A pointwise estimate of the solution to the p-Laplace evolution equation

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Abstract

In the cylinder $(\mathcal{B}_R\backslash\overline{\mathcal{F}})\times(0,T)$ we consider the Dirichlet problem for the p-Laplace evolution equation. Here $\mathcal{F}$ is an open set of diameter d, $\mathcal{B}_R\subset\mathbb{R}^n$ is an open ball of radius $R=R(T,d,p)$, d is small enough, and $p\in(\frac{2n}{n+1},2]$. Using Moser's iterative procedure, we derive the poitnwise estimates for the solution of this problem in terms of the diameter of the set $\mathcal{F}$.

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1 Introduction

The goal of the paper is to derive the pointwise estimates of the solution to the following parabolic boundary value problem:

$$\begin{cases} u_t-\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)=0 \quad \text{in } (\mathcal{B}_R\backslash\overline{\mathcal{F}})\times(0,T);\\ u(x,t)=0 \text{ on } \partial\mathcal{B}_R\times[0,T] \quad \text{and} \quad u(x,t)=k \text{ on } \partial\mathcal{F}\times[0,T];\\ u(x,0)=k\,f(x) \quad \text{in } \mathcal{B}_R\backslash\overline{\mathcal{F}}, \end{cases} \tag{1.1}$$

where $\mathcal{F} \subset \mathcal{B}_d \subset \mathcal{B}_R \subset \mathbb{R}^n$ ($n \geqslant 3$) is an open set with sufficiently smooth boundary; $0 < d < \min\{1, \frac{R}{8}\}$, $T > 0$, $p \in (\frac{2n}{n+1}, 2]$, $k \in \mathbb{R}$ and where, from now on, $\mathcal{B}_\varrho$ denotes an open ball centered at the point zero and of radius ϱ. The value of the radius $R = R(T, d, p, k)$ will be specified later in Theorem 2.2 (see Section 2).

Problem (1.1) is closely related to Skrypnik's method of homogenization of the Dirichlet non–linear parabolic problems in non–periodic strongly perforated domains in the case when the perforations are small disjoint components (domains with fine–grained boundary). In the framework of this method, we construct the asymptotic expansion of the solution of the corresponding non–linear evolution problem in the perforated domain in terms of the model problem like (1.1). Knowing the behavior of the solution of this model problem one can obtain the homogenized problem.

Notice that the homogenization of the Dirichlet boundary value problem in perforated domains was first studied in [15] and then it was revisited by many authors (see, e.g., [8, 17, 25, 1], and the references therein). We also notice that the homogenization of nonlinear elliptic and parabolic equations is a long-standing problem and a number of methods have been developed. There is an extensive literature on the subject. We will not attempt a review of the literature here, but merely mention [7, 20, 31, 2, 3, 4, 5], and the references therein.

The goal of the paper is to derive the pointwise estimates of the solution to the parabolic problem (1.1) under a special choice of the value R and $p \in (\frac{2n}{n+1}, 2]$. Using an approach based on the classical iterative method which was introduced by Moser (see [18]) and following the ideas of [22, 24] we show that the solution of (1.1) satisfies the following pointwise estimate:

$$|u(x,t)| \leqslant C \left(\frac{d}{|x|} \right)^{\frac{n-p}{p-1}} \quad \text{in } (\mathcal{B}_R \backslash \overline{\mathcal{F}}) \times (0, T),$$

where the constant $C = C(k, n, p, T)$. Notice that the pointwise estimates of the fundamental solutions are closely related to the study of their regularity and the qualitative behavior of the solutions in the vicinity of the boundary points. In particular, they are important for the study of removable isolated singularities ([29], [19]), and the Wiener criterion of regularity of the boundary point ([27], [28]). The corresponding estimates for linear and quasilinear elliptic equations were obtained in [14, 12, 13, 10], and for p–Laplacian in [30]. The study of the corresponding problem for the non–linear operators is a more complicated problem (see [9]). The estimates for the potentials for non–linear elliptic operators were obtained by I.V. Skrypnik in [22]. The corresponding result for non–linear parabolic operator was obtained in [23, 24, 26] in the case $p = 2$, using approach developed in [21, 6]. Notice that the proof given in the paper is essentially different from that for equations corresponding to the case of linear growth of the coefficients with respect to $|\nabla u|$. Here we restrict ourselves to the case $\frac{2n}{n+1} < p \leqslant 2$. The case $p > 2$ requires the development of another technique and it is the subject of the forthcoming papers.

The paper is organized as follows. In Section 2 we formulate the assumptions on the data of problem (1.1) and formulate the main result of the paper. It will be proved (for $n \geqslant 3$) in Sections 3–6. First, in Sections 3, 4 we obtain preliminary integral and pointwise estimates for the solution of the boundary value problem (1.1). Then in Section 5, using the results of the previous sections, we obtain the improved pointwise and integral estimates. Finally, in Section 6 we complete the proof of the main result.

Notational convetion. In what follows by C, C_j, c_j we denote generic constants that depend on n, p, T and do not depend on k, d.

2 Statement of the problem and the main result

Let $\mathcal{B}_R \subset \mathbb{R}^n$ $(n \geq 2)$ be an open ball centered at the point zero and of radius R, $T > 0$, $k \in \mathbb{R}$, and $0 < d < \min\{1, \frac{R}{8}\}$.

Let $\mathcal{F}$ be an open subset of $\mathcal{B}_R$ with sufficiently smooth boundary such that $\mathcal{F} \subset \mathcal{B}_d$. We set $\Omega = \mathcal{B}_R \setminus \overline{\mathcal{F}}$ and introduce the following notation:

$$Q_T = \{(x,t) :\ x \in \Omega,\ t \in (0,T)\}, \quad Q_\ell = \{(x,t) :\ x \in \Omega,\ t \in (0,\ell)\}.$$

In the cylinder Q_T we consider the following initial boundary value problem:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = 0 \quad \text{in } Q_T; \\ u(x,t) = 0 \text{ on } \partial\mathcal{B}_R \times (0,T) \text{ and } u(x,t) = k \text{ on } \partial\mathcal{F} \times (0,T); \\ u(x,0) = k\,f(x) \quad \text{in } \Omega, \end{cases} \tag{2.1}$$

where

$$\frac{2n}{n+1} < p \leqslant 2. \tag{2.2}$$

We suppose that the function $f \in W_0^{1,p}(\mathcal{B}_R)$ is such that $f(x) \equiv 1$ in $\mathcal{F}$, and satisfies the bounds:

$$0 \leqslant f \leqslant \min\left\{1,\, C\left(\frac{d}{|x|}\right)^{\frac{n-p}{p-1}}\right\} \text{ in } \mathcal{B}_R \text{ and } \int\limits_{\Omega} |\nabla f|^p\,dx \leqslant C\,d^{n-p}. \tag{2.3}$$

Remark 2.1. The special choice of the function f appears here due to the applications of the pointwise estimates obtained in the paper, in particular, in the homogenization theory (Skrypnik's homogenization approach).

Now, for any p satisfying (2.2) we introduce the Banach spaces (see, e.g., [9]):

$$V^{2,p}(Q_T) = L_\infty(0,T;L_2(\Omega)) \cap L_p(0,T;W^{1,p}(\Omega));$$

$$V_0^{2,p}(Q_T) = L_\infty(0,T;L_2(\Omega)) \cap L_p(0,T;W_0^{1,p}(\Omega)).$$

The norm in these spaces is defined by:

$$\|u\|_{V^{2,p}(Q_T)} = \operatorname{ess\,sup}_{0<t<T}\|u(\cdot,t)\|_{L_2(\Omega)} + \|\nabla u\|_{L_p(Q_T)}.$$

When $p = 2$ we set $V_0^{2,2}(Q_T) \equiv V_0^2(Q_T)$ and $V^{2,2}(Q_T) \equiv V^2(Q_T)$. For the properties of the spaces $V_0^2(Q_T), V^2(Q_T)$ we refer to [11].

Consider now problem (2.1). The weak solution of (2.1) is understood as follows.

Definition 2.1. A locally integrable bounded function function $u = u(x,t)$ is said to be a weak solution to problem (2.1) if:

- **(i)** $u \in V^{2,p}(Q_T)$;
- **(ii)** $u(\cdot,t)\,|_{\partial\Omega}= k\,f(x)$ in the sense of traces of functions in $V_0^{2,p}(Q_T)$;
- **(iii)** for any function $\varphi = \varphi(x,t)$ such that $\varphi \in V_0^{2,p}(Q_T)$ and $\varphi_t \in L^2(0,T;L^2(\Omega))$ and every $\ell \in [0,T]$, the following integral identity holds:

$$\int\limits_{\Omega} u(x,t)\varphi(x,t)\,dx\Big|_0^{\ell} - \int\limits_{Q_\ell} \Big(u\varphi_t - |\nabla u|^{p-2}\,\nabla u\,\nabla\varphi\Big)\,dxdt = 0. \tag{2.4}$$

Remark 2.2. Without lost of generality we suppose that $u_t \in L^2(Q_T)$. If not, we consider the Steklov averages (see, e.g., [9]).

The existence and uniqueness result for the parabolic problem (2.1) is well known (see, e.g., [11, 9]). If $\frac{2n}{n+1} < p \leqslant 2$ the parabolic p–Laplace equation has a unique weak solution for corresponding initial data. That is, the following theorem holds.

Theorem 2.1. *Let $f \in W_0^{1,p}(\mathcal{B}_R)$, where p satisfies (2.2). Then, for any $k \in \mathbb{R}$, $T > 0$ and $0 < d < R/2$, there exists a unique weak solution of problem (2.1). If in addition $k > 0$ and $f \geqslant 0$ in $\mathcal{B}_R$, then there exists a unique positive weak solution of (2.1).*

Remark 2.3. It follows from the maximum principle that

$$0 \leqslant u\,\mathrm{sign}\,k \leqslant |k| \quad \text{in } Q_T, \tag{2.5}$$

where $u = u(x,t)$ is the weak solution of (2.1).

In fact, the weak solution of problem (2.1) possesses higher regularity properties (i.e. higher than it was stated in Theorem 2.1). Namely, it is known from [9], that if u is a bounded solution of the parabolic p–Laplace equation for $p \in (\frac{2n}{n+1}, 2]$, then u is a locally Hölder continuous function with the Hölder exponent and the Hölder norm depending only on p, n and the local L^∞–norm of the solution.

Now we are in position to formulate the main result of the paper.

Theorem 2.2 (Main theorem). *Let $n \geqslant 3$, $k \in \mathbb{R}$, $T > 0$, $0 < d < \min\{1, \frac{R}{8}\}$, p satisfies (2.2),*

$$R = \left(\frac{T^{\frac{p-1}{2-p}}}{(|k|^{p-1} + |k|)d^{n-p}}\right)^{\frac{2-p}{p+n(p-2)}}, \tag{2.6}$$

and $u = u(x,t)$ be a weak solution of problem (2.1). Then there exists a constant $C = C(n,p,T)$ such that the following estimate

$$|u(x_0,t_0)| \leqslant C \min\left\{ |k|, \min\{|k|, (|k| + |k|^{p-1})|k|^{2-p}\} \left(\frac{d}{|x_0|}\right)^{\frac{n-p}{p-1}} \right\} \tag{2.7}$$

is valid for any $(x_0,t_0) \in Q_T$.

Remark 2.4. The definition (2.6) of the radius R shows the conformity between the height and the base of cylinder Q_T. Because the growths of the elliptic and parabolic parts of the equation are essentially different.

Remark 2.5. Notice that using the approach proposed in the paper one can obtain a corresponding estimate for $n = 2$. Namely, in this case the term $\left(\frac{d}{|x_0|}\right)^{\frac{n-p}{p-1}}$ is replaced by $\frac{\ln \frac{1}{|x_0|}}{\ln \frac{1}{d}}$. Since the proofs of the pointwise estimates for $n = 2$ and $n \geqslant 3$ are similar (with corresponding modifications), we restrict ourselves to the case $n \geqslant 3$.

Remark 2.6. Taking into account estimate (2.5), it is enough to prove the following inequality

$$|u(x_0,t_0)| \leqslant C \min\{|k|, (|k| + |k|^{p-1})|k|^{2-p}\} \left(\frac{d}{|x_0|}\right)^{\frac{n-p}{p-1}}.$$

In what follows, without loss of generality, it will be assumed that k is a strictly positive parameter. In fact, if $k < 0$, then one can consider the function $v(x,t) = -u(x,t)$ which is the solution of (2.1) with $-k$ instead of k.

3 Preliminary estimates for the solution of problem (2.1)

In this section we will establish some preliminary integral and pointwise estimates for the solution of the boundary value problem (2.1).

Let ρ be a positive parameter such that $2d < \rho < R$. Then we introduce the following notation:

$$m_\rho = \max\{u(x,t) \,:\, (x,t) \in (\mathcal{B}_R \setminus \mathcal{B}_\rho) \times (0,T)\}; \tag{3.1}$$

$$\mathsf{u}_\rho(x,t) = \max\{0, (u(x,t) - m_\rho)\}; \tag{3.2}$$

$$\mathcal{E}_\rho = \{(x,t) \in Q_T \,:\, \mathsf{u}_\rho(x,t) > 0\}; \tag{3.3}$$

$$[u(x,t)]_\mu = \min\{u, \mu\}; \tag{3.4}$$

$$\mathcal{E}_{\rho,\mu} = \{(x,t) \in Q_T \,:\, 0 \leqslant \mathsf{u}_\rho(x,t) \leqslant \mu\}, \tag{3.5}$$

$$\mathcal{E}_{\rho,\mu}(t) = \{x \in \mathcal{B}_R, t = \tau \,:\, 0 \leqslant \mathsf{u}_\rho(x,\tau) \leqslant \mu\}, \tag{3.6}$$

$$\mathcal{E}^+_{\rho,\mu} = \{(x,t) \in Q_T \,:\, \mathsf{u}_\rho(x,t) > \mu\}. \tag{3.7}$$

First, we obtain integral bounds for the solution of the boundary value problem (2.1). The following result holds.

Lemma 3.1. *Let u be the weak solution of the boundary value problem (2.1). Then, for any $t \in (0,T)$, $k > 0$, $2d < \rho < R$, $\mu \in (0, k - m_\rho)$, u satisfies the bounds:*

$$\int\limits_\Omega u^2(x,t)\,dx + \int\limits_{Q_T} |\nabla u|^p \; dxdt \leqslant c_1 \left(k^2 + k^p\right) d^{n-p}; \tag{3.8}$$

$$\int\limits_\Omega \mathsf{u}_\rho^2(x,t)\,dx + \int\limits_{Q_T} |\nabla \mathsf{u}_\rho|^p \; dxdt \leqslant c_2\,(k + k^{p-1})(k - m_\rho) d^{n-p}; \tag{3.9}$$

$$\int\limits_\Omega [\mathsf{u}_\rho(x,t)]^2_\mu\,dx + \int\limits_{\mathcal{E}_{\rho,\mu}} |\nabla \mathsf{u}_\rho|^p \; dxdt \leqslant c_3\,\mu\,(k + k^{p-1}) d^{n-p}, \tag{3.10}$$

where c_1, c_2, c_3 are constant that does not depend on k, d.

Proof of Lemma 3.1. First we prove estimate (3.8). Plugging the function $\varphi(x,t) = u(x,t) - k\,f(x)$ in the integral identity (2.4) and using (2.3) one obtains:

$$\int\limits_\Omega u^2(x,t)\,dx + \int\limits_{Q_T} |\nabla u|^p \,dxdt \leqslant C\left(k^2\,d^n + T\,k^p d^{n-p}\right) \leqslant C\left(k^2 + k^p\right) d^{n-p}$$

and the desired bound (3.8) is proved.

In a similar way, plugging the test functions

$$\varphi_1(x,t) = \mathsf{u}_\rho(x,t) - \frac{k - m_\rho}{k} u(x,t) \quad \text{and} \quad \varphi_2(x,t) = [\mathsf{u}_\rho(x,t)]_\mu - \frac{\mu}{k - m_\rho}\mathsf{u}_\rho(x,t)$$

for $0 < \mu < k - m_\rho$ in the integral identity (2.4) and using (3.8) one obtains the bounds (3.9) and (3.10), respectively.

Lemma 3.1 is proved.

Using integral estimates from Lemma 3.1, we obtain now a preliminary local estimate for the solution of the boundary value problem (2.1). Later on, in Section 5, we will improve this pointwise estimate and obtain the main result of the paper. The proofs of the lemma and the main theorem are based on the Moser's iteration technique (see for example [18]).

The following result holds.

Lemma 3.2. *Let u be the weak solution of the boundary value problem (2.1) and let (x_0, t_0) be any point of the cylinder Q_T such that $|x_0| = \rho \geqslant 8d$. Then*

$$u(x_0,t_0) \leqslant c_4\left(k + k^{p-1}\right)\frac{d^{n-p}}{\rho^n}, \tag{3.11}$$

where c_4 is a constant that does not depend on k, d.

Proof of Lemma 3.2. First, we introduce the notation. Denote by $\{\rho_j^{(1)}\}, \{\rho_j^{(2)}\}$ the sequences such that:

$$\rho_j^{(1)} = \frac{\rho}{2}(1+2^{-j}), \quad \rho_j^{(2)} = \frac{\rho}{2}(3-2^{-j}) \quad (j=1,2,\ldots). \tag{3.12}$$

Let now $\mathcal{G}_j$ be a set defined by:

$$\mathcal{G}_j = \{x \in \mathbb{R}^n : \rho_j^{(1)} \leqslant |x| \leqslant \rho_j^{(2)}\} \quad (j=1,2,\ldots)$$

and $\xi_j \in C^\infty(\mathbb{R}^n)$ be a cut–off function such that **(i)** $0 \leqslant \xi_j(x) \leqslant 1$ in $\mathbb{R}^n$, **(ii)** $\xi_j(x) = 1$ in $\mathcal{G}_j$ and $\xi_j(x) = 0$ in $\mathbb{R}^n \setminus \mathcal{G}_{j+1}$, **(iii)** $|\nabla \xi_j| \leqslant c_5 2^j/\rho$ in $\mathcal{G}_{j+1} \setminus \mathcal{G}_j$, where c_5 is a constant that does not depend on j and ρ.

We set $\varphi(x,t) = u^l(x,t)\xi_j^m(x)$, where $l, m > 0$. Plugging the function φ in the integral identity (2.4), using Young's inequality, the property **(iii)** of the cut–off function ξ_j, and taking into account that $u \geqslant 0$ in Q_T, we obtain :

$$\int\limits_\Omega u^{l+1}(x,t)\xi_j^m(x)\,dx + \int\limits_{Q_T} |\nabla u(x,t)|^p u^{l-1}(x,t)\xi_j^m(x)\,dxdt \leqslant$$

$$\leqslant C\,(l+m)^p \frac{2^{jp}}{\rho^p} \int\limits_{Q_T} u^{l-1+p}(x,t)\xi_j^{m-p}(x)\,dxdt. \tag{3.13}$$

Let now

$$M_j = \text{ess sup}\{u(x,t) : (x,t) \in \mathcal{G}_j \times (0,T)\}. \tag{3.14}$$

Then applying Moser's iteration procedure to (3.13) we get:

$$M_j^{2+(2-p)\frac{n}{p}} \leqslant C M_{j+1}^{(2-p)\frac{n}{p}} \frac{2^{j(n+p)}}{\rho^{n+p}} \int\limits_0^T \int\limits_{\mathcal{G}_{j+1}} u^p(x,t)\,dxdt. \tag{3.15}$$

It is clear that due to the definition

$$[u(x,t)]_{M_{j+1}} = \min\{u, M_{j+1}\} = u(x,t) \quad \text{in } \mathcal{G}_{j+1} \times (0,T).$$

Now Lemma 3.1, Poincare's inequality, and (3.15) imply that

$$M_j^{2+(2-p)\frac{n}{p}} \leqslant C M_{j+1}^{(2-p)\frac{n}{p}} \frac{2^{j(n+p)}}{\rho^n} \int\limits_{\mathcal{E}_{\rho, M_{j+1}}} |\nabla u(x,t)|^p\,dxdt$$

$$\leqslant C \frac{2^{j(n+p)}}{\rho^{-n}} \left(k + k^{p-1}\right) d^{n-p} M_{j+1}^{1+(2-p)\frac{n}{p}}. \tag{3.16}$$

Iterating inequality (3.16) we obtain that

$$M_1 \leqslant C\,\left(k + k^{p-1}\right) \frac{d^{n-p}}{\rho^n}.$$

This proves Lemma 3.2.

In what follows we obtain some additional integral estimates for the solution of problem (2.1). To this end we consider the following auxiliary elliptic problem:

$$\begin{cases} \operatorname{div}\Big(|\nabla g_\rho(x)|^{p-2}\,\nabla g_\rho(x)\Big)=0 \quad \text{in } \mathcal{B}_\rho\setminus\overline{\mathcal{B}_d}; \\ g_\rho(x)=1 \quad \text{on } \partial\mathcal{B}_d; \\ g_\rho(x)=0 \quad \text{on } \partial\mathcal{B}_\rho, \end{cases} \tag{3.17}$$

where $\rho=|x_0|\geqslant 8d$.

It is easy to see that the solution of problem (3.17) is given by

$$g_\rho(x)=\frac{|x|^{-\frac{n-p}{p-1}}-\rho^{-\frac{n-p}{p-1}}}{d^{-\frac{n-p}{p-1}}-\rho^{-\frac{n-p}{p-1}}},\ x\in\mathcal{B}_\rho\setminus\overline{\mathcal{B}_d}.$$

We extend the function g_ρ to the whole ball $\mathcal{B}_R$ by the formula:

$$g_\rho(x):=\begin{cases} 1 \quad \text{in } \mathcal{B}_d; \\ 0 \quad \text{in } \mathcal{B}_R\setminus\mathcal{B}_\rho \end{cases} \tag{3.18}$$

and keep for it the same notation, then

$$\int\limits_{\mathcal{B}_R} g_\rho^p\,dx\leqslant C\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}\rho^n,\quad \int\limits_{\mathcal{B}_R}|\nabla g_\rho|^p\,dx\leqslant C\,d^{n-p}. \tag{3.19}$$

Now we make use of the function g_ρ and inequality (3.19) to obtain an additional integral estimate for the solution to problem (2.1). First, we introduce the notation. We denote by $\lambda=\lambda(t)$ a smooth function in $\mathbb{R}$ such that **(i)** $0\le\lambda(t)\le 1$ in $\mathbb{R}$, **(ii)** $\lambda(t)=1$ if $|t|\leqslant\frac{1}{2}$, **(iii)** $\lambda(t)=0$ if $|t|\geqslant 1$, and **(iv)** $\left|\frac{d\lambda}{dt}\right|\le 2$ in $\mathbb{R}$.

Lemma 3.3. *Let u and g_ρ be the solutions of problems (2.1) and (3.17), respectively. Then there exist positive constants K_1 and K_2, which do not depend on k,d,ρ, such that for a parameter s satisfying the inequality*

$$\mathsf{K}_1 k^{2-p}\rho^p\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}(2-p)}\leqslant s\leqslant T, \tag{3.20}$$

for any $t,\tau\in[0,T]$, $\tau\geqslant s$, we have:

$$\int\limits_\Omega u^2(x,t)g_\rho^p(x)\lambda_s^p(t-\tau)\,dx+\int\limits_{Q_T}|\nabla u(x,t)|^p g_\rho^p(x)\lambda_s^p(t-\tau)\,dxdt\leqslant$$

$$\leqslant \mathsf{K}_2\,k^p\,s\,d^{n-p}, \tag{3.21}$$

where $\lambda_s(t-\tau)=\lambda((t-\tau)/s)$.

Proof of Lemma 3.3. We set

$$\varphi(x,t) = u(x,t)g_\rho^p(x)\lambda_s^p(t-\tau) - k\,g_\rho^p(x)\lambda_s^p(t-\tau)$$

and plug the function φ in the integral identity (2.4). Using (2.5), the definition of the functions λ, g_ρ, inequalities (3.19), and Young's inequality, one can easily obtain the following estimate:

$$\int\limits_\Omega u^2(x,t)g_\rho^p(x)\lambda_s^p(t-\tau)\,dx + \int\limits_{Q_T} |\nabla u(x,t)|^p g_\rho^p(x)\lambda_s^p(t-\tau)\,dxdt$$

$$\leqslant \frac{C}{s}\int\limits_{Q_T} u^2(x,t)g_\rho^p(x)\lambda_s^{p-1}(t-\tau)\,dxdt + C\,k\int\limits_\Omega u(x,t)g_\rho^p(x)\lambda_s^p(t-\tau)\,dx$$

$$+C\frac{k}{s}\int\limits_{Q_T} u(x,t)g_\rho^p(x)\lambda_s^{p-1}(t-\tau)\,dxdt + \int\limits_{Q_T} u^p(x,t)\,|\nabla g_\rho(x)|^p\,\lambda_s^p(t-\tau)\,dxdt$$

$$\leqslant \frac{C_0}{s}\left(k^p\,d^{n-p} + \frac{k^2}{s}\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}\rho^n\right). \tag{3.22}$$

Now it follows from condition (3.20) that

$$\frac{k^2}{s}\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}\rho^n \leqslant \frac{k^2\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}\rho^n}{\mathsf{K}_1\,k^{2-p}\rho^p\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}(2-p)}} = \frac{k^p}{\mathsf{K}_1}d^{n-p}.$$

This inequality and (3.22) immediately imply the desired inequality (3.21) with $\mathsf{K}_2 = C_0\max\{1,\ \frac{1}{\mathsf{K}_1}\}$ and Lemma 3.3 is proved.

4 Auxiliary integral estimates

In this section we obtain auxiliary estimates for the following integrals:

$$I_\rho(s,\tau) = \operatorname{ess\,sup}_{t\in(0,T)}\int\limits_\Omega \mathsf{u}_\rho^2(x,t)\lambda_s^p(t-\tau)\,dx + \int\limits_{\mathcal{E}_\rho} |\nabla u(x,t)|^p\,\lambda_s^p(t-\tau)\,dxdt,$$

$$I_{\rho,\mu}(s,\tau) = \operatorname{ess\,sup}_{t\in(0,T)}\int\limits_{\mathcal{E}_{\rho,\mu}(t)} [\mathsf{u}_\rho(x,t)]_\mu^2\lambda_s^p(t-\tau)dx + \int\limits_{\mathcal{E}_{\rho,\mu}} |\nabla u(x,t)|^p\,\lambda_s^p(t-\tau)\,dxdt,$$

$$J_{\rho,\mu}(s,\tau) = \frac{\mu}{s}\int\limits_{\mathcal{E}^+_{\rho,\mu}} (\mathsf{u}_\rho-\mu)\lambda_s^{p-1}(t-\tau)\,dxdt$$

where u is the solution of problem (2.1), u_ρ is defined in (3.2) with $r=\rho$, the function λ_s is defined in Lemma 3.3, $[\mathsf{u}_\rho(x,t)]_\mu = \min\{\mathsf{u}_\rho(x,t),\mu\}$, and the sets $\mathcal{E}_\rho$, $\mathcal{E}_{\rho,\mu}$ $\mathcal{E}_{\rho,\mu}(t)$, $\mathcal{E}^+_{\rho,\mu}$ are defined in (3.3), (3.5)–(3.7), respectively.

First, we estimate the integral $I_\rho(s,\tau)$. The following result holds.

Lemma 4.1. *Let s be a parameter satisfying (3.20). Then*

$$I_\rho(s,\tau) \leqslant \frac{c_7}{s} \int\limits_{\mathcal{E}_\rho} \mathsf{u}_\rho^2(x,t)\lambda_{2s}^p(t-\tau)\,dxdt + c_7(k-m_\rho)k^{p-1}sd^{n-p}, \tag{4.1}$$

where c_7 is a constant that does not depend on s, k, d.

Proof of Lemma 4.1. In the integral identity (2.4) we set:

$$\varphi(x,t) = \mathsf{u}_\rho\lambda_s^p(t-\tau) - (k-m_\rho)g_\rho^p(x)\lambda_s^p(t-\tau).$$

Then using Lemma 3.3 we obtain:

$$I_\rho(s,\tau) \leqslant \frac{C}{s} \int\limits_{\mathcal{E}_\rho} \mathsf{u}_\rho^2\lambda_s^p\,dxdt + C(k-m_\rho)\left(\int\limits_{Q_T} |\nabla u|^p\, g_\rho^p\lambda_s^p\,dxdt\right)^{\frac{p-1}{p}}$$

$$\times\left(s\int\limits_{\Omega} |\nabla g_p|^p\,dx\right)^{\frac{1}{p}} \leqslant \frac{c_7}{s}\int\limits_{Q_T} \mathsf{u}_\rho^2\lambda_{2s}^p\,dxdt + c_7(k-m_\rho)k^{p-1}sd^{n-p}. \tag{4.2}$$

Lemma 4.1 is proved.

Now, we estimate the integral $I_{\rho,\mu}(s,\tau)$. The following result holds.

Lemma 4.2. *Let s be a parameter satisfying (3.20). Then*

$$I_{\rho,\mu}(s,\tau) \leqslant \frac{c_8}{s}\int\limits_{\mathcal{E}_{\rho,\mu}^+} [\mathsf{u}_\rho]_\mu^2\lambda_{2s}^p(t-\tau)\,dxdt + c_8 J_{\rho,\mu}(s,\tau)$$

$$+c_8\frac{\mu}{k-m_\rho}(I_\rho(s,\tau)+I_\rho(2s,\tau)), \tag{4.3}$$

where c_8 is a constant that does not depend on s, k, d.

Proof of Lemma 4.2. In the integral identity (2.4) we set:

$$\varphi = [\mathsf{u}_\rho(x,t)]_\mu\lambda_s^p(t-\tau) - \frac{\mu}{k-m_\rho}\mathsf{u}_\rho(x,t)\lambda_s^p(t-\tau)$$

and immediately obtain (4.3). Lemma 4.2 is proved.

To estimate the second term on the right hand side of (4.3) we make use of the lemma.

Lemma 4.3. *Let s be a parameter satisfying (3.20) Let a, q be parameters such that*

$$1 < q < p, \quad 2-p < a < 1. \tag{4.4}$$

Then

$$J_{\rho,\mu}(s,\tau) \leqslant \frac{c_9\mu}{k-m_\rho}\left[I_\rho(s,\tau)+I_\rho(2s,\tau)\right]+$$

$$+c_9\,\mathsf{A}_1 \int\limits_{\mathcal{E}^+_{\rho,\mu}} |x|^{\frac{p(1-a)}{p-1}}\,\mathsf{u}_\rho^{\frac{q}{p-1}}\,\lambda^p_{2s}(t-\tau)\,dxdt, \tag{4.5}$$

where $\mathsf{A}_1=\mathsf{A}_1(s,\mu,d,\rho,k)$ *is given by*

$$\mathsf{A}_1=\frac{\mu^{\frac{p-q}{p-1}}}{s^{\frac{p-a}{p-1}}}\left[k\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}\right]^{\frac{2-p}{p-1}(1-a)}\left[\frac{\rho^n}{k^{p-1}sd^{n-p}}\right]^{\frac{2-p}{p-1}} \tag{4.6}$$

and c_9 *is a constant that does not depend on* s,k,d.

The **Proof of Lemma 4.3** is of a technical character and is given in the Appendix.

Corollary 4.1. *Let* s *be the parameter defined in (3.20). Parameters* a *and* q *satisfy conditions (4.4). Then*

$$I_{\rho,\mu}(s,\tau)\leqslant\frac{c_{10}}{s}\int\limits_{\mathcal{E}_{\rho,\mu}}[\mathsf{u}_\rho]^2_\mu\lambda^p_{2s}(t-\tau)\,dxdt+c_{10}\mathsf{A}_1\int\limits_{\mathcal{E}^+_{\rho,\mu}}|x|^{\frac{p(1-a)}{p-1}}\,\mathsf{u}_\rho^{\frac{q}{p-1}}\,\lambda^p_{2s}\,dxdt$$

$$+c_{10}\frac{\mu}{k-m_\rho}(I_\rho(s,\tau)+I_\rho(2s,\tau)), \tag{4.7}$$

c_{10} *is a constant that does not depend on* s,k,d.

The **Proof of Corollary 4.1** immediately follows from Lemmas 4.1–4.3.

5 Improvement of the integral and pointwise estimates

In this Section we will show how to obtain more accurate pointwise estimate for the solution of the boundary value problem (2.1) and improve the bounds for the integrals $I_\rho, I_{\rho,\mu}$ defined in the beginning of Section 4. To this end, first, we introduce the notation:

$$\mathsf{R}_{\rho,\mu}(C,s)=Cs\,(k^{p-1}+k)\mu\,d^{n-p}+C\mathsf{A}_1 s\left[\frac{(k^{p-1}+k)sd^{n-p}}{\rho^n}\right]^{\frac{q}{p-1}}\rho^{n+\frac{p(1-a)}{p-1}},$$

where C denotes an arbitrary positive constant and A_1 is defined in (4.6).

The following result holds.

Lemma 5.1. *Let* $u=u(x,t)$ *be the solution of problem (2.1). Then there exist positive constants* $\mathsf{K}_1,\mathsf{K}_3,\mathsf{K}_4,\mathsf{K}_5$ *depending on* R, T, n, *such that if, for every* $\rho\in[2d,\rho']$, $\mu\in(0,k-m_\rho),\tau\in[0,T]$,

$$u(x,t)\leqslant\mathsf{K}_3(k^{p-1}+k)\frac{s'd^{n-p}}{|x|^n}\quad\forall\,(x,t)\in Q_T,\ \ |x|\le\rho', \tag{5.1}$$

$$I_\rho(s',\tau) \leqslant \mathsf{K}_4 s'(k-m_\rho)(k^{p-1}+k)d^{n-p}, \tag{5.2}$$

$$I_{\rho,\mu}(s',\tau) \leqslant \mathsf{R}_{\rho,\mu}(\mathsf{K}_5, s'). \tag{5.3}$$

where s', ρ' *are such that*

$$\mathsf{K}_1(k^{p-1}+k)^{\frac{2-p}{p-1}}(\rho')^p\left(\frac{d}{\rho'}\right)^{\frac{n-p}{p-1}(2-p)} \leqslant s' \leqslant T, \tag{5.4}$$

then, for every $\rho \in [2d,\rho'']$, $\mu \in (0, k-m_\rho), \tau \in [0,T]$, u *satisfies the following estimates*

(i)

$$u(x,t) \leqslant \mathsf{K}_3(k^{p-1}+k)\frac{s'' d^{n-p}}{|x|^n} \quad \forall\,(x,t) \in Q_T,\ |x| \leqslant \rho'', \tag{5.5}$$

(ii)

$$I_\rho(s'',\tau) \leqslant \mathsf{K}_4 s''(k-m_\rho)(k^{p-1}+k)d^{n-p}, \tag{5.6}$$

(iii)

$$I_{\rho,\mu}(s'',\tau) \leqslant \mathsf{R}_{\rho,\mu}(\mathsf{K}_5, s''), \tag{5.7}$$

where $s'' = s'/4$, $\rho'' = \rho'/2$.

Proof of Lemma 5.1. Statement **(i)** of Lemma 5.1 is evident.

Let us prove statement **(ii)** of the lemma. First, we notice that the estimate for $I_\rho(s'',\tau)$ with $s'' = T$ is given by (3.9).

Let now $s := s'' < T$. Consider the right-hand side of (4.1). Since, for any $t \in (0,T)$, $\mathcal{E}_\rho \subset \mathcal{B}_\rho$, then using (5.1), (5.4), and Poincare's inequality we get:

$$\begin{aligned}
\frac{c_7}{s''}\int\limits_{\mathcal{E}_\rho} \mathsf{u}_\rho^2\lambda_{2s''}^p(t-\tau)\,dxdt &\leqslant \frac{c_7}{s''}(\mathsf{K}_3(k^{1-p}+k)s'd^{n-p})^{2-p}\int\limits_{\mathcal{E}_\rho}|x|^{-n(2-p)}\mathsf{u}_\rho^p\lambda_{2s''}^p\,dxdt \\
&\leqslant \frac{C}{s''}(\mathsf{K}_3(k^{p-1}+k)s''d^{n-p})^{2-p}\rho^{-n(2-p)}\int\limits_{\mathcal{E}_\rho}\mathsf{u}_\rho^p\lambda_{2s''}^p\,dxdt \\
&\leqslant \frac{C}{s''}(\mathsf{K}_3(k^{p-1}+k)s''d^{n-p})^{2-p}\rho^{p-n(2-p)}\int\limits_{\mathcal{E}_\rho}|\nabla u|^p\lambda_{2s'}^p\,dxdt \\
&\leqslant \frac{C(k^{p-1}+k)^{2-p}}{(s'')^{p-1}}(\mathsf{K}_3 d^{n-p})^{2-p}\rho^{p-n(2-p)}I_\rho(s',\tau) \leqslant \\
&\leqslant C_{11}\frac{\mathsf{K}_3^{2-p}}{\mathsf{K}_1^{p-1}}I_\rho(s',\tau).
\end{aligned} \tag{5.8}$$

Then from (4.1), (5.8), and (5.2) we deduce that

$$I_\rho(s'',\tau) \leqslant c_7\mathsf{K}_2 s''(k-m_\rho)k^{p-1}d^{n-p} + c_{11}\frac{\mathsf{K}_3^{2-p}}{\mathsf{K}_1^{p-1}}I_\rho(s',\tau)$$

$$\leqslant c_7\mathsf{K}_2 s''(k-m_\rho)k^{p-1}d^{n-p} + c_{11}\frac{\mathsf{K}_3^{2-p}}{\mathsf{K}_1^{p-1}}\mathsf{K}_4 s'(k-m_\rho)k^{p-1}d^{n-p}. \tag{5.9}$$

Now let $\mathsf{K}_2,\ \mathsf{K}_3,\ \mathsf{K}_4$ be a positive constant such that

$$c_7\frac{\mathsf{K}_2}{\mathsf{K}_4} + 4c_{11}\frac{\mathsf{K}_3^{2-p}}{\mathsf{K}_1^{p-1}} \leqslant 1.$$

Then inequality (5.9) immediately implies (5.6).

Now let us prove statement **(iii)** of the lemma. We will apply Corollary 4.1. Consider the first integral on the right-hand side of (4.7) with $s := s''$. As in the proof of the inequality (5.8), from Poincare's inequality, (5.4), and (5.1) we obtain:

$$\frac{1}{s''}\int\limits_{\mathcal{E}_{\rho,\mu}} [\mathsf{u}_\rho]_\mu^2 \lambda_{2s''}^p(t-\tau)\,dxdt \leqslant c_{12}\frac{\mathsf{K}_3^{2-p}}{\mathsf{K}_1^{p-1}} I_{\rho,\mu}(2s'',\tau). \tag{5.10}$$

For the second term on the right-hand side of (4.7) we have:

$$c_{10}\mathsf{A}_1 \int\limits_{\mathcal{E}^+_{\rho,\mu}} |x|^{\frac{p(1-a)}{p-1}} \mathsf{u}_\rho^{\frac{q}{p-1}} \lambda_{2s''}^p\,dxdt \leqslant$$

$$\leqslant c_{13}\mathsf{K}_1^{\frac{q}{q-1}}\mathsf{A}_1 s''\left[(k^{p-1}+1)s''d^{n-p}\right]^{\frac{q}{p-1}} \int\limits_\rho^R \xi^{\frac{p(1-a)}{p-1}-\frac{nq}{p-1}+n-1}\,d\xi \leqslant$$

$$\leqslant c_{13}\mathsf{K}_1^{\frac{q}{q-1}}\mathsf{A}_1 s''\left[\frac{(k^{p-1}+k)s''d^{n-p}}{\rho^n}\right]^{\frac{q}{p-1}} \rho^{\frac{p(1-a)}{p-1}+n}. \tag{5.11}$$

Here it was supposed that:

$$\frac{p(1-a)}{p-1} - \frac{nq}{p-1} + n < 0. \tag{5.12}$$

It is easy to see that there exist a, q that satisfy (5.12) along with (4.4). For example, $a = 2 - \frac{2n}{n+1},\ q = 2\frac{p}{n}(\frac{2n}{n+1}-1)+p-1$.

Now statement **(iii)** of the lemma follows from (4.7), (5.3), and (5.10)–(5.12). Lemma 5.1 is proved.

6 Proof of the main pointwise estimate

First, we introduce the sequences $\rho_j^{(1)}$, $\rho_j^{(2)}$ defined in (3.12) and the cut-off function $\xi_j(x)$ (see the beginning of the proof of Lemma 3.2). Then we introduce the sequences $t_j^{(1)}$ and $t_j^{(2)}$ defined as follows:

$$t_j^{(1)}(s,\tau) = \tau - \frac{s}{16} - \theta_j,\ \ t_j^{(2)}(s,\tau) = \tau + \frac{s}{16} + \theta_j,$$

where $\tau \in [0,T]$, $\rho \in [8d, R]$ and

$$\theta_j = (1-2^{1-2j})\rho^p \left(\frac{s(k^{p-1}+k)d^{n-p}}{\rho^n}\right)^{2-p}.$$

Here s is the parameter satisfying (5.4) with the constant K_1 from Lemma 3.3. Finally, we introduce the cut-off functions $\chi_j(t,s,\tau)$ $(j=1,2,\dots)$ such that: **(i)** $0 \leqslant \chi_j(t,s,\tau) \leqslant 1$; **(ii)** $\chi_j(t,s,\tau)=1$ if $t \in [t_j^{(1)}(s,\tau), t_j^{(2)}(s,\tau)]$; **(iii)**$\chi_j(t,s,\tau)=0$ if $t \notin [t_{j+1}^{(1)}(s,\tau), t_{j+1}^{(2)}(s,\tau)]$; **(iv)** $\left|\frac{d\chi_j(t,s,\tau)}{dt}\right| \leqslant \frac{2^{2j}}{\rho^2}$.

Plugging the function $\varphi(x,t) = \mathsf{u}_\rho^l(x,t)\xi_j^m(x)\chi_j^m(t,s,\tau)$, $l, m > 0$, in the integral identity (2.4) we obtain:

$$M_j^2 \leqslant \gamma_j M_{j+1}^{(2-p)\frac{n}{p}} \left(\frac{M_{j+1}^{2-p}}{\theta_j} + \frac{1}{\rho^p}\right)^{\frac{n+p}{p}} \int\limits_{\mathcal{E}_{\rho,M_{j+1}}} [\mathsf{u}_\rho]_{M_{j+1}}^p \chi_j(t,s,\tau)\,dxdt$$

$$\leqslant C\rho^p \left(\frac{M_{j+1}^{2-p}}{\theta_j} + \frac{1}{\rho^p}\right)^{\frac{n+p}{p}} \int\limits_{\mathcal{E}_{\rho,M_{j+1}}} |\nabla u|^p \chi_j(t,s,\tau)\,dxdt \leqslant C\rho^{-n}\left(\frac{M_{j+1}^{2-p}}{\theta_j}\rho^p + 1\right)^{\frac{n+p}{p}}$$

$$\times I_{\rho,M_{j+1}}(s,\tau).$$

Here M_j is defined by (3.14). Then, using condition (5.4) we get

$$s^{-\frac{1-a}{p-1}} \rho^{p\frac{1-a}{p-1}} \left((k^{p-1}+k)^{\frac{1}{p-1}} \left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}\right)^{(2-p)\frac{1-a}{p-1}} \leqslant \mathsf{K}_1^{-\frac{1-a}{p-1}}.$$

From these inequalities analogously to the proof of Lemma 5.1, we obtain

$$M_j^2 \leqslant C\rho^{-n}\left(\frac{M_{j+1}^{2-p}}{\theta_j}\rho^p + 1\right)^{\frac{n+p}{p}}$$

$$\times \mathsf{K}_5 \left\{ s(k^{p-1}+k)d^{n-p}M_{j+1} + \mathsf{K}_1^{-\frac{1-a}{p-1}} M_{j+1}^{\frac{p-q}{p-1}} \left[\frac{(k^{p-1}+k)sd^{n-p}}{\rho^n}\right]^{\frac{q+p-2}{p-1}} \rho^n \right\}. \quad (6.1)$$

We set:

$$z_j = M_j \frac{\rho^n}{s(k^{p-1}+k)d^{n-p}}.$$

Then inequality (6.1) implies that

$$z_j^2 \leqslant C\mathsf{K}_5 \left(z_{j+1}^{2-p} \left(\frac{s(k^{p-1}+k)d^{n-p}}{\rho^n}\right)^{2-p} \frac{\rho^p}{\theta_j} + 1\right)^{\frac{n+p}{p}} \left\{ z_{j+1} + \mathsf{K}_1^{-\frac{1-a}{p-1}} z_{j+1}^{\frac{p-q}{p-1}} \right\}.$$

Finally, we get:

$$z_j^2 \leqslant c_{14}\mathsf{K}_5 \left(z_{j+1}^{2-p} \frac{1}{1-2^{1-2j}} + 1 \right)^{\frac{n+p}{p}} \left\{ z_{j+1} + \mathsf{K}_1^{-\frac{1-a}{p-1}} z_{j+1}^{\frac{p-q}{p-1}} \right\}. \tag{6.2}$$

Now we make use of the following result (see Lemma 1.6 in in [25] Ch.8).

Lemma 6.1. *Let* $\{z_j\} \subset \mathbb{R}$, *be a bounded sequence such that*

$$0 < z_j \leqslant c^{(1)} a^j \sum_{i=1}^{I} A_i \, z_{i+1}^{\sigma_i}, \tag{6.3}$$

for every $j = 1, 2, \dots,$ *where* $c^{(1)}$, $A_1, \dots, A_I$, a *are positive constants,* $\sigma_i \in (0,1)$ $(i = 1, \dots, I)$*. Let* B *be a positive number. Then there exist* $\beta > 0$*, depending on* B, $c^{(1)}$, a, $\sigma_1, \dots, \sigma_I$, I, *only, such that the condition*

$$\sum_{i=1}^{I} A_i \leqslant \beta$$

implies that $z_1 \leqslant B$.

Applying Lemma 6.1 to (6.2) and choosing the appropriate constants K_1, K_5 we derive: $z_1 < \mathsf{K}_3$. This means that

$$M_1 \leqslant \mathsf{K}_3 s (k^{p-1} + k) \frac{d^{n-p}}{\rho^n}. \tag{6.4}$$

We choose

$$R_0 = T^{\frac{1}{p}}, \ R = \left[(k^{p-1} + k)^{-\frac{2-p}{p(p-1)}} R_0 d^{-\frac{n-p}{p-1}\frac{2-p}{p}} \right]^{\frac{p(p-1)}{p+n(p-2)}}.$$

We denote by ρ_i the following decreasing sequence of numbers

$$\rho_i = \left[(k^{p-1} + k)^{-\frac{2-p}{p(p-1)}} \frac{R_0}{2^i} d^{-\frac{n-p}{p-1}\frac{2-p}{p}} \right]^{\frac{p(p-1)}{p+n(p-2)}}, \ \ i = 0, 1, \dots I.$$

Then

$$(k^{p-1} + k)^{2-p} \rho_i^p \left(\frac{d}{\rho_i} \right)^{\frac{n-p}{p-1}(2-p)} = \frac{R_0^p}{2^{ip}}.$$

Let us denote by

$$s_i = \frac{R_0^p}{2^{ip}}, \ \ i = 0, 1, \dots, I.$$

For $s_0 = T$, $\rho_0 = R$, estimates (5.1)–(5.3) are valid. Let I be such that $\rho_I \leqslant 8d < \rho_{I-1}$ and (x_0, t_0) be an arbitrary point from Q_T such that $\rho_I < |x_0| < \rho_0$. We introduce i_0, $1 \leqslant i_0 \leqslant I - 1$ such that

$$\rho_{i_0+1} \leqslant |x_0| \leqslant \rho_{i_0}.$$

Then from (6.4) and definition of s_{i_0} we have

$$u(x_0,t_0) \leqslant \mathsf{K}_3(k^{p-1}+k)s_{i_0}\frac{d^{n-p}}{\rho^n_{i_0}} = \mathsf{K}_3(k^{p-1}+k)\frac{R_0^p}{2^{i_0 p}}\frac{d^{n-p}}{\rho^n_{i_0}} \leqslant \mathsf{K}_3(k^{p-1}+k)\frac{R_0^p}{2^{i_0 p}}\frac{d^{n-p}}{|x_0|^n}$$

$$\leqslant C\mathsf{K}_3\frac{k^{2-p}(k^{p-1}+k)}{T}\frac{d^{\frac{n-p}{p-1}(2-p)+n-p}}{|x_0|^{n-\frac{p+n(p-2)}{p-1}}} = C\mathsf{K}_3 k^{2-p}(k^{p-1}+k)\left(\frac{d}{|x_0|}\right)^{\frac{n-p}{p-1}}.$$

Let us prove the desired inequality for $|x_0| < \rho_I$. From the comparison principle we obtain:

$$u(x_0,t_0) \leqslant k = k\left(\frac{d}{|x_0|}\right)^{\frac{n-p}{p-1}}\left(\frac{|x_0|}{d}\right)^{\frac{n-p}{p-1}} \leqslant Ck\left(\frac{d}{|x_0|}\right)^{\frac{n-p}{p-1}}.$$

Let now $|x_0| > \rho_0 = R$. Then from the preliminary pointwise estimate (3.11) we deduce

$$u(x_0,t_0) \leqslant c_4\left(k+k^{p-1}\right)\frac{d^{n-p}}{|x_0|^n} = c_4\left(k+k^{p-1}\right)\frac{d^{n-p}}{|x_0|^{n-\frac{p+n(p-2)}{p-1}}}\frac{1}{|x_0|^{\frac{p+n(p-2)}{p-1}}}$$

$$\leqslant c_4\left(k+k^{p-1}\right)\frac{k^{2-p}}{T}\frac{d^{n-p+\frac{n-p}{p-1}(2-p)}}{|x_0|^{n-\frac{p+n(p-2)}{2}}} \leqslant C\left(k+k^{p-1}\right)k^{2-p}\left(\frac{d}{|x_0|}\right)^{\frac{n-p}{p-1}}$$

and Theorem 2.2 is proved.

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7 Appendix I. Proof of Lemma 4.3

Consider the integral $J_{\rho,\mu}(s,\tau)$. Let

$$\mathfrak{f}(x) = |x|^{1-a}\left[\frac{k^{p-1}sd^{n-p}}{\rho^n}\right]^{\frac{2-p}{p}}\left[k\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}\right]^{\frac{(p-2)(1-a)}{p}}\frac{\mu^{\frac{q}{p}}}{s^{\frac{a}{p}}}.$$

Then applying Young's inequality we have:

$$J_{\rho,\mu}(s,\tau) = \frac{\mu}{s}\int\limits_{\mathcal{E}^+_{\rho,\mu}}(\mathsf{u}_\rho-\mu)\lambda_s^{p-1}(t-\tau)\,\frac{\mathfrak{f}(x)}{\mathfrak{f}(x)}\,dxdt \leqslant$$

$$\leqslant \mathsf{A}_1\int\limits_{\mathcal{E}^+_{\rho,\mu}}|x|^{\frac{p(1-a)}{p-1}}\mathsf{u}_\rho^{\frac{q}{p-1}}\lambda^p_{2s}(t-\tau)\,dxdt+$$

$$+C\mathsf{A}_2\rho^{-ap}\int\limits_{\mathcal{E}^+_{\rho,\mu}}|x|^{-p(1-a)}\mathsf{u}_\rho^{-q}(\mathsf{u}_\rho-\mu)^p\lambda_s^p(t-\tau)\,dxdt, \tag{7.1}$$

where

$$\mathsf{A}_2\equiv\mathsf{A}_2(s,\mu,d,\rho,k)=\frac{\mu^q}{s^a}\left[k\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}\right]^{(p-2)(1-a)}\left[\frac{k^{p-1}sd^{n-p}}{\rho^n}\right]^{2-p}\rho^{ap}$$

For the second integral on the right–hand side of the last inequality, using Poincaré's inequality we derive:

$$\int\limits_{\mathcal{E}^+_{\rho,\mu}}|x|^{-p(1-a)}\mathsf{u}_\rho^{-q}(\mathsf{u}_\rho-\mu)^p\lambda_s^p(t-\tau)\,dxdt\leqslant$$

$$\leqslant C\,\rho^{ap}\int\limits_{\mathcal{E}^+_{\rho,\mu}}\mathsf{u}_\rho^{-q}\,|\nabla u|^p\,\lambda_s^p\,dxdt. \tag{7.2}$$

Consider now the integral on the right–hand side of (7.2). In the integral identity (2.4) we set

$$\varphi(x,t)=\left(\frac{1}{\mu^{q-1}}-\frac{1}{[\max\{\mathsf{u}_\rho,\mu\}]^{q-1}}\right)\lambda_s^p(t-\tau)-$$

$$-\left(\frac{1}{\mu^{q-1}}-\frac{1}{(k-m_\rho)^{q-1}}\right)\frac{1}{k-m_\rho}\mathsf{u}_\rho(x,t)\lambda_s^p(t-\tau).$$

Then we obtain

$$\int\limits_{\mathcal{E}^+_{\rho,\mu}}\mathsf{u}_\rho^{-q}\,|\nabla u|^p\,\lambda_s^p(t-\tau)\,dxdt\leqslant C\frac{\mu^{1-q}}{k-m_\rho}\,[I_\rho(s,\tau)+I_\rho(2s,\tau)]+$$

$$+\frac{C}{s}\mu^{1-q}\int\limits_{\mathcal{E}^+_{\rho,\mu}}(\mathsf{u}_\rho-\mu)\lambda_s^{p-1}(t-\tau)\,dxdt. \tag{7.3}$$

Multiplying (7.3) by A_2 we get:

$$\mathsf{A}_2\int\limits_{\mathcal{E}^+_{\rho,\mu}}\mathsf{u}_\rho^{-q}\,|\nabla u|^p\,\lambda_s^p(t-\tau)\,dxdt\leqslant C\mathsf{A}_3\frac{\mu}{k-m_\rho}\,[I_\rho(s,\tau)+I_\rho(2s,\tau)]+$$

$$+C\frac{\mu^{1-q}}{s}\mathsf{A}_2\int\limits_{\mathcal{E}^+_{\rho,\mu}}(\mathsf{u}_\rho-\mu)\lambda_s^{p-1}(t-\tau)\,dxdt. \tag{7.4}$$

Moreover, due to (3.20) and (3.7),

$$\mathsf{A}_3\equiv\mathsf{A}_3(s,\mu,d,\rho,k)=\frac{1}{s^{a+p-2}}\left[k\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}\right]^{(2-p)(a+p-2)}\rho^{ap+p(p-2)}\leqslant\mathsf{K}_1^{-(a+p-2)}.$$

It is also follows (3.20) that

$$s^{-ap}\left[k\left(\frac{d}{\rho}\right)^{\frac{n-p}{p-1}}\right]^{p(2-p)(a-1)}\left(k^{p-1}sd^{n-p}\rho^{-n}\right)^{(2-p)p}\rho^{ap^2}\leqslant \mathsf{K}_1^{2(2-p)-ap}.$$

Then applying Young's and Poincaré's inequalities, from (7.4), we get:

$$\mathsf{A}_2\int\limits_{\mathcal{E}^+_{\rho,\mu}}\mathsf{u}_\rho^{-q}\,|\nabla u|^p\,\lambda_s^p(t-\tau)\,dxdt\leqslant C\frac{\mu}{k-m_\rho}\left[I_\rho(s,\tau)+I_\rho(2s,\tau)\right]$$
$$+C\mathsf{A}_1\int\limits_{\mathcal{E}^+_{\rho,\mu}}|x|^{\frac{p(1-a)}{p-1}}\,u_\rho^{\frac{\overline{q}}{p-1}}\,\lambda_{2s}^p(t-\tau)\,dxdt \tag{7.5}$$

The statement of our Lemma follows from (7.1), (7.2), (7.5).

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