ISSN 1055-1344, Siberian Advances in Mathematics, 2009, Vol. 19, No. 3, pp. 179–191. © Allerton Press, Inc., 2009.

Compactness Theory and Mappings with Finite Length Distortion

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Received November 28, 2007; in final form, February 19, 2009

Abstract—The present paper is devoted to the study of mappings with finite length distortion introduced in 2004 by O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov. It is proved that the locally uniform limit of homeomorphisms with finite length distortion is a homeomorphism or a constant provided that the so-called inner dilatations of the sequence of homeomorphisms are almost everywhere (a.e.) majorized by a locally integrable function.

In particular, it is studied the pointwise behavior of the so-called outer dilatations. For these dilatations, the pointwise semicontinuity and semicontinuty in the mean are proved. It is also proved some theorems on the convergence of matrix dilatations.

It is proved that the class of homeomorphisms with finite length distortion is closed in the space of all homeomorphisms, under minimal conditions on dilatations of the direct and inverse mappings.

The results of the paper can be applied to various classes of spatial mappings.

DOI: `10.3103/S1055134409010040`

Key words: *modulus, spatial mapping, quasiconformal mapping, homeomorphism with finite distortion, the outer and matrix dilatations.*

1. INTRODUCTION

The mappings of finite length distortion form one of the most broad classes of spatial mappings including the class of mappings with bounded distortion (see [16], 4.7 in [10], and Theorem 8.2 in [12]) and the class of mappings with bounded length distortion (see [13]). Recall some definitions. Let $x \in E \subset \mathbb{R}^n$ and $\varphi : E \to \mathbb{R}^n$. Set

$$L(x, \varphi) = \limsup_{y \to x,\, y \in E} \frac{|\varphi(x) - \varphi(y)|}{|y - x|},$$

$$l(x, \varphi) = \liminf_{y \to x,\, y \in E} \frac{|\varphi(x) - \varphi(y)|}{|y - x|}.$$

Below we assume that D is a domain in $\mathbb{R}^n$, $n \geq 2$, and all the mappings $f : D \to \mathbb{R}^n$ under consideration are continuous. Recall that a mapping $f : D \to \mathbb{R}^n$ is said to be of *finite metric distortion* ($f \in FMD$ for brevity) if f possesses the Lusin (N)-property and

$$0 < l(x, f) \leq L(x, f) < \infty \quad \text{a.e.}$$

(for example, see Section I in [10] and relations 8.1–8.3 in [12]). Recall that a mapping $f : X \to Y$ between measurable spaces (X, Σ, μ) and (X', Σ', μ') is said to have *(N)-property* if $\mu'(f(S)) = 0$ whenever $\mu(S) = 0$. Similarly, f possesses *(N^{-1})-property* if $\mu(S) = 0$ whenever $\mu'(f(S)) = 0$.

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A *path* γ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$, where Δ is an interval in $\mathbb{R}$. Its image $\gamma(\Delta)$ is denoted by $|\gamma|$. We say that a property P holds for *almost every* (*a.e.*) path γ from a family Γ if the subfamily of all the paths from Γ for which P fails has the below-defined modulus zero.

If $\gamma : \Delta \to \mathbb{R}^n$ is a locally rectifiable path then there is the unique increasing length function l_γ from Δ onto a length interval $\Delta_\gamma \subset \mathbb{R}$, with the prescribed normalization $l_\gamma(t_0) = 0 \in \Delta_\gamma$, $t_0 \in \Delta$, such that $l_\gamma(t)$ is equal to the length of the subpath $\gamma|_{[t_0,t]}$ of γ if $t > t_0$, $t \in \Delta$, and $l_\gamma(t)$ is equal to $-l(\gamma|_{[t,t_0]})$ if $t < t_0$, $t \in \Delta$. Let $g : |\gamma| \to \mathbb{R}^n$ be a continuous mapping. Suppose that the path $\tilde{\gamma} = g \circ \gamma$ is also locally rectifiable. Then there is a unique increasing function $L_{\gamma,\,g} : \Delta_\gamma \to \Delta_{\tilde{\gamma}}$ such that (for example, see 8.1 in [12].)

$$L_{\gamma,\,g}(l_\gamma(t)) = l_{\tilde{\gamma}}(t) \qquad \text{for all } \ t \in \Delta\,.$$

We say that a mapping $f : D \to \mathbb{R}^n$ possesses *(L)-property* if the following two conditions hold:

(L_1) for a.e. path γ in D, $\tilde{\gamma} = f \circ \gamma$ is locally rectifiable and the function $L_{\gamma,\,f}$ possesses (N)-property;

(L_2) for a.e. path $\tilde{\gamma}$ in $f(D)$, each lifting γ of $\tilde{\gamma}$ is locally rectifiable and the function $L_{\gamma,\,f}$ possesses (N^{-1})-property.

Recall that a path γ in D is called a *lifting* of a path $\tilde{\gamma}$ in $\mathbb{R}^n$ under $f : D \to \mathbb{R}^n$ if $\tilde{\gamma} = f \circ \gamma$. Note that condition (L_2) applies only to paths $\tilde{\gamma}$ which have a (maximal) lifting.

We say that a mapping $f : D \to \mathbb{R}^n$ is of *finite length distortion* ($f \in FLD$ for brevity) if f is FMD and possesses (L)-property. It is known that (L)-property implies ACL-property, moreover, implies ACP property in the straight and in the inverse directions (see Remark 4.1 and Proposition 4.3 in [10], and Ch. 8 in [12]). There are examples of FMD mappings which have no (L)-property (see [1]).

Recall that a mapping $f : D \to \mathbb{R}^n$ is said to be of *finite distortion* if $f \in W^{1,n}_{loc}(D)$ and

$$\|f'(x)\|^n \ \leq \ K(x) \ \cdot J(x,f) \ \text{ a.e. } x \in D \tag{1.1}$$

for some finite-valued function $K(x) : D \to [1\,,\infty]$ (see [2], [3], [5], [7], and [8]). According to Corollary 4.9 in [10], every homeomorphism of finite distortion is a mapping of finite length distortion whenever $K(x)$ in inequality (1.1) belongs to L^{n-1}_{loc}.

2. PRELIMINARIES

In what follows, we use the following notation: D is a domain in $\mathbb{R}^n$, $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$, and $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$. Let a mapping $f : D \to \mathbb{R}^n$ be differentiable a.e. in D. Recall that $f'(x)$ denotes the Jacobian matrix of f and $J(x,f)$ is its determinant, $\|f'(x)\|$ is the operator norm of $f'(x)$, i.e.,

$$\|f'(x)\| \ = \ \max_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|}$$

and

$$l\,(f'(x)) \ = \ \min_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|}\ .$$

The *outer dilatation* of f at x is defined by the relations

$$K_O(x,f) \ = \ \frac{\|f'(x)\|^n}{|J(x,f)|}$$

if $J(x,f) \neq 0$,

$$K_O(x,f) = 1$$

if $f'(x) = 0$, and at the rest points, we set

$$K_O(x,f) = \infty.$$

Similarly, the *inner dilatation* of f at the points x is defined as

$$K_I(x,f) \ = \ \frac{|J(x,f)|}{l\,(f'(x))^n}$$

if $J(x,f) \neq 0$,

$$K_I(x,f) = 1$$

if $f'(x) = 0$, and

$$K_I(x,f) = \infty$$

otherwise.

We write $f \in W^{1,n}_{loc}(D)$ if all the coordinate functions of $f = (f_1, \ldots, f_n)$ have distributional derivatives which are locally integrable to the power n in D. Note that the homeomorphisms of the Sobolev class $W^{1,n}_{loc}$ in $\mathbb{R}^n$, $n \geq 2$, are a.e. differentiable and possess the Lusin(N)-property (see [17]–[18]). Moreover, if $K_O(x,f) \in L^{n-1}_{loc}$ then $f^{-1} \in W^{1,n}_{loc}$ (see [5]). Thus f possesses $(N^{-1})-$property which is equivalent to the condition $J(x,f) \neq 0$ a.e. (see [15]).

Note that mappings with finite metric distortion and, consequently, mappings with finite length distortion are a.e. differentiable, with $J(x,f) \neq 0$ a.e. (for example, see Proposition 3.7 in [10] and Proposition 8.3 in [12]).

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let $dm(x)$ be the Lebesgue measure in $\mathbb{R}^n$, and let $Q : D \to [1, \infty]$ be a measurable function. We say that a homeomorphism $f : D \to \mathbb{R}^n$ is a Q-*homeomorphism* if

$$M(f\Gamma) \leq \int\limits_D Q(x) \cdot \rho^n(x) \ dm(x) \tag{2.1}$$

for every family Γ of paths in D and every admissible function ρ for Γ. The notion of Q-homeomorphism is closely related with the weighted Sobolev classes (for example, see [22]).

Given a family of paths Γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ ($\rho \in \operatorname{adm} \Gamma$ for brevity) if

$$\int\limits_\gamma \rho(x)|dx| \geq 1$$

for each path $\gamma \in \Gamma$. As usually, the notation $\int\limits_\gamma \rho(x)|dx|$ denotes the line integral of ρ along γ. The *modulus* $M(\Gamma)$ of Γ is defined as

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x) dm(x)$$

interpreted as $+\infty$ if $\operatorname{adm}\Gamma = \varnothing$. Thus, every family Γ which contains a constant path, is of infinite modulus.

By Theorem 6.10 in [10], every homeomorphism with finite length distortion is a Q-homeomorphism, with $Q = K_I(x, f)$ (see also Theorem 8.6 in [12]).

In what follows, h is a *chordal (spherical)* distance between x and y in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$:

$$h(x,y) = |\pi(x) - \pi(y)|\,, \tag{2.2}$$

where π is a stereographic projection of $\overline{\mathbb{R}^n}$ on $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$:

$$h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}\,,$$

$$h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \qquad x \neq \infty \neq y,$$

(see 12.1 in [21]). Note that

$$h(x,y) \le 1$$

and

$$h(x,y) \le |x-y|\,.$$

The following value is said to be a *chordal diameter* of $E \subseteq \overline{\mathbb{R}^n}$:

$$h(E) = \sup_{x\,,y\,\in E} h(x,y)\,.$$

3. SEMICONTINUITY OF OUTER DILATATIONS

The following statement is a generalization of the well-known Weierstrass theorem on the locally uniform convergence of analytic functions.

Theorem 3.1. *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$, and let $f_m : D \to \mathbb{R}^n$ be a sequence of homeomorphisms with finite length distortion, converging locally uniformly to a mapping f. If*

$$K_I(x, f_m) \le Q(x) \in L^1_{loc} \tag{3.1}$$

then f is either a homeomorphism or $f \equiv const$ in D.

Proof. As the locally uniform limit of continuous mappings, f is continuous. Let $f \not\equiv const$.

First we show that f is a discrete mapping. Indeed, let f is not to be discrete. Then there is a point $x_0 \in D$ and a sequence $x_k \in D$, $x_k \neq x_0$, $k = 1, 2\ldots$, such that $x_k \to x_0$ as $k \to \infty$, with $f(x_k) = f(x_0)$. Since f is continuous, the set $E_0 = \{x \in D : f(x) = f(x_0)\}$ is closed in the topology of D, i.e., if $x_k \in E_0$ and $x_k \to x \in D$ as $k \to \infty$ then $x \in E_0$. Note also that E_0 does not coincide with D because $f \not\equiv const$. Thus, we can replace x_0 by a boundary non-isolated point x_0' of the set E_0, i.e., every neighborhood U of x_0' contains at least one point $a_0 \in E_0$ and at least one point $b_0 \in D \setminus E_0$, where $a_0 \neq x_0' \neq b_0$. For the convenience, in the last case, we will use the notation x_0 as well.

Without loss of generality, we may assume that $x_0 = 0$, $f_m(0) = f(0) = 0$, $\overline{\mathbb{B}^n} \subset D$, and there is at least one point $z_0 \in \mathbb{B}^n$ where $f(z_0) \neq 0$. By continuity of the chordal metric,

$$h\left(f_m\left(z_0\right), 0\right) \ \geq\ \delta_0/2 \qquad \forall\, m\ \geq\ M_0,$$

where $\delta_0 = h(f(z_0), 0) > 0$. Since $\overline{\mathbb{B}^n}$ is a compact in D and $f_m \to f$ uniformly in $\overline{\mathbb{B}^n}$,

$$h\left(\overline{\mathbb{R}^n} \backslash f_m\left(\overline{\mathbb{B}^n}\right)\right) \ \geq\ \delta_*/2 \qquad \forall\, m\ \geq\ M_*,$$

where $\delta_* = h\left(\overline{\mathbb{R}^n} \backslash f\left(\overline{\mathbb{B}^n}\right)\right)$. Setting $\delta = \min\{\delta_0/2,\ \delta_*/2\}$ and $M = \max\{M_0, M_*\}$, we have by Theorem 6.10 in [10] and by Theorem 3.7 in [11] that

$$|f_m(x)| \ \geq\ \psi(|x|) \qquad \forall\, m\ \geq\ M$$

for all $x \in B(0, r)$ and $r = \min\{\frac{|z_0|}{2}, 1 - |z_0|\}$, where ψ is an increasing nonnegative function with $\psi(0) = 0$ depending only $\|Q\|_1$, n and δ. Thus,

$$|f(x)| \ \geq\ \psi(|x|) \qquad \forall\, x\ \in\ B\left(0, r\right). \tag{3.2}$$

Then, in particular,

$$0\ =\ |f(x_k)| \ \geq\ \psi(|x_k|) \qquad \forall\, k\ \geq\ k_0,$$

and, consequently, $\psi(r_k) = 0$ for $r_k = |x_k| \neq 0,\ k \geq k_0$. This contradiction shows that f is discrete.

We now show that f is injective in D . Indeed, assume that there exist $x_1, x_2 \in D,\ x_1 \neq x_2$, with $f(x_1) = f(x_2)$. Let $x_2 \notin \overline{B(x_1, t)} \subset D$ for all $t \in (0, t_0]$. Then every $f_m\left(\partial B(x_1, t)\right),\ t \in (0, t_0]$, separates $f_m(x_1)$ from $f_m(x_2)$, i.e., for every path γ joining $f_m(x_1)$ and $f_m(x_2)$, there exists a point z on γ which belongs to $f_m\left(\partial B(x_1, t)\right)$. The distances $h(f_m(x_1), f_m\left(\partial B(x_1, t)\right)$ and $h\left(f_m(x_1), f_m(x_2)\right)$ can be understood as Euclidean distances between correspondent projections on the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$ (see (2.2)). Note that every path joining the projections $\pi(f_m(x_1))$ and $\pi(f_m(x_2))$ on $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ intersects the set $\pi\left(f_m\left(\partial B(x_1, t)\right)\right)$. So,

$$h(f_m(x_1), f_m(\partial B(x_1, t)) \ <\ h\left(f_m(x_1), f_m(x_2)\right) \tag{3.3}$$

because the geodesic joining $\pi(f_m(x_1))$ and $\pi(f_m(x_2))$ on $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ is longer than its part joining $\pi(f_m(x_1))$ with $\pi\left(f_m\left(\partial B(x_1, t)\right)\right)$. Let $h\left(f_m(x_1), f_m\left(\partial B(x_1, t)\right)\right) = h\left(f_m(x_1), f_m(x_{m,t})\right)$. Since a boundary of a ball in $\mathbb{R}^n$ is compact, there exists a subsequence $x_{m_k,t} \to x_t\ \in \partial B(x_1, t)$. Since the locally uniform convergence of continuous functions in a metric space implies the continuous convergence (see Theorem 3 in [9], p. 229), we obtain$h(f_{m_k}(x_{m_k,t}), f(x_t)) \to 0$ as $k \to \infty$. Hence, by (3.3),

$$h\left(f(x_1), f(\partial B(x_1, t)\right) \ \leq\ h\left(f(x_1), f(x_2)\right). \tag{3.4}$$

Since $f(x_1) = f(x_2)$, it follows from (3.4) that, for every $t \in (0, t_0]$, there is a point $x_t \in \partial B(x_1, t)$ such that $f(x_t) = f(x_1)$. The latter contradicts to discreteness of the mapping f. Continuity of the inverse mapping f^{-1} also follows from (3.2). Thus, f is a homeomorphism.

Since $K_I(x, g) \leq K_O^{n-1}(x, g)$, for the conclusion of Theorem 3.1, it suffices to suppose that $K_O(x, f_m)\ \leq\ K(x) \in\ L_{loc}^{n-1}$ instead of (3.1). Let

$$P_O\left(x, f\right)\ =\ \left(K_O\left(x\,,\ f\right)\right)^{\frac{1}{n-1}}. \tag{3.5}$$

Lemma 3.1. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $f_j : D \to \mathbb{R}^n$, $j = 1, 2, \ldots$, a sequence of FLD homeomorphisms in D converging locally uniformly to a mapping $f : D \to \mathbb{R}^n$. Then, at each point x_0 of differentiability of the mapping f,*

$$P_O(x_0, f) \ \leq \ \liminf_{h \to 0} \liminf_{j \to \infty} \frac{1}{h^n} \int\limits_{C(x_0,h)} P_O(y, f_j) dm(y), \tag{3.6}$$

where $C(x_0, h)$ denotes the cube in $\mathbb{R}^n$ centered at x_0 with edges oriented along the principal axes of quadratic form $(f'(x_0)z, f'(x_0)z)$ and have the length h.

In particular, the right hand of (3.6) equals to infinity in points x_0 where $f'(x) \neq 0$ and $J(x, f) = 0$. The proof of Lemma 3.1 is similar to the proof of Lemma 4.7 in [4] for the mappings with bounded distortion.

Applying to (3.6) Jensen's inequality, we obtain the following conclusion.

Corollary 3.1. *Under the conditions and terms of Lemma* 3.1*,*

$$\Phi\left(P_O\left(x_0, f\right)\right) \ \leq \ \liminf_{h \to 0} \liminf_{j \to \infty} \frac{1}{h^n} \int\limits_{C(x_0,\,h)} \Phi\left(P_O\left(y, f_j\right)\right) dm(y) \tag{3.7}$$

for every increasing convex function $\Phi(t) : [1, +\infty] \to [0, +\infty]$.

In particular, for $\Phi(t) \ = \ t^{n-1}$, we have the next conclusion.

Corollary 3.2. *Under the assumptions of Lemma* 3.1*,*

$$K_O(x_0, f) \ \leq \ \liminf_{h \to 0} \liminf_{j \to \infty} \frac{1}{h^n} \int\limits_{C(x_0,h)} K_O(y, f_j) dm(y)\,. \tag{3.8}$$

Theorem 3.2. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $f_m : D \to \mathbb{R}^n$ be a sequence of FLD homeomorphisms converging locally uniformly to a FLD homeomorphism f. If*

$$K_O\left(x, f_m\right) \ \leq \ K\left(x\right) \, \in L^1_{loc}, \ \ m = 1, 2, 3, \ldots, \tag{3.9}$$

then

$$K_O\left(x, f\right) \, \leq \, \limsup_{j \to \infty} \, K_O\left(x, f_j\right) \quad a.e. \tag{3.10}$$

Proof. Let a mapping $g : D \to \mathbb{R}^n$ be differentiable a.e. Note that the corresponding function $K_O(x, g)$ is measurable. In fact, $|J(x, g)|$ is Borel function (see 24.2 and 24.4 in [21]), and $\|g'(x)\|$ is Borel, too (see Theorem 5.1 in [21]). So, $K_O(x, g)$ is measurable by its definition. Applying Corollary 3.2 and the theorem on term-by-term integration, we have that

$$K_O\left(x, f\right) \ \leq \liminf_{h \to 0} \frac{1}{h^n} \int\limits_{C(x,h)} \limsup_{j \to \infty} K_O(y, f_j) dm(y)\,. \tag{3.11}$$

Now, by the theorem on differentiability of the indefinite Lebesgue integral and (3.9), we obtain a.e. the equality

$$\lim_{h \to 0} \frac{1}{h^n} \int\limits_{C(x,h)} \limsup_{j \to \infty} K_O\left(y, f_j\right) dm(y) \ = \limsup_{j \to \infty} K_O\left(x, f_j\right)\,. \tag{3.12}$$

Combining (3.11) and (3.12), we come to (3.10).

Theorem 3.3. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $f_j : D \to \mathbb{R}^n$, $j = 1, 2, \ldots$, be a sequence of FLD homeomorphisms converging locally uniformly to an FLD homeomorphism f. Suppose that $\Phi : [1, +\infty] \to [0, \infty]$ is a convex increasing function and*

$$P_O(x, f_j) \leq K(x) \quad a.e., \tag{3.13}$$

where

$$\Phi\,(K(x)) \in L^1_{loc}. \tag{3.14}$$

Then

$$\int\limits_E \Phi\,(P_O\,(x, f))\,dm(x) \;\leq\; \liminf_{j\to\infty} \int\limits_E \Phi\,(P_O\,(x, f_j))\,dm(x) \tag{3.15}$$

for every measurable set $E \subset D$, with $\operatorname{mes} E \,<\, \infty$.

Proof. By Corollary 3.1 and (3.13) we get that $\Phi\,(P_O\,(x, f)) \,\leq\, \Phi\,(K\,(x))$ a.e. and hence, $\Phi\,(P_O\,(x, f)) \,\in\, L^1_{loc}\,(D)$ by (3.14). Consequently, by the theorem of differentiability of the indefinite integral a.e.,

$$\lim_{h\to 0}\; \frac{1}{h^n} \int\limits_{C(x,h)} \Phi\,(P_O(y, f))\,dm(y) \;=\; \Phi\,(P_O(x, f))\,. \tag{3.16}$$

Let E_0 be the set of all $x \in D$ where either f is not differentiable or (3.16) does not hold. Note that $\operatorname{mes} E_0 = 0$. By Corollary 3.1,

$$\Phi\,(P_O(x, f)) \;\leq\; \liminf_{h\to 0}\, \liminf_{j\to\infty}\; \frac{1}{h^n} \int\limits_{C(x,h)} \Phi\,(P_O(y, f_j))\,dm(y)\;\;,\;\; \forall\; x \in D\backslash E_0.$$

Hence we have that $\forall\, \varepsilon > 0 : \exists\, \delta \,=\, \delta\,(x, \varepsilon) : \forall\, h \,<\, \delta$,

$$h^n \cdot \Phi\,(P_O(x, f)) \;\leq\; \liminf_{j\to\infty} \int\limits_{C(x,h)} \Phi\,(P_O(y, f_j))\,dm(y) \,+ \varepsilon h^n\,, \tag{3.17}$$

and by (3.16),

$$\int\limits_{C(x,h)} \Phi\,(P_O(y, f))\,dm(y) \;\leq\; \liminf_{j\to\infty} \int\limits_{C(x,h)} \Phi\,(P_O(y, f_j))\,dm(y) \,+ \varepsilon h^n \tag{3.18}$$

for $h \,<\, \delta \,= \delta(x, \varepsilon)\,.$

Let $\Omega \subset D$ be an open set. The system of cubes $C(x, h)$, $x \in \Omega\backslash E_0$, forms the Vitali covering of the set $\Omega\backslash E_0$. Thus, by the Vitali theorem (see (3.1) in Ch. IV in [20]), there is a sequence of nonintersecting cubes $C_m \,=\, C(x_m, h_m) \,\subseteq \Omega$ such that

$$\operatorname{mes}\Big(\Omega\backslash \bigcup C_m\Big) \;=\; 0\,. \tag{3.19}$$

Applying (3.19) to (3.18), we obtain that

$$\int\limits_{\Omega} \Phi\,(P_O(y, f))\,dm(y) \;\leq\; \liminf_{j\to\infty} \int\limits_{\Omega} \Phi\,(P_O(y, f_j))\,dm(y) \,+ \varepsilon \operatorname{mes} \Omega\,, \tag{3.20}$$

and since $\varepsilon > 0$ is arbitrary, inequality (3.15) is proved for an arbitrary open set $\Omega \subset D$, with $\mathrm{mes}\Omega < \infty$.

Now, let E be a measurable set in D, with $\mathrm{mes}E < \infty$. Then, for every $\varepsilon > 0$, there is an open set $\Omega = \Omega_\varepsilon \supseteq E$, with $\mathrm{mes}\,(\Omega_\varepsilon \backslash E) < \varepsilon$ (see (6.6), Ch. III in [20]). From inequality (3.15), for the set Ω, we have that

$$\int\limits_E \Phi\left(P_O(y,f)\right) dm(y) \ \le \ \liminf_{j\to\infty} \int\limits_E \Phi\left(P_O(y,f_j)\right) dm(y) + \int\limits_{\Omega_\varepsilon \backslash E} \Phi\left(K(\,y\,)\right) dm(y).$$

In view of (3.14), we finally obtain (3.15) by absolute continuity of the Lebesgue integral.

4. CONVERGENCE OF MATRIX DILATATIONS

Let $f : D \to \mathbb{R}^n$ be a mapping with finite metric distortion. Recall that the *matrix dilatation of f at $x \in D$* is the normalized Jacobian matrix, namely,

$$M_f\,(x) \ = \ \frac{f'\,(x)}{|J\,(x,\,f)\,|^{1/n}} \tag{4.21}$$

if x is a regular point of the mapping f, where f is differentiable, with $J(x,f) \neq 0$ and $M_f\,(x) = \mathbf{I}$ (the unit matrix) in the contrary case. The *dilatation tensor of f at $x \in D$* is the symmetrized normalized Jacobian matrix, i.e.,

$$G_f\,(x) \ = \ M_f^*\,(x)\,M_f\,(x)\,, \tag{4.22}$$

where M^* denotes the transpose of M. It is clear that $|\mathrm{det}M_f\,(x)\,| = 1 = \mathrm{det}G_f\,(x)$ and that $K_O\,(x,f) = \|M_f\,(x)\,\|^n$ a.e.

Theorem 4.1. *Let $f, f_j : D \to \mathbb{R}^n$ be FLD homeomorphisms such that $f_j \to f$ as $j \to \infty$ locally uniformly in D, let M and M_j be their matrix dilatations, and let $K\,(x,f_j) \le K\,(x) \in L^1_{loc}$ a.e. Assume that*

$$U_j(x)M_j(x) \to M_0(x) \qquad a.e. \tag{4.23}$$

as $j \to \infty$ for some sequence of orthogonal matrices $U_j(x)$. Then

$$M\,(x) = U\,(x)\,M_0\,(x) \qquad a.e. \tag{4.24}$$

for some orthogonal matrix $U(x)$.

Proof. Let $A_m : \mathbb{R}^n \to \mathbb{R}^n$ be some enumeration of all the matrices with rational elements which satisfy the condition $\mathrm{det}A_m = 1$. Note that each matrix A_m defines a quasiconformal linear mapping $x \to A_m x$, where $x \in \mathbb{R}^n$ is interpreted as a vector-column. Let N and N_j be matrix dilatations of the mappings $f \circ A_m^{-1}$ and $f_j \circ A_m^{-1}$, correspondingly. Then, by the composition rule,

$$N\,(y) = M\left(A_m^{-1}y\right)A_m^{-1}, \quad N_j\,(y) = M_j\left(A_m^{-1}y\right)A_m^{-1}. \tag{4.25}$$

Note that the mappings $f \circ A_m^{-1}$ and $f_j \circ A_m^{-1}$ are also FLD homeomorphisms such that $f_j \circ A_m^{-1} \to f \circ A_m^{-1}$ locally uniformly as $j \to \infty$, and

$$K_O\left(y, f_j \circ A_m^{-1}\right) \le K_O\left(A_m^{-1}y, f_j\right) K_O\left(A_m^{-1}\right) \le K\left(A_m^{-1}y\right) \cdot c_m^n \in L^1_{loc},$$

where $c_m = \|A_m^{-1}\|$ is a matrix norm of A_m^{-1} (see (2.11), Ch. I in [16]). Thus, for each fixed $m \in \mathbf{N}$, the mappings $f \circ A_m^{-1}$ and $f_j \circ A_m^{-1}$ satisfy the conditions of Theorem 3.2. By Theorem 3.2, for each fixed $m = 1, 2, \dots,$

$$\|M\left(A_m^{-1}y\right)A_m^{-1}\| \le \limsup_{j\to\infty} \|M_j\left(A_m^{-1}y\right)A_m^{-1}\| \qquad \text{a.e.} \tag{4.26}$$

because $K_O(x, f) = \|M_f(x)\|^n$. Consequently, for each fixed $m = 1, 2, \dots,$ we have also

$$\|M(x)A_m^{-1}\| \le \limsup_{j\to\infty} \|M_j(x)A_m^{-1}\| \qquad \text{a.e.} \tag{4.27}$$

Since the collection of matrices A_m is countable, (4.27) holds for a.e. $x \in D$ and all $m = 1, 2, \dots.$ Note that the set of matrices $\{A_m\}_{m=1}^{\infty}$ is dense in the space of all matrices with $\det A = 1$, and it follows from (4.23) that $\det M_0(x) = 1$ a.e. Thus, we have from (4.27) that

$$\|M(x)M_0^{-1}(x)\| \le \limsup_{j\to\infty} \|M_j(x)M_0^{-1}(x)\| \qquad \text{a.e.} \tag{4.28}$$

Since U_j are orthogonal matrices and $U_j(x)M_j(x) \to M_0(x)$ a.e., we have from (4.28) that

$$\|M(x)M_0^{-1}(x)\| \le 1 \qquad \text{a.e.} \tag{4.29}$$

On the other hand (for example, see (2.6), Ch. I in [16])

$$\|M(x)M_0^{-1}(x)\| \ge 1. \tag{4.30}$$

Thus,

$$\|M(x)M_0^{-1}(x)\| = 1 \qquad \text{a.e.} \tag{4.31}$$

Finally, taking the relation $\det M(x)M_0^{-1}(x) = 1$ a.e. into account, from (4.31) we have that $M(x)M_0^{-1}(x)$ is an orthogonal matrix $U(x)$, i.e., $M(x) = U(x)M_0(x)$ a.e. The proof is over.

By Lemma 3.18 from [4], the relation $G_j \to G$ is valid iff $U_jM_j \to M$ for some orthogonal matrices U_j. Thus, we have the following

Corollary 4.1. *Let $f, f_j : D \to \mathbb{R}^n$ be FLD homeomorphisms such that $f_j \to f$ locally uniformly as $j \to \infty$. Let G and G_j be the dilatation tensors of f and f_j, respectively, and let $K_O(x, f_j) \le K(x) \in L^1_{loc}$ a.e. If*

$$G_j(x) \to G_0(x) \qquad \text{a.e.} \tag{4.32}$$

as $j \to \infty$, then

$$G(x) = G_0(x) \qquad \text{a.e.} \tag{4.33}$$

5. CLOSED SPACES OF HOMEOMORPHISMS

Let (X, d) and (X', d') be metric spaces with distances d and d', respectively. A family of continuous mappings $\mathfrak{F} : X \to X'$ is said to be a *normal* if every sequence of mappings $f_m(x) \in \mathfrak{F}$ has subsequence f_{m_k} converging uniformly on each compact set $C \subset X$ to a continuous mapping f. If, in addition, $f \in \mathfrak{F}$ then the family $\mathfrak{F}$ ia said to be *compact*. In other words, family $\mathfrak{F}$ is compact if and only if $\mathfrak{F}$ is normal and closed. Criteria of normality for Q-homeomorphisms which can be applicable to mappings with finite length distortion, were obtained in [19].

Lemma 5.1. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let $f : D \to \mathbb{R}^n$ be an ACL homeomorphism of finite metric distortion, and let the following conditions hold*:

$$K_O\left(x, f\right) \leq K(x) \in L_{loc}^{n-1}, \tag{5.1}$$

$$K_O\left(x, f^{-1}\right) \leq Q(x) \in L_{loc}^{n-1}. \tag{5.2}$$

Then $f \in W_{loc}^{1,n}(D)$ and $g = f^{-1} \in W_{loc}^{1,n}\left(f\left(D\right)\right)$. Moreover, for each compact $C \subset D$,

$$\int\limits_{C} \|f'(x)\|^n dm(x) \leq \int\limits_{f(C)} Q^{n-1}(y) dm(y),$$

$$\int\limits_{f(C)} \|g'(y)\|^n dm(y) \leq \int\limits_{C} K^{n-1}(x) dm(x). \tag{5.3}$$

In particular, f is a mapping with finite length distortion.

Proof. Let C be a compact in D. The condition $f \in FMD$ is equivalent to the condition that f is a.e. differentiable, with $J(x, f) \neq 0$, and the mapping f^{-1} is a.e. differentiable, with $J\left(x, f^{-1}\right) \neq 0$ (see Corollary 3.14 in [10]). Then

$$f'\left(x\right) = \left(f^{-1\prime}\left(f\left(x\right)\right)\right)^{-1} \text{ a.e.} \tag{5.4}$$

and, moreover, the change of variables formula holds (for example, see Proposition 3.7 in [10] and Proposition 8.3 in [12]). Thus,

$$\int\limits_{C} \|f'(x)\|^n dm(x) = \tag{5.5}$$

$$= \int\limits_{f(C)} K_I\left(y, f^{-1}\right) dm(y) \leq \int\limits_{f(C)} Q^{n-1}(y) dm(y) < \infty,$$

and, consequently, $f \in W_{loc}^{1,n}$ because $f \in ACL$ (for example, see Theorem 2 on p. 9 in [14]) Thus, by (5.1), we obtain that $f^{-1} \in W_{loc}^{1,n}$ (see [5]). Inequality (5.3) can be obtained by analogy. Thus, $f \in W_{loc}^{1,n}$. Consequently, $f \in FLD$ by Theorem 4.6 in [10].

Corollary 5.1. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let $f : D \to \mathbb{R}^n$ be a homeomorphism of finite length distortion and let conditions* (5.1) *and* (5.2) *hold. Then f and f^{-1} belong to the space $W_{loc}^{1,n}$.*

Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $K : D \to [1, \infty]$ and $Q : D' \to [1, \infty]$ be measurable functions. Denote by $H_{K,Q}$ the class of all ACL homeomorphisms $f : D \to D'$ such that

$$K_O\left(x, f\right) \leq K(x) \text{ a.e.} \tag{5.6}$$

and

$$K_O\left(y, f^{-1}\right) \leq Q(y) \text{ a.e.} \tag{5.7}$$

Theorem 5.1. *If $K \in L_{loc}^{n-1}(D)$ and $Q \in L_{loc}^{n-1}(D')$ then $H_{K,Q} \subset FLD$ and $H_{K,Q}$ forms a closed subspace of the space H of all homeomorphisms.*

Proof. The inclusion $H_{K,Q} \subset FLD$ is one of the statements of Lemma 5.1. Let $f_m, m = 1, 2, \ldots,$, be a sequence of homeomorphisms of the class $H_{K,Q}$ converging locally uniformly to some homeomorphism $f \in H$ as $m \to \infty$. By Lemma 5.1, for each compact $C \subset D$ and for each $\varepsilon \in (0, \rho(f(C), \partial D'))$,

$$\int\limits_{C} \|f_m'(x)\|^n dx \leq \int\limits_{f_m(C)} Q^{n-1}(y)\, dy \leq \int\limits_{C_\varepsilon} Q^{n-1}(y)\, dy,$$

where

$$C_\varepsilon = \{y : \rho(y, f(C)) \leq \varepsilon\}.$$

By Lemma 3.5 of Chapter III in [16], we have $f \in W_{loc}^{1,n}$, and by Theorem 3.2, $K_O(x, f) \leq K(x)$ a.e. Since f_m and f are homeomorphisms, $f_m^{-1} \to f^{-1}$ locally uniformly. By analogy, we can prove that $f^{-1} \in W_{loc}^{1,n}$ and $K_O(x, f^{-1}) \leq Q(x)$. By theorem 4.6 in[12], $f \in FLD$.

Taking Theorem 3.1 into account, we obtain the next statement.

Corollary 5.2. *Let D and D' be domains in $\mathbb{R}^n$,$n \geq 2$, and $f_m : D \to D'$ be a sequence of homeomorphisms with finite length distortion, converging locally uniformly in D to a mapping f. If, for all $m = 1, 2, \ldots$ and a.e. $x \in D$,*

$$K_O(x, f_m) \leq K(x) \in L_{loc}^{n-1},$$

$$K_O\left(x, f_m^{-1}\right) \leq Q(x) \in L_{loc}^{n-1}$$

then either f is homeomorphism with finite length distortion satisfying conditions (5.6) *and* (5.7)*, or $f \equiv const$.*

Following [6], we say that a function $\varphi : D \to \mathbb{R}$ has *finite mean oscillation* at point $x_0 \in D$ (for brevity, $\varphi \in FMO$ at x_0) if

$$\overline{\lim_{\varepsilon \to 0}} \;⨍_{B(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon|\, dm(x) < \infty, \tag{5.8}$$

where

$$\overline{\varphi}_\varepsilon = ⨍_{B(x_0,\varepsilon)} \varphi(x)\, dm(x).$$

Note that if (5.8) holds, it is possible that $\overline{\varphi_\varepsilon} \to \infty$ as $\varepsilon \to 0$. We say that a function $\varphi : D \to \mathbb{R}$ has finite mean oscillation in the domain D (for brevity, $\varphi \in FMO(D)$ or $\varphi \in FMO$) if φ has finite mean oscillation at every point $x \in D$.

Bellow $q_{x_0}(r)$ denotes the mean integral value of $K(x)$ over the sphere $|x - x_0| = r$.

Let $K : \mathbb{R}^n \to [1, \infty]$ and $Q : \mathbb{R}^n \to [1, \infty]$, $n \geq 2$, be functions of the class L_{loc}^{n-1}. Denote by $\Re_{K,Q}$ the class of all ACL homeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(0) = 0$, $f(I) = I$, where $I = (1, 0, \ldots, 0)$ and

$$K_O(x, f) \leq K(x), \quad K_O\left(x, f^{-1}\right) \leq Q(x).$$

The following results are from [19].

Proposition 5.1. *The class* $\mathfrak{R}_{K,Q}$ *is normal if at least one of the following conditions holds*:

I. *The relation*

$$\int\limits_0^{\infty} \frac{dr}{r q_{x_0}^{\beta}(r)} = \infty$$

is valid for some $\beta \geq 1/(n-1)$ *at every point* $x_0 \in \mathbb{R}^n$.

II. *At each point* $x_0 \in \mathbb{R}^n$,

$$q_{x_0}(r) = O\left[\log \frac{1}{r}\right]^{n-1}$$

as $r \to 0$.

IIa. *In particular, relation* II *is valid if*

$$Q(x) = O\left[\log \frac{1}{|x - x_0|}\right]^{n-1}$$

as $x \to x_0$ *for all* $x_0 \in \mathbb{R}^n$.

III. $K(x) \in FMO$ *in* $\mathbb{R}^n$.

IIIa. *In particular, relation* III *is valid if*

$$\overline{\lim_{\varepsilon \to 0}} \, ⨍_{B(x_0,\varepsilon)} K(x)\,dm(x) < \infty.$$

We now deduce the following consequence of Corollary 5.2.

Corollary 5.3. *The class* $\mathfrak{R}_{K,Q}$ *is compact if at least one of the conditions of Proposition* 5.1 *holds*.

REFERENCES

1. F. W. Gehring, "The definitions and exceptional sets for quasiconformal mappings," Ann. Acad. Sci. Fenn. Math., Ser. A **281** (1960).
2. F. W. Gehring and T. Iwaniec, "The limit of mappings with finite distortion," Ann. Acad. Sci. Fenn. Math. **24**, 253–264 (1999).
3. A. L. Golberg, "Distortion of p-modulo of separating sets under mappings with finite integral dilatations," Bull. Soc. Sci. Lett. Lodz, Ser. Rech. Deform. **40**, 41–51 (2003).
4. V. Ya. Gutlyanskiĭ, O. Martio, V. I. Ryazanov, and M. Vuorinen, "On convergence theorems for space quasiregular mappings," Forum Math. **10**, 353–375 (1998).
5. J. Heinonen and P. Koskela, "Sobolev mappings with integrable dilatations," Arch. Ration. Mech. Anal. **125** (1), 81–97 (1993).
6. A. Ignat'ev and V. Ryazanov, "To the theory of the boundary behavior of space mappings," Ukr. Mat. Vis. **3** (2), 199–211 (2006) [Ukrain. Math. Bull. **3** (2), 189–201 (2006)].
7. T. Iwaniec and G. Martin, *Geometrical Function Theory and Non-Linear Analysis* (Clarendon Press, Oxford, 2001).
8. T. Iwaniec and V. Sverák, "On mappings with integrable dilatation," Proc. Amer. Math. Soc. **118**, 181–188 (1993).
9. K. Kuratowski, *Topology*, I (Academic Press, New York and London, 1966).
10. O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, "Mappings with finite length distortion," J. Anal. Math. **93**, 215–236 (2004).
11. O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, "On Q-homeomorphisms," Ann. Acad. Sci. Fenn. Math. **30**, 49–69 (2005).

12. O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, *Moduli in modern mapping theory* (Springer, New York, 2009).
13. O. Martio and J. Vaisala, "Elliptic equations and maps of bounded length distortion," Math. Ann. **282** (3), 423–443 (1988).
14. V. Maz'ya, *Sobolev Classes* (Springer, Berlin–New York, 1985).
15. S. P. Ponomarev, "The N^{-1}-property of mappings, and Lusin's (N) condition," Mat. Zametki **58** (3), 411–418 (1995) [Math. Notes **58** (3), 960–965 (1995)].
16. Yu. G. Reshetnyak, *Space Mappings with Bounded Distortion* (Transl. of Math. Monographs **73**, AMS, 1989).
17. Yu. G. Reshetnyak, "Distributional derivatives and a.e.- differentiability," Mat. Sb. **75** (3), 323–334 (1968) [Sb. Math. **4** (3), 293–302 (1968)].
18. Yu. G. Reshetnyak, "Some geometrical properties of functions and mappings with generalized derivatives," Sibirsk. Mat. Zh. **7** (4), 886–919 (1966) [Siberian Math. J. **7** (4), 704–732 (1966)].
19. V. Ryazanov and E. Sevost'yanov, "Equicontinuous classes of ring Q–homeomorphisms," Sibirsk. Mat. Zh. **48** (6), 1361–1376 (2007) [Siberian Math. J. **48** (6), 1093–1105 (2007)].
20. S. Saks, *Theory of the Integral* (Dover Publ., New York, 1964).
21. J. Väisälä, *Lectures on n-dimensional quasiconformal mappings* (Lecture Notes in Math. **229**, Springer–Verlag, Berlin, 1971).
22. S. K. Vodop'yanov and A. D. Ukhlov, "Weighted Sobolev spaces and quasiconformal mappings," Dokl. Akad. Nauk **403** (5), 583–588 (2005) [Dokl. Math. **72** (1), 561–566 (2005)].