ELSEVIER
Nonlinear Analysis 55 (2003) 641 – 656
Nonlinear Analysis
www.elsevier.com/locate/na

On the stability of invariant sets of functional differential equations☆

Stephen R. Bernfeld[a,✠], Constantin Corduneanu[a,*],
Alexander O. Ignatyev[b]

[a] University of Texas at Arlington, Arlington, TX 76019-0408, USA
[b] Institute for Applied Mathematics & Mechanics, R. Luxemburg Street, 74, Donetsk-83114, Ukraine

Received 17 December 2002; accepted 13 August 2003

Abstract

A system of functional differential equations with delay $\mathrm{d}z/\mathrm{d}t = Z(t, z_t)$, where $z = (x, y), x \in R^n$, $y \in R^m$, and Z is the vector-valued functional, is considered. It is supposed that this system has a positive invariant set $x = 0$. Definitions of its stability, asymptotic stability, and uniform asymptotic stability are given. Theorems on the uniform asymptotic stability are formulated and proved.

MSC: primary 34K20

Keywords: Functional differential equations; Lyapunov functionals; Uniform asymptotic stability

---

1. Introduction

Let $t \in R_+ = [0, \infty)$, $x = (x^1, \dots, x^n) \in R^n$, $|x| = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$, $y = (y^1, \dots, y^m)$, $|y| = \sqrt{(y^1)^2 + \cdots + (y^m)^2}$, $z = (x, y) = (z^1, \dots, z^{n+m}) \in R^{n+m}$. For a given $h > 0$, C denotes the space of continuous functions mapping $[-h, 0]$ into R^{n+m}. Let $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^{n+m}) = (\psi, \lambda) \in C$, where $\psi = (\psi^1, \dots, \psi^n), \lambda = (\lambda^1, \dots, \lambda^m)$. Denote

$$\|\psi\| = \sup(|\psi^i(\theta)| \text{ under } -h \leqslant \theta \leqslant 0,\ 1 \leqslant i \leqslant n),$$

$$\|\lambda\| = \sup(|\lambda^j(\theta)| \text{ under } -h \leqslant \theta \leqslant 0,\ 1 \leqslant j \leqslant m),$$

☆ This work is being supported by COBASE Grant of the National Academies (UTA-3501-02).

* Corresponding author.

E-mail addresses: cordun@uta.edu (C. Corduneanu), ignat@iamm.ac.donetsk.ua (A.O. Ignatyev).

✠ Deceased.

0362-546X/$ - see front matter
doi:10.1016/j.na.2003.08.002

$$\|\varphi\| = \max(\|\psi\|, \|\lambda\|),$$

$$C_H = \{\varphi \in C : \|\psi\| \leqslant H,\ \|\lambda\| < +\infty\}.$$

If z is a continuous function of u defined on $-h \leqslant u < A, A > 0$, and if t is a fixed number satisfying $0 \leqslant t < A$, then z_t denotes the restriction of z to the segment $[t-h, t]$ so that $z_t = (z_t^1, \dots, z_t^{n+m}) = (x_t, y_t)$ is an element of C defined by $z_t(\theta) = z(t+\theta)$ for $-h \leqslant \theta \leqslant 0$.

Consider a system of functional differential equations

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = Z(t, z_t). \tag{1.1}$$

In this system $\mathrm{d}z/\mathrm{d}t$ denotes the right-hand derivative of z at t, t is time, and $Z(t,\varphi) = (X(t,\varphi), Y(t,\varphi)) \in R^{n+m}$ is defined on $R_+ \times C_H$; $X \in R^n, Y \in R^m$. Such systems were studied in [1,3–6,8,10–15,17,18,20–27].

According to Burton [3], we denote by $z(t_0,\varphi) = (x(t_0,\varphi), y(t_0,\varphi))$ a solution of (1.1) with initial condition $\varphi \in C_H$, where $z_{t_0}(t_0,\varphi) = \varphi$ and we denote by $z(t,t_0,\varphi)$ the value of $z(t_0,\varphi)$ at t and $z_t(t_0,\varphi) = z(t+\theta, t_0, \varphi)$, $-h \leqslant \theta \leqslant 0$.

It is assumed that the vector-valued functional $Z(t,\varphi)$ is continuous on $R_+ \times C_H$ so that a solution will exist for each continuous initial condition. We suppose that each solution $z(t_0,\varphi)$ is defined for those $t \geqslant t_0$, such that $\|x_t(t_0,\varphi)\| < H$.

Let $V(t,\varphi)$ be a continuous functional defined for $t \geqslant 0, \varphi \in C_H$. The upper right-hand derivative of V along solutions of (1.1) is defined to be [3,17,23,24]

$$\dot{V}(t, z_t(t_0,\varphi)) = \frac{\mathrm{d}V(t, z_t(t_0,\varphi))}{\mathrm{d}t} = \overline{\lim_{\Delta t \to +0}} \{V(t+\Delta t, z_{t+\Delta t}(t_0,\varphi)) - V(t, z_t(t_0,\varphi))\} \frac{1}{\Delta t}.$$

If V satisfies a Lipschitz condition in the second argument, then this limit is finite.

In [7,19,29,31–35] the partial stability results were obtained for ordinary differential equations, and in [9,21,22,34,35] the partial stability results were obtained for functional differential equations with delay. The aim of this paper is to prove analogous results for the stability of the set $x = 0$.

2. Main definitions. Theorem on the asymptotic stability

Let $M \subset C$ be any set. The value $\rho(M,\xi) = \inf_{\varphi \in M} \|\varphi - \xi\|$ is the distance between any function $\xi \in C$ and M. The set

$$S(M, r) := \{\xi \in C : \rho(M,\xi) < r\}$$

is called a neighbourhood of M with the radius r.

Definition 2.1. A set $M \subset C$ is called a positive invariant set of system (1.1) if $t_0 \in R_+$, $\varphi \in M$ implies $z_t(t_0,\varphi) \in M$ for each $t \geqslant t_0$.

In the next definitions it is assumed that M is a positive invariant set of system (1.1).

Definition 2.2. A set $M \subset C$ is called a stable set of system (1.1) if for every $\varepsilon > 0$ and $t_0 \geqslant 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\varphi \in C$, $\rho(M, \varphi) < \delta$ implies $\rho(M, z_t(t_0, \varphi)) < \varepsilon$ for all $t \geqslant t_0$.

Definition 2.3. If δ does not depend on t_0 in Definition 2.2 (i.e., $\delta = \delta(\varepsilon)$), then the set M is called uniformly stable.

Definition 2.4. A set $M \subset C$ is called an attractive set of system (1.1) if for every $t_0 \geqslant 0$ there exists $\eta = \eta(t_0) > 0$, and for every $\varepsilon > 0$ and $\varphi \in S(M, \eta)$ there exists $\sigma = \sigma(\varepsilon, t_0, \varphi) > 0$ such that $\rho(M, z_t(t_0, \varphi)) < \varepsilon$ for any $t \geqslant t_0 + \sigma$. In this case we say that the domain of attraction of M at t_0 contains the set $S(M, \eta)$. In other words, a set M is called attractive if

$$\lim_{t \to \infty} \rho(M, z_t(t_0, \varphi)) = 0. \tag{2.1}$$

Definition 2.5. A set $M \subset C$ is called uniformly attractive if for some $\eta > 0$ and any $\varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon) > 0$ such that $\rho(M, z_t(t_0, \varphi)) < \varepsilon$ for all $t_0 \geqslant 0$, $\varphi \in S(M, \eta)$ and $t \geqslant t_0 + \sigma$. In other words, a set M is called uniformly attractive if (2.1) holds uniformly in $t_0 \in R_+, \varphi \in S(M, \eta)$.

Definition 2.6. A set M is called

- asymptotically stable if it is stable and attractive;
- uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

Let $\mathscr{K}$ denote the class of Hahn's functions [16,30], that is $r \in \mathscr{K}$ if $r : R_+ \to R_+$ is a continuous monotonically increasing function such that $r(0) = 0$. Note that in [4,5,18] these functions are called *wedges*.

Henceforth we shall consider the set M of the form

$$M := \{\varphi \in C : \|\psi\| = 0, \|\lambda\| < \infty\}; \tag{2.2}$$

then $\rho(M, z_t) = \|x_t\|$, $\rho(M, \varphi) = \|\psi\|$. If $X(t, \varphi) \equiv 0$ for $\varphi \in M$, then the solution $z_t(t_0, \varphi)$ of system (1.1) satisfies the condition $\|x_t(t_0, \varphi)\| \equiv 0$. In other words, M is an invariant set of system (1.1). In this case the following theorem holds.

Theorem 2.1. *Let functional differential equations* (1.1) *be such that there is a continuous functional* $V : R_+ \times C_H \to R$ *which satisfies*

$$a(\|x_t\|) \leqslant V(t, z_t) \leqslant b(\|x_t\|), \quad a, b \in \mathscr{K}, \tag{2.3}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant -c(\|x_t\|), \quad c \in \mathscr{K}. \tag{2.4}$$

along solutions of system (1.1). *Then* M *is a positive invariant and asymptotically stable set of system* (1.1), *and there exists* $H_0 > 0$ ($H_0 < H$) *such that the domain* *of attraction of M contains the set C_{H_0}. The identities*

$$X_i(t,0,\lambda) \equiv 0 \quad (1 \leqslant i \leqslant n) \tag{2.5}$$

are also valid.

Proof. If $\varphi \in M$, then $V(t, z_t(t_0, \varphi)) \equiv 0$ in view of (2.3), (2.4), whence it follows that $\|x_t\| = 0$ for $t \geqslant t_0$, and this proves the positive invariance of M. Note that conditions $\|x_t(t_0, \varphi)\| = 0$ and (2.5) are equivalent if $\varphi \in M$.

Pick any $\varepsilon_1 > 0$ ($\varepsilon_1 < H$), and choose $\delta = b^{-1}(a(\varepsilon_1))$, where b^{-1} is the inverse of the function b. If $\|\psi\| < \delta$, then from properties (2.3), (2.4) we have

$$a(\|x_t\|) \leqslant V(t, z_t) \leqslant V(t_0, \varphi) \leqslant b(\|\psi\|) < b(b^{-1}(a(\varepsilon_1))) = a(\varepsilon_1),$$

whence it follows $\|x_t\| < \varepsilon_1$ for $t > t_0$. This proves the uniform stability of M.

The uniform stability of M implies that for every positive ε_2 ($\varepsilon_2 < H$) there exists $H_0 = H_0(\varepsilon_2) > 0$ such that for any $t_0 \in R_+$, $\varphi \in C_{H_0}$ the inequality $\|x_t\| < \varepsilon_2$ holds for $t > t_0$. Consider the trajectory $z(t_0, \varphi)$ of Eqs. (1.1). Since $\varphi \in C_{H_0}$, then $\|\psi\| \leqslant H_0$, whence

$$V(t_0, \varphi) \leqslant b(\|\psi\|) \leqslant b(H_0). \tag{2.6}$$

Let ε be any sufficiently small positive number. Denote $T(\varepsilon) := b(H_0)/c(b^{-1}(a(\varepsilon)))$. Let us show that there exists $\sigma \in [0, T]$ such that

$$V(t_0 + \sigma, z_{t_0+\sigma}) < a(\varepsilon). \tag{2.7}$$

Suppose the opposite: for any $\sigma \in [0, T]$ the inequality $V(t_0 + \sigma, z_{t_0+\sigma}) \geqslant a(\varepsilon)$ holds, whence we have

$$\|x_t\| \geqslant b^{-1}(V(t, z_t)) \geqslant b^{-1}(a(\varepsilon)) \tag{2.8}$$

for $t_0 \leqslant t \leqslant t_0 + T$. From inequalities (2.4), (2.8) we derive that

$$\frac{\mathrm{d}V(t, z_t)}{\mathrm{d}t} \leqslant -c(b^{-1}(a(\varepsilon))),$$

whence $0 < a(\|x_t\|) \leqslant V(t, z_t) \leqslant V(t_0, \varphi) - c(b^{-1}(a(\varepsilon)))(t - t_0)$. For $t - t_0 = T$ we have $V(t_0, \varphi) - b(H_0) > 0$, but it contradicts relations (2.6). This proves the existence of $\sigma \in [0, T]$ such that inequality (2.7) is valid. Since V does not increase along the solution $z(t_0, \varphi)$, then $V(t, z_t) < a(\varepsilon)$ for $t \geqslant t_0 + \sigma$. This implies $\|x_t(t_0, \varphi)\| < \varepsilon$ for $t \geqslant t_0 + \sigma$. Hence M is uniformly attractive, and its domain of attraction contains the set C_{H_0}. This completes the proof. □

3. The converse theorem

Let us show that Theorem 2.1 is invertible. The following lemma is needed in the proof of such statement.

Lemma. *Suppose that functionals* X_i $(1 \leqslant i \leqslant n)$ *of system* (1.1) *satisfy conditions*

$$|X_i(t,\psi_1,\lambda_1) - X_i(t,\psi_2,\lambda_2)| \leqslant L\|\psi_1 - \psi_2\| \quad L = const \tag{3.1}$$

in the domain $R_+ \times C_H$. *Let* $t_0 \geqslant 0$ *be the initial moment of time, and* $\varphi_1 = (\psi_1,\lambda_1)$, $\varphi_2 = (\psi_2,\lambda_2)$ *be the initial functions satisfying conditions* $\varphi_1 \in C_H$, $\varphi_2 \in C_H$. *For those values* $t \geqslant t_0$ *for which* $z_t(t_0,\varphi_1)$ *and* $z_t(t_0,\varphi_2)$ *belong to* C_H, *the inequality*

$$\|x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2)\| \leqslant \|\psi_1 - \psi_2\| e^{L(t-t_0)} \tag{3.2}$$

holds.

Proof. Along solutions of system (1.1) we have

$$\frac{dx^i(t,t_0,\varphi_1)}{dt} = X_i(t,x_t(t_0,\varphi_1),y_t(t_0,\varphi_1)) \quad (i = 1,\dots,n),$$

$$\frac{dx^i(t,t_0,\varphi_2)}{dt} = X_i(t,x_t(t_0,\varphi_2),y_t(t_0,\varphi_2)) \quad (i = 1,\dots,n)$$

for $t > t_0$, whence we derive

$$\begin{aligned} &x^i(t+\theta,t_0,\varphi_1) - x^i(t+\theta,t_0,\varphi_2) \\ &\quad = x^i(t_0+\theta,t_0,\varphi_1) - x^i(t_0+\theta,t_0,\varphi_2) + \int_{t_0+\theta}^{t+\theta} [X_i(\tau,x_\tau(t_0,\varphi_1),y_\tau(t_0,\varphi_1)) \\ &\qquad - X_i(\tau,x_\tau(t_0,\varphi_2),y_\tau(t_0,\varphi_2))]\, d\tau, \quad \theta \in [-h,0]; \end{aligned}$$

$$\begin{aligned} &|x^i(t+\theta,t_0,\varphi_1) - x^i(t+\theta,t_0,\varphi_2)| \\ &\quad \leqslant \|\psi_1 - \psi_2\| + \int_{t_0+\theta}^{t+\theta} |X_i(\tau,x_\tau(t_0,\varphi_1),y_\tau(t_0,\varphi_1)) \\ &\qquad - X_i(\tau,x_\tau(t_0,\varphi_2),y_\tau(t_0,\varphi_2))|\, d\tau. \end{aligned}$$

In view of

$$\sup_{\substack{\theta\in[-h,0]\\ 1\leqslant i\leqslant n}} |x^i(t+\theta,t_0,\varphi_1) - x^i(t+\theta,t_0,\varphi_2)| = \|x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2)\|$$

and property (3.1), we obtain

$$\|x_t(t_0,\varphi_1) - x_t(t_0,\varphi_2)\| \leqslant \|\psi_1 - \psi_2\| + L\int_{t_0+\theta}^{t+\theta} \|x_\tau(t_0,\varphi_1) - x_\tau(t_0,\varphi_2)\|\, d\tau.$$

After applying Gronwall–Bellman's lemma to this inequality, we derive relation (3.2) for each $t \geqslant t_0$. The proof of the Lemma is complete. □

Theorem 3.1. *Let functionals* $X_i(t,\varphi)$ *satisfy condition* (3.1), M *be uniformly asymptotically stable, and its domain of attraction contain* C_H. *Then there exists a continuous functional* $V : R_+ \times C_H \to R$ *such that inequalities* (2.3), (2.4) *along solutions* *of* (1.1) *are valid. The functional V also satisfies condition*

$$|V(t,\psi_1,\lambda_1)-V(t,\psi_2,\lambda_2)| \leqslant L_0\|\psi_1-\psi_2\|, \quad L_0 = const. \tag{3.3}$$

In the case $Z(t,\varphi)$ is periodic in t with period ω, then functional V is also a periodic function of t with period ω; if $Z(t,\varphi)$ does not depend on t, then V also does not depend on t.

Proof. We shall follow the method used in [24]. From uniform asymptotic stability of M there exists a scalar monotonically decreasing continuous function $f : R_+ \to R_+$ such that

$$\|x_t(t_0,\varphi)\| \leqslant f(t-t_0) \quad \text{for } t \geqslant t_0, \quad \varphi \in C_H \quad \text{and } \lim_{t\to\infty} f(t) = 0. \tag{3.4}$$

In [28] the existence of f has been proved as well as the existence of continuously differentiable function G such that $G \in \mathscr{K}$, $G' \in \mathscr{K}$, and

$$\int_0^\infty G(f(\tau))\,\mathrm{d}\tau = N_1 < \infty, \tag{3.5}$$

$$\int_0^\infty G'(f(\tau))\mathrm{e}^{L\tau}\,\mathrm{d}\tau = N_2 < \infty, \tag{3.6}$$

and

$$G'(f(\tau))\mathrm{e}^{L\tau} \leqslant N_3 < \infty \quad \text{under all } \tau \geqslant 0. \tag{3.7}$$

Here N_1, N_2, N_3 are constants. Let us show that the functional

$$V(t,\varphi) = \int_t^\infty G(\|x_\tau(t,\varphi)\|)\,\mathrm{d}\tau + \sup_{\tau\in[t,+\infty)} G(\|x_\tau(t,\varphi)\|) \tag{3.8}$$

satisfies the conditions of Theorem 3.1. Integral (3.5) converges, hence from conditions (3.4) and $G \in \mathscr{K}$ we have the convergence of the integral in the right-hand side of (3.8) for $t \in R_+$, $\varphi \in C_H$. In view of positiveness of function G and (3.8) we get

$$V(t,\varphi) \geqslant \sup_{\tau\in[t,+\infty)} G(\|x_\tau(t,\varphi)\|) \geqslant G(\|\psi\|). \tag{3.9}$$

From (3.8), (3.4), (3.5) it follows

$$V(t,\varphi) < \int_0^\infty G(f(\tau))\,\mathrm{d}\tau + G(f(0)) = N_1 + G(f(0)) = N_4 = \text{const},$$

for all $t \in R_+, \varphi \in C_H$, i.e. the functional $V(t,\varphi)$ is uniformly bounded in the domain $R_+ \times C_H$. Now let us show that the functional $V(t,\varphi)$ satisfies conditions (3.3). Indeed,

$$|V(t,\varphi_1) - V(t,\varphi_2)|$$
$$= \left|\int_t^\infty [G(\|x_\tau(t,\varphi_1)\|) - G(\|x_\tau(t,\varphi_2)\|)]\,\mathrm{d}\tau\right.$$

$$+\sup_{\tau\in[t,+\infty)} G(\|x_\tau(t,\varphi_1)\|)-\sup_{\tau\in[t,+\infty)} G(\|x_\tau(t,\varphi_2)\|)\Bigg|$$

$$\leqslant \int_t^\infty |G(\|x_\tau(t,\varphi_1)\|)-G(\|x_\tau(t,\varphi_2)\|)|\,d\tau$$

$$+\left|\sup_{\tau\in[t,+\infty)} G(\|x_\tau(t,\varphi_1)\|)-\sup_{\tau\in[t,+\infty)} G(\|x_\tau(t,\varphi_2)\|)\right|$$

$$\leqslant \int_t^\infty G_f'(\sup(\|x_\tau(t,\varphi_1)\|,\ \|x_\tau(t,\varphi_2)\|))\ \|x_\tau(t,\varphi_1)-x_\tau(t,\varphi_2)\|\,d\tau$$

$$+\sup_{\tau\in[t,+\infty)} |G(\|x_\tau(t,\varphi_1)\|)-G(\|x_\tau(t,\varphi_2)\|)|.$$

Using the above Lemma and applying to the second summand the Mean Value Theorem, by means of relations (3.6), (3.7), we obtain

$$|V(t,\varphi_1)-V(t,\varphi_2)|$$

$$\leqslant \|\psi_1-\psi_2\|\left[\int_t^\infty G_f'(f(\tau-t))e^{L(\tau-t)}\,d\tau+\sup_{\tau\in[t,+\infty)} G_f'(f(\tau-t))e^{L(\tau-t)}\right]$$

$$=(N_2+N_3)\|\psi_1-\psi_2\|. \tag{3.10}$$

This proves inequality (3.3). Properties (3.9), (3.10) imply that inequalities (2.3) are valid. Now, let us verify that functional $V(t,\varphi)$ continuously depends on $t\in R_+,\ \varphi\in C_H$. In view of proved property (3.3) it suffices to show that functional $V(t,\varphi)$ continuously depends on t for a fixed function φ.

First we prove that the functional $V(t,\varphi)$ has uniformly bounded right upper and lower Dini derivatives [2,30] along solutions of (1.1). After using the property

$$x_\tau(t,z_t(t_0,\varphi))=x_\tau(t_0,\varphi)\quad \text{under } t_0\leqslant t\leqslant\tau,$$

we get

$$D^+V(t,z_t)|_{t=t_0}=\lim_{\Delta t\to+0}\sup\frac{\Delta V}{\Delta t}\bigg|_{t=t_0}=\frac{d}{dt}\int_t^\infty G(\|x_\tau(t_0,\varphi)\|)\,d\tau\bigg|_{t=t_0}$$

$$+\lim_{\Delta t\to+0}\sup\left[\sup_{t_0+\Delta t\leqslant\tau<\infty} G(\|x_\tau(t_0+\Delta t,z_{t_0+\Delta t}(t_0,\varphi))\|)\right.$$

$$\left.-\sup_{t_0\leqslant\tau<\infty} G(\|x_\tau(t_0,z_{t_0}(t_0,\varphi))\|)\right].$$

Since $\sup_{t_0+\Delta t\leqslant\tau<\infty} G(\|x_\tau(t_0+\Delta t, z_{t_0+\Delta}(t_0,\varphi))\|)\leqslant \sup_{t_0\leqslant\tau<\infty} G(\|x_\tau(t_0, z_{t_0}(t_0,\varphi))\|)$, then

$$D^+V|_{t=t_0}\leqslant -G(\|x_{t_0}(t_0,\varphi)\|)=-G(\|\psi\|).$$

Let us estimate the right lower Dini derivative along solutions of system (1.1):

$$\begin{aligned} D_+V|_{t=t_0} &= \lim_{\Delta t\to+0}\inf\frac{\Delta V}{\Delta t}=-G(\|\psi\|)\\ &\quad+\lim_{\Delta t\to+0}\inf\left[\sup_{t_0+\Delta t\leqslant\tau<\infty} G(\|x_\tau(t_0+\Delta t, z_{t_0+\Delta t})\|)-\sup_{t_0\leqslant\tau<\infty} G(\|x_\tau(t_0,\varphi)\|)\right]\\ &\geqslant -G(\|\psi\|)-G'_f\sup_{1\leqslant i\leqslant n}(|X_i(t_0,\varphi)|). \end{aligned}$$

Thus it is proved that functional $V(t,z_t)$ has uniformly bounded right upper and right lower Dini derivatives along the trajectory $z(t_0,\varphi)$ in the domain $R_+\times C_H$.

Let us show that functional $V(t,\varphi)$ depends continuously on t and φ. Because of inequality (3.3), to prove this, all we need is to show that functional $V(t,\varphi)$ depends on t continuously for fixed function φ on the fixed solution $z(t_0,\varphi)$. Let $\Delta t:=t-t_0>0$. Then

$$|V(t_0,\varphi)-V(t,\varphi)|\leqslant|V(t_0,\varphi)-V(t,z_t)|+|V(t,\varphi)-V(t,z_t)|.$$

The uniform boundedness of right upper and lower Dini derivatives along the solution $z(t_0,\varphi)$ implies

$$|V(t_0,\varphi)-V(t,z_t(t_0,\varphi))|\leqslant N_5\Delta t,\qquad N_5=const,$$

and from (3.3) it follows

$$|V(t,\varphi)-V(t,z_t)|\leqslant L_0\|\psi-x_t(t_0,\varphi)\|\leqslant L_0\sup_{1\leqslant i\leqslant n}(|X_i(t,z_t)|)\Delta t,$$

whence we obtain

$$|V(t_0,\varphi)-V(t,\varphi)|\leqslant N_6\Delta t,\qquad N_6=const. \tag{3.11}$$

Inequalities (3.3), (3.11) imply that functional $V(t,\varphi)$ depends continuously on t and φ, that is, for every $\varepsilon>0$ there exists $\delta>0$ such that $|V(t_0,\varphi_0)-V(t,\varphi)|<\varepsilon$ if $|t-t_0|<\delta$, $\|\varphi_0-\varphi\|<\delta$.

Suppose that right-hand sides of system (1.1) is periodic in t with the period ω. We show that in this case functional (3.8) is also a periodic function of t with the period ω. Since, $Z(t+\omega,\varphi)\equiv Z(t,\varphi)$ then

$$\sup_{\tau\in[t+\omega,\infty)} G(\|x_\tau(t+\omega,\varphi)\|)=\sup_{\tau\in[t,\infty)} G(|x_\tau(t,\varphi)|). \tag{3.12}$$

We now show

$$\int_{t+\omega}^{\infty} G(\|x_\tau(t+\omega,\varphi)\|)\,\mathrm{d}\tau \equiv \int_{t}^{\infty} G(\|x_\tau(t,\varphi)\|)\,\mathrm{d}\tau. \tag{3.13}$$

In fact, using the property $x_{\tau+\omega}(t+\omega,\varphi)\equiv x_\tau(t,\varphi)$ we find

$$\begin{aligned}\int_{t+\omega}^{\infty} G(\|x_\tau(t+\omega,\varphi)\|)\,\mathrm{d}\tau &= \int_{t}^{\infty} G(\|x_{\tau+\omega}(t+\omega,\varphi)\|)\,\mathrm{d}\tau\\ &= \int_{t}^{\infty} G(\|x_\tau(t,\varphi)\|)\,\mathrm{d}\tau.\end{aligned}$$

Relations (3.12), (3.13) imply the property $V(t+\omega,\varphi)\equiv V(t,\varphi)$. In the case when right-hand side of (1.1) does not depend on t, the value ω can be taken arbitrary, i.e. V does not depend on t. Consequently, the functional (3.8) satisfies all conditions of the theorem. □

4. Stability in functional differential equations under perturbations

Along with Eqs. (1.1) consider the system

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = Z(t,w_t)+R(t,w_t), \tag{4.1}$$

where $w(t)=(u(t),v(t))$, $u\in R^n$, $v\in R^m$, $w_t=(u_t,\,v_t)=(w_t^1,\dots,w_t^{n+m})$; $u_t=(u_t^1,\dots,u_t^n)$, $v_t=(v_t^1,\dots,v_t^m)$. Suppose that functionals $Z(t,\varphi)$ and $R(t,\varphi)$ are uniformly bounded in the domain $R_+\times C_H$ and satisfy the condition

$$\begin{aligned}&|Z_i(t,\varphi_1)-Z_i(t,\varphi_2)| \leqslant L^*\|\psi_1-\psi_2\|,\\ &|R_i(t,\varphi_1)-R_i(t,\varphi_2)| \leqslant L^*\|\psi_1-\psi_2\|, \quad L^*=const,\ 1\leqslant i\leqslant n.\end{aligned}$$

We also assume that

$$Z_i(t,\varphi)=0, \quad R_i(t,\varphi)=0 \ \ \varphi_i=0 \ (i=1,\dots,n), \tag{4.2}$$

so Eqs. (4.1) have the solution w such that

$$u_t=0, \tag{4.3}$$

and M is an invariant set of system (4.1).

Theorem 4.1. *Assume M is uniformly asymptotically stable for system* (1.1), *and its domain of attraction contains the set C_H. If $R(t,\varphi)$ satisfies*

$$\lim_{t\to\infty}\int_t^{t+\tau} R_i(t,\varphi)\,\mathrm{d}t = 0, \quad 1\leqslant i\leqslant n \tag{4.4}$$

uniformly in $\tau > 0$, $\varphi \in C_H$, *then* M *is uniformly asymptotically stable for system* (4.1), *and there exists positive* $\eta(\eta < H)$ *such that* C_η *is contained in the domain of attraction of* M.

Proof. According to the assumptions of Theorem 4.1, in view of Theorem 3.1, it follows that there is a functional $V : R_+ \times C_H$ such that inequality (3.3) holds. Hence, if $L = \max\{L_0, L^*\}$, then

$$|Z_i(t,\varphi^{(1)}) - Z_i(t,\varphi^{(2)})| \leqslant L\|\psi^{(1)} - \psi^{(2)}\|,$$
$$|R_i(t,\varphi^{(1)}) - R_i(t,\varphi^{(2)})| \leqslant L\|\psi^{(1)} - \psi^{(2)}\|,$$
$$|V(t,\varphi^{(1)}) - V(t,\varphi^{(2)})| \leqslant L\|\psi^{(1)} - \psi^{(2)}\|, \quad 1 \leqslant i \leqslant n$$

for all $t \in R_+, \varphi^{(1)}, \varphi^{(2)} \in C_H$. Conditions (2.3), (2.4) are also valid along solutions of system (1.1).

Pick any $\varepsilon > 0$ such that $\varepsilon < H$. Denote $\xi := b^{-1}(\frac{1}{2}a(\varepsilon))$. Then in view of inequalities (2.3) we have

$$\inf_{\|\psi\|=\varepsilon} V(t,\varphi) \geqslant a(\varepsilon), \quad \sup_{\|\psi\|\leqslant\xi} V(t,\varphi) \leqslant b(\xi) = \tfrac{1}{2}\, a(\varepsilon). \tag{4.5}$$

Let us show that each trajectory $w(t_1,\bar{\varphi}) = w(t_1,\bar{\psi},\bar{\lambda})$ of Eqs. (4.1) satisfies $\|u_t\| < \varepsilon$ for $t > t_1$ where t_1 is sufficiently large and $\|\bar{\psi}\| = \xi$. Suppose not: there is a system of functional differential Eqs. (4.1), satisfying the above conditions, which has a solution

$$w(t_1,\bar{\varphi}), \tag{4.6}$$

satisfying conditions $w_{t_1} = \bar{\varphi}$, $\|u_{t_1}\| = \xi$, $\|u_{t_2}\| = \varepsilon$. We also assume that the trajectory (4.6) satisfies

$$\|u_t\| \geqslant \xi, \quad \|u_t\| \leqslant \varepsilon$$

for $t \in [t_1,t_2]$. Let us divide the segment $[t_1,t_2]$ into p equal segments of length τ by the points $\theta_k = t_1 + k\tau$ $(k = 1,2,\dots,p-1)$; $\theta_0 = t_1$, $\theta_p = t_2$. On the kth segment $[\theta_k,\theta_{k+1}]$ we denote $z_{\theta_k} = w(\theta_k + \theta, t_1, \bar{\varphi})$ $-h \leqslant \theta \leqslant 0$ (i.e. $z_{\theta_k} = w_{\theta_k}$). We designate $z_{\theta_{k+1}}$ an element of the trajectory $z(\theta_k, z_{\theta_k})$ of equations (1.1) corresponding to the time moment θ_{k+1}, i.e. $z_{\theta_{k+1}} = z(\theta_{k+1} + \theta, \theta_k, z_{\theta_k})$ $-h \leqslant \theta \leqslant 0$.

Let us consider

$$\begin{aligned}\Delta V &= V(t_2, w_{t_2}) - V(t_1, w_{t_1})\\ &= \sum_{k=0}^{p-1} [V(\theta_{k+1}, w_{\theta_{k+1}}) - V(\theta_k, w_{\theta_k})]\\ &= \sum_{k=0}^{p-1} \{[V(\theta_{k+1}, z_{\theta_{k+1}}) - V(\theta_k, z_{\theta_k})] + [V(\theta_{k+1}, w_{\theta_{k+1}})\\ &\quad - V(\theta_{k+1}, z_{\theta_{k+1}})] + [V(\theta_k, z_{\theta_k}) - V(\theta_k, w_{\theta_k})]\}.\end{aligned}$$

Note that $V(\theta_k, z_{\theta_k}) - V(\theta_k, w_{\theta_k}) = 0$, because $z_{\theta_k} = w_{\theta_k}$. We assume that $t_1 \geqslant t_* = t_*(\varepsilon)$, where the value t_* will be given below. We suppose $t_* > t_0 + 2h$. Under above assumptions, $x_t(t_1, \varphi)$ and $u_t(t_1, \varphi)$ satisfy Lipschitz conditions in t with constants L_1 and L_2, respectively [24], where L_1 and L_2 do not depend on corresponding solutions and depend only on constants which estimate $Z_i(t, \varphi)$, $Z_i(t, \varphi)+R_i(t, \varphi)$ when $1 \leqslant i \leqslant n$, $t \in R_+, \varphi \in C_H$.

Notice if

$$J_1 = \sum_{k=0}^{p-1} [V(\theta_{k+1}, z_{\theta_{k+1}}) - V(\theta_k, z_{\theta_k})]$$

and if

$$J_2 = \sum_{k=0}^{p-1} [V(\theta_{k+1}, w_{\theta_{k+1}}) - V(\theta_{k+1}, z_{\theta_{k+1}})],$$

then

$$J_1 \leqslant \sum_{k=0}^{p-1} \int_{\theta_k}^{\theta_{k+1}} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.1)} \mathrm{d}t \leqslant -\, pc(\xi)\tau \tag{4.7}$$

and

$$|J_2| \leqslant \sum_{k=0}^{p-1} |V(\theta_{k+1}, w_{\theta_{k+1}}) - V(\theta_{k+1}, z_{\theta_{k+1}})|. \tag{4.8}$$

Hence

$$\begin{aligned} &|V(\theta_{k+1}, w_{\theta_{k+1}}) - V(\theta_{k+1}, z_{\theta_{k+1}})| \\ &\quad \leqslant L\|u_{\theta_{k+1}} - x_{\theta_{k+1}}\| = L \sup_{1 \leqslant i \leqslant n} |u^i_{\theta_{k+1}} - x^i_{\theta_{k+1}}|. \end{aligned} \tag{4.9}$$

Thus

$$u^i_{\theta_{k+1}} - x^i_{\theta_{k+1}} = \int_{\theta_k}^{\theta_{k+1}} [X_i(t, w_t) - X_i(t, z_t)]\,\mathrm{d}t + \int_{\theta_k}^{\theta_{k+1}} R_i(t, w_t)\,\mathrm{d}t, \qquad 1 \leqslant i \leqslant n$$

implying

$$\begin{aligned} |u^i_{\theta_{k+1}} - x^i_{\theta_{k+1}}| \leqslant{}& \int_{\theta_k}^{\theta_{k+1}} |X_i(t, w_t) - X_i(t, z_t)|\,\mathrm{d}t + \left|\int_{\theta_k}^{\theta_{k+1}} R_i(t, w_{\theta_k})\,\mathrm{d}t\right| \\ &+ \int_{\theta_k}^{\theta_{k+1}} |R_i(t, w_t) - R_i(t, w_{\theta_k})|\,\mathrm{d}t. \end{aligned} \tag{4.10}$$

Now

$$|X_i(t, w_t) - X_i(t, z_t)| \leqslant L\|u_t - x_t\| \leqslant L(L_1 + L_2)\tau \quad \text{for} \quad \theta_k \leqslant t \leqslant \theta_{k+1}$$

implies

$$\int_{\theta_k}^{\theta_{k+1}} |X_i(t,w_t) - X_i(t,z_t)|\,\mathrm{d}t \leqslant L(L_1+L_2)\tau^2. \tag{4.11}$$

Similarly,

$$\int_{\theta_k}^{\theta_{k+1}} |R_i(t,w_t) - R_i(t,w_{\theta_k})|\,\mathrm{d}t \leqslant L\int_{\theta_k}^{\theta_{k+1}} \|w_t - w_{\theta_k}\|\,\mathrm{d}t \leqslant LL_2\tau^2,$$
$$1 \leqslant i \leqslant n. \tag{4.12}$$

Since limit relations (4.4) hold uniformly in $\tau > 0$, $\varphi \in C_H$, then there exists a positive monotonically decreasing function $\beta : R_+ \to R_+$ such that

$$\lim_{t\to+\infty} \beta(t) = 0, \qquad \left|\int_t^{t+\tau} R_i(s,\varphi)\,\mathrm{d}s\right| \leqslant \beta(t) \quad (\varphi \in C_H,\ 1 \leqslant i \leqslant n),$$

whence we get the estimate

$$\left|\int_{\theta_k}^{\theta_{k+1}} R_i(t,w_{\theta_k})\,\mathrm{d}t\right| \leqslant \beta(\theta_k) < \beta(t_*).$$

Estimates (4.7)–(4.12) imply

$$\begin{aligned}\Delta V &\leqslant -pc(\xi)\tau + pL[L(L_1+L_2)\tau^2 + LL_2\tau^2 + \beta(t_*)]\\ &= p\left\{-\tfrac{1}{2}c(\xi)\tau + [L^2(L_1+2L_2)\tau^2 - \tfrac{1}{2}c(\xi)\tau + L\beta(t_*)]\right\}.\end{aligned}$$

Choose τ satisfying

$$\tau^2 - 2\alpha\tau + \gamma < 0, \tag{4.13}$$

where

$$\alpha = \frac{c(\xi)}{4L^2(L_1+2L_2)}, \qquad \gamma = \gamma(t_*) = \frac{\beta(t_*)}{L(L_1+2L_2)}.$$

This inequality is correct if $\tau \in (\tau_1,\tau_2)$ where $\tau_1 = \alpha - \sqrt{\alpha^2-\gamma}$, $\tau_2 = \alpha + \sqrt{\alpha^2-\gamma}$. We suppose the value t_* to be so large that inequality

$$\beta(t_*) < \frac{c^2(\xi)}{16L^3(L_1+2L_2)} \tag{4.14}$$

holds. This guarantees the validity of the relations $\alpha^2 - \gamma > 0$, $\tau_2 > \tau_1 > 0$. Let us show that there exists a natural number p and $\tau \in (\tau_1,\tau_2)$ such that the inequality

$$p\tau = t_2 - t_1 \tag{4.15}$$

is valid. Inequalities $\tau_1 < \tau < \tau_2$ and (4.15) imply

$$\frac{t_2-t_1}{\tau_1} > p > \frac{t_2-t_1}{\tau_2}. \tag{4.16}$$

The existence of a natural number p satisfying inequalities (4.16) is guaranteed by the inequality

$$\frac{t_2 - t_1}{\tau_1} - \frac{t_2 - t_1}{\tau_2} > 1. \tag{4.17}$$

Let Δt be a lower bound of the differences $t_2 - t_1$ such that $\|u_{t_1}\| = \xi(\varepsilon)$, $\|u_{t_2}\| = \varepsilon$. It is clear that $\Delta t = \Delta t(\varepsilon) > 0$. Inequality (4.17) is valid if

$$\Delta t(\tau_1^{-1} - \tau_2^{-1}) > 1. \tag{4.18}$$

It is easy to see that (4.18) is valid for

$$\gamma < \gamma_1 = 2\left[\sqrt{(\Delta t)^4 + \alpha^2} - (\Delta t)^2\right].$$

Note that γ_1 depends only on ε because Δt and α depend only on ε as well as constants L, L_1, L_2, and do not depend on a trajectory. Thus, if t_* satisfies (4.14) and inequality $t_* \geqslant \beta^{-1}(L(L_1 + 2L_2)\gamma_1(\varepsilon))$, then $\Delta V < -\frac{1}{2}c(\xi)(t_2 - t_1)$. This contradicts (4.5) and implies that there do not exist t_1, t_2 ($t_2 > t_1 \geqslant t_*(\varepsilon)$) such that $\|u_{t_1}\| = \xi(\varepsilon)$, $\|u_{t_2}\| = \varepsilon$ for (4.1).

Each solution $w(t_0, \varphi)$ of functional differential equations (4.1) depends continuously on initial conditions. Consequently, in view of assumption (4.2), there is a $\delta > 0$ such that for all $t_0 \in [0, t_*]$, $\varphi \in C_\delta$ we have $\|u_{t_*}(t_0, \varphi)\| \leqslant \xi$. Since ξ and t_* depend only on ε, then δ also depends only on ε. This proves that M is uniformly stable for (4.1). Now let us show that M is uniformly asymptotically stable. Let q be any fixed positive number ($q < H$). We have proved there exists $\eta(q) > 0$ such that each trajectory $w(t_0, \varphi)$ of system (4.1) satisfying the initial condition $\|u_{t_0}\| < \eta$, satisfies inequality $\|u_t\| < q$ for all $t > t_0 \geqslant 0$. Let us show that for every $\rho > 0$ ($\rho < q$) there is a $\sigma = \sigma(\rho) > 0$ such that the inequality $\|u_t(t_0, \varphi)\| < \rho$ holds for arbitrary $t_0 \in R_+, \varphi \in C_\eta, t \geqslant t_0 + \sigma$.

Let $0 < \rho < q$; we have proved there exists $\delta = \delta(\rho) > 0$ such that $u_{T_0} \in C_\delta$ implies $u_t \in C_\rho$ for every $t > T_0 \geqslant 0$. Let us estimate the time for which the element of the trajectory w_t will lie in the domain $C_q \setminus C_\delta$.

Similarly to the above, one can show that

$$\Delta V = V(t, w_t) - V(T_1, w_{T_1}) \leqslant -\tfrac{1}{2}c(\delta)(t - T_1) \tag{4.19}$$

for $t \geqslant T_1$, where T_1 depends only on $\delta(\rho)$, i.e. $T_1 = T_1(\rho)$. Then for $t \geqslant T_1$ inequality (4.19) implies

$$t - T_1 \leqslant \frac{2[V(T_1, w_{T_1}) - V(t, w_t)]}{c(\delta)} \leqslant 2\,\frac{V(T_1, w_{T_1})}{c(\delta)} \leqslant 2\frac{b(\delta(\rho))}{c(\delta(\rho))} = T_2(\rho).$$

Setting $\sigma(\rho) = T_1(\rho) + T_2(\rho)$, we obtain that the inequality $\|u_t(t_0, \varphi)\| < \rho$ is valid for all $t_0 \in R_+$, $\varphi \in C_\eta$, $t \geqslant t_0 + \sigma$. Hence the invariant set M of system (4.1) is uniformly asymptotically stable, and its domain of attraction contains C_η. This completes the proof of Theorem 4.1.

Example 4.1. Consider two systems of functional differential equations with delay:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -a(t)x(t) + b(t)x(t-2), \tag{4.20}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = f(t, y_t) \tag{4.21}$$

and

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -a(t)x(t) + b(t)x(t-2) + \sin t^2 \sin x(t) \cos(t-1), \tag{4.22}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = f(t, y_t) + g(t, x_t, y_t) \tag{4.23}$$

where

$$a(t) := \begin{cases} 1 & \text{if } t \in \bigcup\limits_{k\in N} [2k, 2k+1], \\ 0 & \text{otherwise,} \end{cases}$$

$$b(t) := \begin{cases} 1 & \text{if } t \in \bigcup\limits_{k\in N} [8k+2, 8k+3], \\ 0 & \text{otherwise.} \end{cases}$$

Here f and g are functionals for which there exist continuous solutions of equations (4.21), (4.23) such that for any $t_0 \in R_+, \lambda \in R$, and some H we have that $y_{t_0}(t_0, \lambda) = \lambda, \|x_t\| < H$ for $t \geqslant t_0$. These systems have the positive invariant set

$$x_t = 0. \tag{4.24}$$

Hatvani [18] showed that (4.24) is a uniformly asymptotically stable set of system (4.20), (4.21). Since $\lim_{t\to\infty} \int_t^{t+\tau} \sin t^2 = 0$ uniformly in $\tau > 0$, then, by Theorem 4.1, the positive invariant set (4.24) is also uniformly asymptotically stable set of system (4.22), (4.23).

5. Concluding remark—causal operators

Some natural questions to ask are: Can we extend our work to the infinite delay case, or to the case of integral equations or to integro differential equations? To answer these questions we will consider the general case of causal operators (see [10]) which include the above cases. We will analyze the theory and stability properties of causal operators presented in [10] and try to extend this to the cases mentioned above. We are presently investigating this.

References

[1] Z.S. Athanassov, Families of Liapunov–Krasovskii functionals and stability for functional differential equations, Ann. Math. Pure Appl. 176 (4) (1999) 145–165.

[2] A. Bacciotti, L. Rosier, Liapunov Functions and Stability in Control Theory, in: Lecture Notes in Control and Information Sciences, Vol. 267, Springer, London, 2001.

[3] T.A. Burton, Uniform Asymptotic Stability in Functional Differential Equations, Proc. of the Amer. Math. Soc. 68 (2) (1978) 195–199.

[4] T.A. Burton, L. Hatvani, On nonuniform asymptotic stability for nonautonomous functional–differential equations, Differential Integral Equations 3 (1974) 285–293.

[5] T.A. Burton, L. Hatvani, Stability theorems for nonautonomous functional-differential equations by Liapunov fuctionals, Tõhoku Math. J. 41 (2) (1989) 65–104.

[6] T.A. Burton, G. Makay, Asymptotic stability for functional-differential equations, Acta Math. Hungar 65 (1994) 243–251.

[7] C. Corduneanu, Some problems concerning partial stability, Symp. Math. Meccanica Nonlineare. Stability, 23–26 febbraio, 1970, Vol. 6, Academic Press, New York, 1971, pp. 141–154.

[8] C. Corduneanu, Some differential equations with delay, Proceedings of EQUADIFF 3, Brno, 1973, pp. 105–114.

[9] C. Corduneanu, On partial stability for delay systems, Ann. Polon. Math. 29 (1975) 357–362.

[10] C. Corduneanu, Functional Equations with Causal Operators, Taylor and Francis, London and New York, 2002.

[11] C. Corduneanu, V. Lakshmikantham, Equations with unbounded delay: a survey, Nonlinear Analysis: TMA 4 (5) (1980) 831–877.

[12] C. Corduneanu, V. Lakshmikantham, Equations with unbounded delay, Avtomat. i Telemekh. 7 (1985) 5–44 (in Russian).

[13] I.V. Gaishun, L.B. Knyazhishche, Nonmonotone Lyapunov functionals. Conditions for the stability of equations with delay, Differential Equations 30 (1994) 1195–1200.

[14] I. Győri, F. Hartung, Stability in delay perturbed differential and difference equations, Fields Inst. Comm. 29 (2001) 181–194.

[15] I. Győri, F. Hartung, J. Turi, Preservation of stability in delay equations under delay perturbations, J. Math. Anal. Appl. 220 (1998) 290–312.

[16] W. Hahn, Stability of Motion, Springer, New York, Berlin, Heidelberg, 1967.

[17] J. Hale, Theory of Functional Differential Equations, Springer, New York, Heidelberg, Berlin, 1977.

[18] L. Hatvani, On the asymptotic stability for nonautonomous functional differential equations by Lyapunov functionals, Trans. Amer. Math. Soc. 354 (9) (2002) 3555–3571.

[19] A.O. Ignatyev, Partial stability under persistent disturbances, Izv. Vyssh. Uchebn. Zaved. Mat. 2 (345) (1991) 55–60 (in Russian).

[20] A.O. Ignatyev, On the asymptotic stability in functional differential equations, Proc. Amer. Math. Soc. 127 (6) (1999) 1753–1760.

[21] A.O. Ignatyev, On the partial equiasymptotic stability in functional differential equations, J. Math. Anal. Appl. 268 (2002) 615–628.

[22] M.A. Kalistratova, On the partial stability of the systems with delay, Avtimat. i Telemekh. 47 (1986) 32–37 (in Russian).

[23] V.B. Kolmanovskii, V.R. Nosov, Stability of Functional Differential Equations, Academic Press, New York, 1986.

[24] N.N. Krasovskii, Stability of Motion, Stanford University Press, Stanford, California, 1963.

[25] V. Lakshmikantham, S. Leela, S. Sivasundaram, Liapunov functions on product spaces and stability theory of delay differential equations, J. Math. Anal. Appl. 154 (1991) 391–402.

[26] X. Liu, Stability in terms of two measures for functional differential equations, Differential Integral Equations 3 (1989) 257–261.

[27] X. Liu, D.Y. Xu, Uniform asymptotic stability of abstract functional-differential equations, J. Math. Anal. Appl. 216 (1997) 626–643.

[28] I.G. Malkin, Stability of Motion, Nauka, Moscow, 1966 (in Russian).

[29] K. Peiffer, N. Rouche, Liapunov's second method applied to partial stability, J. Mecanique 6 (2) (1969) 20–29.

[30] N. Rouche, P. Habets, M. Laloy, Stability Theory by Liapunov's Direct Method, Springer, New York, 1977.

[31] V.V. Rumyantsev, A.S. Oziraner, Partial Stability and Stabilization of Motion, Nauka, Moscow, 1987 (in Russian).

[32] A.Ya. Savchenko, A.O. Ignatyev, Some Problems of Stability Theory, Naukova Dumka, Kiev, 1989 (in Russian).

[33] E.D. Sontag, Y. Wang, Lyapunov characterizations of input to output stability, SIAM J. Control Optim. 39 (1) (2001) 226–249.

[34] V.I. Vorotnikov, Partial Stability and Control, Birkhauser, Boston, 1998.

[35] V.I. Vorotnikov, V.V. Rumyantsev, Stability and Control with Respect to a Part of the Phase Coordinates of Dynamic Systems: Theory, Methods, and Applications, Scientific World, Moscow, 2001 (in Russian).