}}{l}\sum_{i\in I_s''}(\mathcal C_p(\mathcal F_i^{(s)})+(r_i^{(s)})^n)\le\frac{K_{15}}{l},$$

where l is a fixed number such that inequality (5.7) holds. Then

$$\lim_{\ell\to\infty}J_4^{(s,\ell)}=0, \tag{5.9}$$

uniformly with respect to $s\in\mathbb N$.

In view of the given partition $\{\Omega_\nu\}_{\nu=1}^{\Upsilon(\ell)}$ of the domain Ω, if there exists a constant $N>0$ and a number $s_2=s_2(\ell)$ so that, for $|t|\le N,\ s\ge s_2,\ \nu=1,\dots,\Upsilon(\ell)$, then we have

$$\left|\sum_{i\in I_s(\Omega_\nu)}\int\limits_\Omega|\nabla v_i^{(s)}(x,t)|^pdx-\int\limits_{\Omega_\nu}c(x,t)dx\right|<\frac{1}{\Upsilon(\ell)\ell} \tag{5.10}$$

according to condition (**C**). Then

$$J_3^{(s,\ell)}=-\sum_{\nu=1}^{\Upsilon(\ell)}\sum_{i\in I_s(\Omega_\nu)}\int\limits_\Omega|\nabla v_i^{(s)}(x,\vartheta_{\ell,\nu})|^p\,dx=J_5^{(s,\ell)}+J_6^{(s,\ell)}, \tag{5.11}$$

where

$$J_5^{(s,\ell)}:=-\sum_{\nu=1}^{\Upsilon(\ell)}\int\limits_{\Omega_\nu}c(x,\vartheta_{\ell,\nu})dx,$$

$$J_6^{(s,\ell)}:=-\sum_{\nu=1}^{\Upsilon(\ell)}\sum_{i\in I_s(\Omega_\nu)}\int\limits_\Omega|\nabla v_i^{(s)}(x,\vartheta_{\ell,\nu})|^p\,dx+\sum_{\nu=1}^{\Upsilon(\ell)}\int\limits_{\Omega_\nu}c(x,\vartheta_{\ell,\nu})\,dx.$$

Inequality (5.10) implies that

$$\lim_{\ell\to\infty} J_6^{(s,\ell)} = 0 \tag{5.12}$$

where the convergence is uniform with respect to $s \in \mathbb{N}$. Let us represent the integral $J_5^{(s,\ell)}$ as follows

$$J_5^{(s,\ell)} = -\int\limits_\Omega c(x,\vartheta(x))dx + J_7^{(\ell)} + J_8^{(\ell)}, \tag{5.13}$$

where

$$J_7^{(\ell)} := \sum_{\nu=1}^{\Upsilon(\ell)} \int\limits_{\Omega_\nu} (-c(x,\vartheta_{\ell,\nu}) + c(x,\vartheta_\ell(x))dx,$$

$$J_8^{(\ell)} := \sum_{\nu=1}^{\Upsilon(\ell)} \int\limits_{\Omega_\nu} (-c(x,\vartheta_\ell(x)) + c(x,\vartheta(x)))dx.$$

From the continuity of the function $c(x,t)$ with respect to t we obtain

$$\lim_{l\to\infty} J_7^{(\ell)} = 0, \ \lim_{l\to\infty} J_8^{(\ell)} = 0. \tag{5.14}$$

Taking into account (5.2), (5.9), (5.12), (5.14), from (5.1), (5.8), (5.11), (5.13) we derive (3.12).

5.2. Proof of Lemma 4.1

Let us investigate the following integral

$$\int\limits_\Omega |\nabla u_k^{(s)}|^p dx = \int\limits_\Omega \left|\nabla\left(u_k + \sum_{m=1}^3 q_{s,k}^{(m)}\right)\right|^p dx = \int\limits_\Omega |\nabla u_k|^p dx + J_s, \tag{5.15}$$

where

$$J_s := \int\limits_\Omega \left\{\left|\nabla\left(u_k + \sum_{m=1}^3 q_{s,k}^{(m)}\right)\right|^p - |\nabla u_k|^p\right\} dx.$$

Let us show that

$$\lim_{s\to\infty} J_s = -\int\limits_\Omega c(x,u_k(x))dx. \tag{5.16}$$

It follows in the standard way that

$$J_s = \int\limits_\Omega \int\limits_0^1 \frac{d}{d\mu}\left|\nabla\left(u_k + \mu\sum_{m=1}^3 q_{s,k}^{(m)}\right)\right|^p d\mu dx$$

$$= p\int\limits_\Omega \int\limits_0^1 \left|\nabla\left(u_k + \mu\sum_{m=1}^3 q_{s,k}^{(m)}\right)\right|^{p-2} \left(\left(\nabla u_k, \nabla\left(\sum_{m=1}^3 q_{s,k}^{(m)}\right)\right)\right.$$

$$+\mu\left|\nabla\left(\sum_{m=1}^{3} q_{s,k}^{(m)}\right)\right|^{2}\Big)d\mu dx=\int_{\Omega}\left|\nabla q_{s,k}^{(3)}(x)\right|^{p}dx+J_{9}^{(s)}+J_{10}^{(s)}+J_{11}^{(s)}+J_{12}^{(s)},\quad(5.17)$$

where

$$J_{9}^{(s)}:=p\int_{\Omega}\int_{0}^{1}\left|\nabla\left(u_k+\mu\sum_{m=1}^{3} q_{s,k}^{(m)}\right)\right|^{p-2}(\nabla u_k,\nabla\left(\sum_{m=1}^{3} q_{s,k}^{(m)}\right))d\mu dx,$$

$$J_{10}^{(s)}:=p\int_{\Omega}\int_{0}^{1}\mu\left|\nabla\left(u_k+\mu\sum_{m=1}^{3} q_{s,k}^{(m)}\right)\right|^{p-2}\left(\left|\nabla\left(\sum_{m=1}^{3} q_{s,k}^{(m)}\right)\right|^{2}-\left|\nabla q_{s,k}^{(3)}\right|^{2}\right)d\mu dx,$$

$$J_{11}^{(s)}:=p\int_{\Omega}\int_{0}^{1}\mu\left(\left|\nabla\left(u_k+\mu\sum_{m=1}^{3} q_{s,k}^{(m)}\right)\right|^{p-2}-\left|\nabla\left(\mu\sum_{m=1}^{3} q_{s,k}^{(m)}\right)\right|^{p-2}\right)\left|\nabla q_{s,k}^{(3)}\right|^{2}dxd\mu,$$

$$J_{12}^{(s)}:=\int_{\Omega}\left(\left|\nabla\left(\sum_{m=1}^{3} q_{s,k}^{(m)}\right)\right|^{p-2}-\left|\nabla q_{s,k}^{(3)}\right|^{p-2}\right)\left|\nabla q_{s,k}^{(3)}\right|^{2}dx.$$

From the strong convergence of $\left\{q_{s,k}^{(m)}\right\}_{s=1}^{\infty}$, $m=1,2$, and the weak convergence of $\left\{q_{s,k}^{(3)}\right\}_{s=1}^{\infty}$ to zero in $W^{1,p}(\Omega)$ as $s\to\infty$ (Lemma 3.5, Lemma 3.6), using Hölder's inequality, we derive

$$\lim_{s\to\infty}J_{10}^{(s)}=0,\quad \lim_{s\to\infty}J_{12}^{(s)}=0.\qquad(5.18)$$

Using the definition of $q_{s,k}^{(3)}$ and the Hölder inequality, we obtain

$$\lim_{s\to\infty}J_{9}^{(s)}=0,\quad \lim_{s\to\infty}J_{11}^{(s)}=0.\qquad(5.19)$$

Indeed,

$$J_{9}^{(s)}\le p\int_{0}^{1}\left(\int_{\Omega}\left|\nabla\left(u_k+\mu\sum_{m=1}^{3} q_{s,k}^{(m)}\right)\right|^{p}dx\right)^{\frac{p-2}{p}}\left(\int_{\Omega}\left(\nabla u_k,\nabla\left(\sum_{m=1}^{3} q_{s,k}^{(m)}\right)\right)^{\frac{p}{2}}dx\right)^{\frac{2}{p}}d\mu$$

$$\le K_{16}\left(\int_{\Omega}|\nabla u_k|^{\frac{p}{2}}|\nabla\left(\sum_{m=1}^{3} q_{s,k}^{(m)}\right)|^{\frac{p}{2}}dx\right)^{\frac{2}{p}}$$

$$\le K_{17}\left(\int_{\Omega}|\nabla u_k|^{p}dx\right)^{\frac{1}{2}}\left(\int_{\Omega}|\nabla\left(q_{s,k}^{(1)}+q_{s,k}^{(2)}\right)|^{p}dx\right)^{\frac{1}{2}}+K_{17}\left(\int_{\Omega}|\nabla u_k|^{\frac{p}{2}}|\nabla q_{s,k}^{(3)}|^{\frac{p}{2}}dx\right)^{\frac{2}{p}}.$$

The first term on the right-hand side tends to zero as $s\to\infty$ due to Lemma 3.5. We consider the second integral term

$$\int_{\Omega}|\nabla u_k|^{\frac{p}{2}}|\nabla q_{s,k}^{(3)}|^{\frac{p}{2}}dx\le K_{18}\sum_{i\in I_s''}\int_{\mathcal{D}_i^{(s)}}|\nabla u_k|^{\frac{p}{2}}|\nabla(\varphi_i^{(s)}v_i^{(s)}(x,u_{k,s,i}))|^{\frac{p}{2}}dx$$

$$\le K_{19}\sum_{i\in I''_s}\Big(\int\limits_{\mathcal{D}_i^{(s)}}|\nabla u_k|^{\frac{p}{2}}|\nabla v_i^{(s)}(x,u_{k,s,i})|^{\frac{p}{2}}dx$$

$$+\frac{1}{(\rho_i^{(s)})^{\frac{p}{2}}}\int\limits_{\mathcal{K}_i^{(s)}}|\nabla u_k|^{\frac{p}{2}}|v_i^{(s)}(x,u_{k,s,i})|^{\frac{p}{2}}dx\Big). \tag{5.20}$$

Using Hölder's inequality and (3.2) we have

$$\sum_{i\in I''_s}\int\limits_{\mathcal{D}_i^{(s)}}|\nabla u_k|^{\frac{p}{2}}|\nabla v_i^{(s)}(x,u_{k,s,i})|^{\frac{p}{2}}dx$$

$$\le\sum_{i\in I''_s}\left(\int\limits_{\mathcal{D}_i^{(s)}}|\nabla u_k|^p dx\right)^{\frac{1}{2}}\left(\int\limits_{\mathcal{D}_i^{(s)}}|\nabla v_i^{(s)}(x,u_{k,s,i})|^p dx\right)^{\frac{1}{2}}$$

$$\le K_{20}\left(\sum_{i\in I''_s}\int\limits_{\mathcal{D}_i^{(s)}}|\nabla u_k|^p dx\right)^{\frac{1}{2}}\left(\sum_{i\in I''_s}\int\limits_{\mathcal{D}_i^{(s)}}|\nabla v_i^{(s)}(x,u_{k,s,i})|^p dx\right)^{\frac{1}{2}}$$

$$\le K_{21}\left(\int\limits_{\bigcup\limits_{i\in I''_s}\mathcal{D}_i^{(s)}}|\nabla u_k|^p dx\right)^{\frac{1}{2}}\left(\sum_{i\in I''_s}(\mathcal{C}_p(\mathcal{F}_i^{(s)})+(r_i^{(s)})^n)\right)^{\frac{1}{2}}$$

$$\le K_{22}\left(\int\limits_{\bigcup\limits_{i\in I''_s}\mathcal{D}_i^{(s)}}|\nabla u_k|^p dx\right)^{\frac{1}{2}}. \tag{5.21}$$

Here $meas\bigcup_{i\in I''_s}\mathcal{D}_i^{(s)}$ tends to zero as $s\to\infty$ due to (3.6). The convergence to zero on the right-hand side of (5.21) as $s\to\infty$ follows from the property of absolutely continuity of integrals. Now we consider the second term on the right-hand side of (5.20):

$$\sum_{i\in I''_s}\frac{1}{(\rho_i^{(s)})^{\frac{p}{2}}}\int\limits_{\mathcal{K}_i^{(s)}}|\nabla u_k|^{\frac{p}{2}}|v_i^{(s)}(x,u_{k,s,i})|^{\frac{p}{2}}dx$$

$$\le\sum_{i\in I''_s}\left(\int\limits_{\mathcal{K}_i^{(s)}}|\nabla u_k|^p dx\right)^{\frac{1}{2}}\left(\frac{1}{(\rho_i^{(s)})^p}\int\limits_{\mathcal{K}_i^{(s)}}|v_i^{(s)}(x,u_{k,s,i})|^p dx\right)^{\frac{1}{2}}$$

$$\leq K_{23} \left(\sum_{i \in I_s''} \int_{\mathcal{K}_i^{(s)}} |\nabla u_k|^p dx \right)^{\frac{1}{2}} \left(\sum_{i \in I_s''} \frac{1}{(\rho_i^{(s)})^p} \int_{\mathcal{K}_i^{(s)}} |v_i^{(s)}(x, u_{k,s,i})|^p dx \right)^{\frac{1}{2}}$$

$$\leq K_{24} \left(\sum_{i \in I_s''} \frac{1}{(\rho_i^{(s)})^p} \int_{\mathcal{K}_i^{(s)}} \left(\frac{\mathcal{C}_p(\mathcal{F}_i^{(s)})}{|x - x_i^{(s)}|^{n-p}} \right)^{\frac{p}{p-1}} dx \right)^{\frac{1}{2}}$$

$$\leq K_{25} \left(\sum_{i \in I_s''} \left(\frac{\mathcal{C}_p^p(\mathcal{F}_i^{(s)})}{(\rho_i^{(s)})^{n-p}} \right)^{\frac{1}{p-1}} \right)^{\frac{1}{2}}. \qquad (5.22)$$

We have the convergence to zero analogously to the proof of (5.4). From (5.20), (5.21), (5.22) follows that $J_9^{(s)}$ tends to zero as $s \to \infty$. Let us prove the second equality of (5.19):

$$J_{11}^{(s)} = p \int_0^1 \mu \int_\Omega \int_0^1 \frac{d}{d\eta} \left(\left| \nabla \left(\eta u_k + \mu \sum_{m=1}^{3} q_{s,k}^{(m)} \right) \right|^{p-2} \right) d\eta |\nabla q_{s,k}^{(3)}|^2 \, dx d\mu$$

$$\leq K_{26} \int_\Omega |\nabla q_{s,k}^{(3)}| \, |\nabla u_k| \left(|\nabla \left(\sum_{m=1}^{3} q_{s,k}^{(m)} \right) | + |\nabla u_k| \right)^{p-2} dx$$

$$\leq K_{27} \left(\int_\Omega \left(|\nabla \left(\sum_{m=1}^{3} q_{s,k}^{(m)} \right) | + |\nabla u_k| \right)^p dx \right)^{\frac{p-2}{p}} \left(\int_\Omega |\nabla q_{s,k}^{(3)}|^{\frac{p}{2}} |\nabla u_k|^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}}.$$

The right-hand side of the last inequality tends to zero due to the absolute continuity of integrals and (5.20). This proves (5.19). Finally, from (5.17)-(5.19) and Lemma 3.7 we get (5.16). Then, by (5.15), (5.16) we obtain (4.6). Using the same arguments and Lemmas 3.5, 3.6, we derive (4.7).

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