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Backward motion and waiting time phenomena for degenerate parabolic equations with nonlinear gradient absorption

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Abstract. We study energy solutions of a Cauchy problem for the p-Laplace evolution equation with nonlinear gradient absorption and nonnegative compactly supported initial data. We obtain the sufficient local asymptotic conditions on initial data that imply the backward motion and waiting time phenomena.

1. Introduction

We consider the qualitative properties of nonnegative solutions of the Cauchy problem for the viscous Hamilton-Jacoby equation:

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |\nabla u|^q = 0 \text{ in } Q_T, \tag{1}$$

$$u(0,x) = u_0(x), \quad x \in R^N, \tag{2}$$

where $Q_T = (0,T) \times R^N$, $T > 0$, $N \geq 3$, and $u_0 \in L^2(R^N)$ is a compactly supported non-negative function. That kind of equations appear as the viscosity approximation to the first order partial differential equations of Hamilton-Jacoby type, in different physical settings.

From here on we assume that

$$p > 2 \text{ and } q > 0. \tag{3}$$

The case of $p > 2$ corresponds to the slow p-Laplacian diffusion. It is well-known that the behaviour of solution to problem (1), (2) strongly depends on the value of the parameter $q > 0$. The non-negative and integrable solution to the Hamilton-Jacoby equations with the standard linear diffusion was investigated by many authors (see, for example, [10] and references therein). In the special case $p = 2$, the problem (1), (2) was extensively studied (in detail, see [6,16]).

The long-time behaviour of solution for the evolution equation (1) with nonlinear diffusion and gradient absorption was studied in [17] where the infinite waiting time was proven for $1 < q < p - 1$. In this situation the

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effect of localization arises thanks to the influence of the Hamilton-Jacoby term $|\nabla u|^q$. When the parameter $0 < q < 1$ then the nonlinear absorption term becomes dominant and diffusion plays a secondary role for the large times. It looks natural to expect a singular phenomena like the finite time extinction for $0 < q < 1$ (see [4,9]).

The main goal of this paper is to analyse the long-time behaviour of a non-negative weak solution to the problem (1), (2) for all $q > 0$ in terms of the local asymptotic of initial data. In comparison with works [3,4,9, 17] our result are more general. In particular, we obtain the *waiting time phenomenon* (briefly WTP) for every $q > 0$, and not just in the super-linear range $1 < q < p - 1$. This means that for some (finite or infinite) time the solution′s support locally does not expand, or shrink, or both, at a point y of its boundary. In most cases, the reason for the WTP come as a consequence of initial data being sufficiently flat in a neighborhood of y. Moreover, in the case $0 < q < 1$ we show the *backward motion phenomenon* (briefly BMP), and not just the finite time extinction (see, e. g. [5,12]). This means that from some time the solution′s support locally shrinks at the point y, and the reason for the BMP is the flatness of initial data in a neighborhood of y too. These phenomena were studied for the p-Laplace evolution equation and other degenerate parabolic equations in works [2,8,11,14,15,20–23] etc. We reference the reader to the work [11] for more detailed discussion about these phenomena for the doubly nonlinear degenerate parabolic equation with nonlinear (non-gradient) absorption.

The paper is organized as follows. In Section 2 the assumptions and main results are formulated. It turns out, that the appearance of WTP and BMP depends strongly on the local flatness conditions of initial data, i. e. the stronger flatness of initial data the stronger influence of gradient absorption term. For example, if $0 < q < p - 1$ then the WTP is possible for all times (see Theorem 1), For $0 < q < 1$ the BMP appears under the additional restriction on the initial data (see Theorem 2). Some auxiliary integral estimates are shown in Section 3. The main results are proved in Sections 4,5. In Section 4, we show the finite WTP for $q > p - 1$ and the infinite WTP for $0 < q < p - 1$ to problem (1), (2). In Section 5, we obtain the BMP for $0 < q < 1$ that implies the finite time extinction. The Appendix A contains the proofs of several technical lemmas. Some auxiliary results from functional analysis are placed in Appendix B.

2. Main results

We define a weak solution for the problem (1), (2) in the sense of [1].

Definition 1. *A nonnegative function u is a weak solution of the problem (1), (2) if*

(*i*) $u \in C(0,T; L^2(R^N)), \quad |\nabla u|^p, |\nabla u^{\frac{q+1}{q}}|^q \in L^1(Q_T);$

(*ii*) *the following identity holds*

$$\frac{1}{2}\int_{R^N} u^2(t,x)\,\phi(t,x)\,dx - \frac{1}{2}\int\int_{Q_T} u^2(t,x)\,\phi_t(t,x)\,dx\,dt$$
$$+\int\int_{Q_T} |\nabla u|^p \phi\,dx\,dt + \Big(\frac{q}{q+1}\Big)^q \int\int_{Q_T} |\nabla u^{\frac{q+1}{q}}|^q \phi\,dx\,dt$$
$$-\int\int_{Q_T} |\nabla u|^{p-2}\nabla u\,u\,\nabla\phi\,dx\,dt = \frac{1}{2}\int_{R^N} u_0^2(x)\,\phi(0,x)\,dx \qquad (4)$$

for every function $\phi \in C^1(Q_T)$*;*

(*iii*) *the function* u *attains the initial data in the following sense*

$$u(t,\cdot) \to u_0 \text{ in } L^2(R^N) \text{ as } t \to 0$$

where the compactly supported function $u_0 \in L^2(R^N)$*.*

It is worth to mention that the definition above is different from the one introduced in [17] for weak viscosity solutions.

Now we formulate some auxiliary definitions. For $x, y \in R^N$ let

$$x' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N), \quad y' = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_N).$$

For every $s \in R^1,\ i = 1, \ldots, N$, let us denote by

$$\Omega^{(y)}(s) := \big\{x \in R^N : x_i > y_i + d(s + |x' - y'|),\ y \in \partial\{\operatorname{supp} u_0\}\big\} \qquad (5)$$

a cone with vertex at $y \in R^N$, opening angle $d = d(y) > 0$, and symmetry axis parallel to the x_i-axis. Let u be an arbitrary weak solution to (1), (2) in the sense of Definition 1. Using (5), we define the function $\Gamma^{(y)}(t)$ as

$$\Gamma^{(y)}(t) := \inf\{s \in R^1 : \operatorname{supp} u(t,\cdot) \cap \Omega^{(y)}(s) = \emptyset\} \qquad (6)$$

for every $y \in R^N,\ t \in R^+$. At the point $y = 0$ we define

$$\Gamma(t) := \Gamma^{(0)}(t).$$

Our main goal is to study the dependence between the evolution of support of a weak energy solution u and the local asymptotic of initial data u_0. We suppose that u_0 satisfies the following conditions:

$$u_0(x) \geq 0, \qquad (7)$$

$$\operatorname{meas}\{\Omega^{(y)}(s) \cap \operatorname{supp} u_0\} = 0 \ \ \forall\, y \in \partial\{\operatorname{supp} u_0\},\ s \geq 0. \qquad (8)$$

We study the properties of solutions of problem (1), (2) under some extra restriction on flatness of initial function u_0 in a neighborhood of the boundary support. We formulate this assumption in terms of the function $h_0^{(y)}(s)$:

$$h_0^{(y)}(s) := \int\limits_{\Omega^{(y)}(s)} u_0^2(x)\,dx, \qquad (9)$$

for an arbitrary point $y \in \partial\{\operatorname{supp} u_0\}$ and for every $y \in R^N$.

Remark 1. It follows from (9) and (8), that $h_0^{(y)}(s)$ is a nonincreasing non-negative continuous function satisfying the following property:

$$h_0^{(y)}(s) \equiv 0 \ \ \forall\, s \geq 0, \ \ y \in \partial\{\operatorname{supp} u_0\}. \tag{10}$$

Let us assume that the function $h_0^{(y)}(s)$ satisfies the *flatness* conditions. Hence, for every $s:\ s_0 < s < 0$ and $p > 2$, one of the following estimates

$$h_0^{(y)}(s) \leq \chi\,(-s)^{N+\frac{2p}{p-2}} \text{ for } q > 0; \tag{11}$$

$$h_0^{(y)}(s) \leq \chi\,(-s)^{N+\frac{2q+N(q-p+1)}{p-2}} \text{ for } q \geq q_* := \max\left\{1, \frac{N(p-1)}{N+2}\right\}; \tag{12}$$

$$h_0^{(y)}(s) \leq \chi\,(-s)^{N+\frac{2p}{p-q-1}} \text{ for } 0 < q < p-1; \tag{13}$$

holds true for some positive constant χ.

In this paper, we prove the infinite WTP if the condition (13) is satisfied and the finite WTP if one of the conditions (11), (12) is valid.

Remark 2. Note, that the convexity of the support is not required as the conditions (11)–(13) are local at y.

Let us proceed with the formulation of sufficient condition for the WTP.

Theorem 1. *Assume that the initial data u_0 satisfies the conditions (7), (8), the point $y \in \partial\{supp\, u_0\}$ and the function $h_0^{(y)}(s)$ satisfies the condition (10) and one of the flatness conditions (11)–(13).*

Then for an arbitrary weak solution u of problem (1), (2) there exists the time $T^* = T^*(\chi) > 0$ depending on the known parameters only such that

$$\operatorname{supp} u(t,\cdot) \cap \Omega^{(y)}(0) = \emptyset \ \forall\, 0 < t < T^*, \tag{14}$$

where χ is the constant from the flatness conditions. Note, that $T^* \to +\infty$ as $\chi \to 0$.

Moreover, if $0 < q < p-1$ and there exists a constant $\kappa_1 > 0$ depending on the known parameters only (see (54)), such that the constant χ of the flatness condition (13) satisfies the following inequality

$$\chi < \kappa_1. \tag{15}$$

Then the property (14) holds for all $t > 0$.

Remark 3. We do not give the proof of the finite WTP. This phenomenon follows from [8, Theorem 2.1] applied with $(w,p,q,s,l,k) = (u,p,2,p,1,1)$ for $q > 0$, and $(w,p,q,s,l,k) = (u^{\frac{q+1}{q}}, q, \frac{2q}{q+1}, \frac{pq}{q+1}, 1, 1)$ for $q \geq q_*$ directly.

Now we illustrate the conditions of Theorem 1 for $0 < t < T^*$ and stronger regularity of initial data.

Example 1. Let the initial function $u_0 \in C(R^N)$ satisfies one of the following properties:

1) if $0 < q < 1$ then we require that

$$\sup_{x\in\Omega^{(y)}(s)} u_0(x) \le \chi\,(-s)^{\frac{p}{p-q-1}};$$

2) if $q_* \le q < q^* := p - \frac{N}{N+2}$ then we require that

$$\sup_{x\in\Omega^{(y)}(s)} u_0(x) \le \chi\,(-s)^{\frac{2q+N(q-p+1)}{2(p-2)}};$$

3) if $q \ge q^*,\ p > 2$ or $1 \le q < q_*,\ p > \frac{2(N+1)}{N}$ then we require that

$$\sup_{x\in\Omega^{(y)}(s)} u_0(x) \le \chi\,(-s)^{\frac{p}{p-2}}.$$

The requirements $1) - 3)$ will imply the flatness conditions (11)–(13) and hence guarantee the WTP, i. e. the property (14) for solutions of problem (1), (2).

Under the proposed conditions on the parameters p and q, the influence of the diffusive term in equation (1) is more stronger than nonlinear absorption term for the large times for $p > 2$ and $q \ge 1$. On the other hand, the nonlinear absorption term dominates in the case $0 < q < 1$. The upper and lower bounds on the waiting time for the solution to the Cauchy problem for doubly nonlinear parabolic equations were found in [8, 13, 20, 21], and they coincide with our result for the case 3) when the diffusive term dominates, i. e. for $q \ge q^*,\ p > 2$ or $1 \le q < q_*,\ p > \frac{2(N+1)}{N}$.

Now we formulate the main result of the present paper about the sufficient condition of backward motion. Let us denote by γ the following constant:

$$\gamma := N + \frac{2p}{p-q-1}, \tag{16}$$

which is the critical exponent from the right-hand side of estimate (13). Let $H_0^{(y)}(s)$ be a majorant of function $h_0^{(y)}(s)$, which satisfies condition (13) for all $s : s_0 < s < 0$, i. e.

$$0 \le h_0^{(y)}(s) \le H_0^{(y)}(s) \le \chi(-s)^{\gamma}. \tag{17}$$

Moreover, the following condition of the qualified monotonicity:

$$H_0^{(y)}\big(s + K\,(H_0^{(y)}(s))^{\frac{1}{\gamma}}\big) \ge k\,H_0^{(y)}(s)\ \forall\, s \in R^1, \tag{18}$$

holds true for some constants $1 < K < \infty$ and $0 < k < 1$ depending on the known parameters only, and γ defined by (16).

Let us denote by f the following auxiliary function

$$f(s) := |s|^{1-\mu}(H_0^{(y)}(s))^{\sigma}, \tag{19}$$

where

$$\mu := 1 - \frac{2p + N(p-2)}{2p + N(p-q-1)}, \ \sigma := \frac{p(1-q)}{2p + N(p-q-1)}.$$

This function characterizes the speed of backward motion.

The nonlinear absorption becomes strongly dominant when the parameter $q \in (0,1)$. In this case, the solution of the problem (1), (2) is not Lipschitz continuous. It gives the grounds to expect the singular phenomena such as the finite time extinction. The next theorem describes the condition for the BMP.

Theorem 2. *Assume that for*

$$0 < q < 1 \tag{20}$$

the initial function u_0 *satisfies conditions (7) and (8), the point* $y \in \partial\{supp\, u_0\}$, *and the majorant function* $H_0^{(y)}(s)$ *satisfies flatness condition (17). Let also for the constant* χ *from the flatness condition (17) the following inequality*

$$\chi \leq \kappa_2. \tag{21}$$

be valid.

Then there exists a constant κ_2 *(see (67))*: $0 < \kappa_2 \leq \kappa_1$ *depending on* q, p, N *and* $\|u_0\|_{L^2(\mathbf{R}^N)}$ *only, such that for* $\Gamma^{(y)}(t)$ *the following estimate*

$$\Gamma^{(y)}(t) < 0 \tag{22}$$

is true for all $t > 0$.

Moreover, there exist positive constants κ_3 *and* κ_4 *depending on* N, p, q, $\|u_0\|_{L^2(R^N)}$ *only, such that the following estimate of the shrinking speed of support holds*

$$-\Gamma^{(y)}(t) \geq \kappa_3\, f^{-1}(\kappa_4\, t) \quad \forall\, t > 0, \tag{23}$$

where $f^{-1}(\cdot)$ *is the inverse function of* $f(\cdot)$ *defined by (19).*

Remark 4. For compactly supported initial data u_0 the finite time extinction follows immediately from the estimate (23).

Remark 5. From the definitions κ_1 and κ_2 are defined by (54) and (67) accordingly, there is an initial function u_0 such that $\kappa_1 = \kappa_2$. Hence the condition (15) for the infinite WTP and the condition (21) for the BMP coincide.

The next theorem describes the behaviour of a weak energy solution to the problem (1), (2) with an initial functions decreasing at infinity.

Theorem 3. *Under the hypotheses of Theorem 2, let the majorant $H_0^{(y)}(s)$ satisfy the flatness condition (17) and*

$$H_0^{(y)}(s) \leq \overline{H} = const < +\infty \ \forall\, s < 0. \tag{24}$$

Then there exists a constant $\kappa_5 > 0$ such that the following estimate

$$-\Gamma^{(y)}(t) \geq \kappa_5 t^{\frac{1}{1-\mu}} \tag{25}$$

is valid for all $t > 0$.

We proceed with examples to illustrate the case of the BMP under different kind of flatness conditions for the function (9). For the critical growth of the initial data, the Example 2 explains the maximum asymptotic speed of backward motion at large times.

Example 2. Let there exists a constant $\kappa_2^{(0)} : 0 < \kappa_2^{(0)} \leq \kappa_2$, such that the following inequality

$$h_0^{(y)}(s) \leq \kappa_2^{(0)} H_0^{(0)}(s) \quad \forall\, s < 0, \tag{26}$$

holds for $H_0^{(0)}(s) := (-s)^{\gamma}$, where the constant γ is the critical exponent defined by (16). Then, in virtue of (23), we have

$$-\Gamma^{(y)}(t) \geq \kappa_6 t^{\beta_1},$$

where $\kappa_6 = \kappa_3\,(\kappa_2^{(0)})^{-\sigma}(\kappa_4)^{\beta_1}$, $\beta_1 := \frac{(p-q-1)(2p+N(p-q-1))}{(p-q-1)(2p+N(p-2))+p(N+2)(1-q)} < 1$.

The next example illustrates the asymptotic behaviour of the solution to the problem (1), (2) for the case of subcritical growth of the initial function.

Example 3. Let there exists $\kappa_2^{(1)} : 0 < \kappa_2^{(1)} \leq \kappa_2$, such that the following inequality

$$h_0^{(y)}(s) \leq \kappa_2^{(1)} H_0^{(1)}(s) \quad \forall\, s < 0, \tag{27}$$

holds for $H_0^{(1)}(s) := \kappa_2^{(1)}(-s)^{\tilde{\gamma}}$, $\tilde{\gamma} > \gamma$, where the constant γ is the critical exponent defined by (16). Then from (13) we obtain

$$-\Gamma^{(y)}(t) \geq \kappa_7 t^{\beta_2},$$

where $\kappa_7 = \kappa_3(\kappa_2^{(1)})^{-\sigma}(\kappa_4)^{\beta_2}$, $\beta_2 := \left(\frac{1}{\beta_1} + \frac{(\tilde{\gamma}-\gamma)p(1-q)(p-q-1)}{2p+N(p-q-1)}\right)^{-1} < \beta_1$.

The following example illustrates the case of slow decreasing of the solution to the problem (1), (2) under the logarithmic correction in the flatness condition for the initial function.

Example 4. Let there exists $\kappa_2^{(2)} : 0 < \kappa_2^{(2)} \le \kappa_2$, such that the following inequality

$$h_0^{(y)}(s) \le \kappa_2^{(2)} H_0^{(2)}(s) \quad \forall\, s < 0, \tag{28}$$

is valid for $H_0^{(2)}(s) := (-s)^\gamma |\ln(-s)|^\vartheta$, $\vartheta \in R^1$, where the constant γ is defined by (16). Then, in virtue of (13), we find

$$-\Gamma^{(y)}(t) \ge \kappa_8 t^{\beta_1} |\ln(\kappa_4\, t)|^{-\vartheta\sigma\beta_1},$$

where $f(s) = (\kappa_2^{(2)})^\sigma (-s)^{1/\beta_1} |\ln(-s)|^{\sigma\vartheta}$, $\kappa_8 = \kappa_3\,(1 + |\vartheta|)(\kappa_2^{(2)})^{-\sigma}(\kappa_4)^{\beta_1}$.

Notation. From here on by $C, C_j, c_j, \tilde{c}_j$ we denote generic constants that depend on N, p, q only.

3. Preliminary integral estimates

Below, without loss of generality, we suppose that $y = 0$ and $i = 1$ in (5). Let us denote by $h_0(s)$ the following function

$$h_0(s) := h_0^{(0)}(s).$$

We define the families of sets:

$$\Omega(s) := \Omega^{(0)}(s), \quad Q_{t_1}^{t_2}(s) := (t_1, t_2) \times \Omega(s), \quad K(s,\delta) := \Omega(s) \setminus \Omega(s+\delta),$$

$$K_{t_1}^{t_2}(s,\delta) := (t_1,t_2)\times K(s,\delta),\ Q_T(s) := Q_0^T(s),\ K_T(s,\delta) := K_0^T(s,\delta). \tag{29}$$

Further, we use the family of basic cut-off functions $\varphi_{s,\delta} \in C^2(R^N)$ from [22,23], which possess the following properties:

$$0 \le \varphi_{s,\delta}(x) \le 1\ \forall\, x \in R^N,\ \operatorname{supp}\varphi_{s,\delta}(x) \subset \Omega(s),$$

$$\varphi_{s,\delta}(x) = \begin{cases} 0 & \forall\, x \notin \Omega(s), \\ 1 & \forall\, x \in \Omega(s+\delta), \end{cases}$$

$$|\nabla\varphi_{s,\delta}| \le \frac{\tilde{c}}{\delta},\ |\Delta\varphi_{s,\delta}| \le \frac{\tilde{c}}{\delta^2} \quad \forall\, x \in K(s,\delta), \tag{30}$$

with the positive constant $\tilde{c}$.

Now let u be an arbitrary weak solution to (1), (2) in the sense of Definition 1. For every $s \in R^1$, $\delta > 0$, $h > 0$, $T > 0$, and τ, $\omega : 0 < \omega < \tau \le T$ we introduce the energy functionals related to this solution:

$$\mathcal{E}_T(s,\tau) := \int\!\!\int_{Q_{T-\tau}^T(s)} u^p\, dxdt, \quad \mathcal{E}_T(s) := \mathcal{E}_T(s,T); \tag{31}$$

$$\mathcal{H}_T(s,\tau,h) := \int_{T-\tau}^T \left(\int_{\Omega(s)} u^2\, dx \right)^h dt,$$

$$\mathcal{H}_T(s,\tau) := \mathcal{H}_T(s,\tau,1), \quad \mathcal{H}_T(s) := \mathcal{H}_T(s,T,1); \tag{32}$$

$$\mathcal{R}_T(s,\delta,\tau,\omega) := \delta^{-p}\mathcal{E}_T(s,\tau) + \omega^{-1}\mathcal{H}_T(s,\tau); \tag{33}$$

$$\mathcal{M}_T(s,\delta,\tau,\omega) := \delta^{-p}\mathcal{E}_T(s,\tau) + \omega^{-1}\mathcal{H}_{T-\tau+\omega}(s,\omega); \tag{34}$$

$$\mathcal{P}_T(s,\delta) := \delta^{-p}\mathcal{E}_T(s) + h_0(s). \tag{35}$$

For every $s \in R^1$, $\delta > 0$, $T > 0$, let us denote by

$$\mathcal{L}_T(s+\delta) := \sup_{t\in(0,T)} \int_{\Omega(s+\delta)} u^2(t,x)\,dx + \frac{1}{T}\int\int_{Q_T(s+\delta)} u^2\,dx\,dt$$

$$+ \int\int_{Q_T(s+\delta)} |\nabla u|^p\,dx\,dt + c_1\int\int_{Q_T(s+\delta)} |\nabla u^{\frac{q+1}{q}}|^q\,dx\,dt, \tag{36}$$

where the constant $c_1 = 2\big(\frac{q}{q+1}\big)^q$ and u is an arbitrary weak solution of the problem (1), (2).

Further, we prove the following preliminary lemmas.

Lemma 1. *Assume that condition (3) is satisfied. Then there exists a positive constant c_2 such that the following estimate*

$$\mathcal{L}_T(s+\delta) \le c_2\mathcal{P}_T(s,\delta) \tag{37}$$

is valid for all $s \in R^1,\ \delta > 0,\ T > 0$.

Proof. Testing integral identity (4) by

$$\phi(x,t) = \varphi_{s,\delta}(x)\exp\left(-t\cdot T^{-1}\right)\ \ \forall T > 0$$

and making the simple transformations, we come to the following inequality

$$\mathcal{L}_T(s+\delta) \le \varepsilon\int\int_{K_T(s,\delta)} |\nabla u|^p\,dx\,dt + \frac{c(\varepsilon)}{\delta^p}\int\int_{K_T(s,\delta)} u^p\,dx\,dt + h_0(s), \tag{38}$$

where $s \in R^1,\ \delta > 0,\ T > 0$. Choosing $\varepsilon > 0$ sufficiently small, e. g. $\varepsilon = 2^{-1-p}$, and iterating inequality (38), we obtain (37) for $c_2 = \max\{e, \frac{(p-1)^{p-1}(2e\tilde{c})^p}{2^{1-p^2}p^p}\}$.

Lemma 2. *Assume that condition (3) is satisfied. Let u be an arbitrary weak solution of the problem (1), (2). Then there exists a positive constant c_3 such that the following estimate*

$$\sup_{t\in(0,T)} \int_{\Omega(s+\delta)} u^2(t,x)\,dx + \int\int_{Q^T_{T-\tau}(s+\delta)} \left|\nabla u\right|^p\,dx\,dt$$

$$+c_1\int\int_{Q^T_{T-\tau}(s+\delta)} \left|\nabla u^{\frac{q+1}{q}}\right|^q\,dx\,dt \le c_3\mathcal{M}_T(s,\delta,\tau,\omega) \tag{39}$$

is valid for all $s \in R^1,\ \delta > 0$, and $\tau,\,\omega : 0 < \omega < \tau \le T$.

Proof. Let us define by $\psi \in C^1(R^1)$ the cut-off function with the following properties:

$$\psi(t) = \begin{cases} 0 & \text{for } t \le T - \tau - \omega, \\ 1 & \text{for } t \ge T - \tau, \end{cases} \quad 0 \le \psi(t) \le 1 \quad \forall\, t > 0,$$

$$\Big|\frac{\partial \psi(t)}{\partial t}\Big| \le \frac{\tilde{\tilde{c}}}{\omega} \quad \forall\, 0 < \omega < \tau < T,\ T > \tau + \omega. \tag{40}$$

The function $\varphi_{s,\delta}(x)$ is defined by (30). Substituting the function $\phi(x,t) = \varphi_{s,\delta}(x)\psi(t)$ into the integral identity (4), after simple transformations we obtain (39) for $c_3 = \max\{\tilde{\tilde{c}}, \frac{(p-1)^{p-1}(2\tilde{c})^p}{2^{1-p^2}p^p}\}$.

Lemma 3. *Let $\mathcal{H}_T(s,\tau,h)$ be the function defined by (32). Then there exists a positive constant c_4 such that the following inequality*

$$\mathcal{H}_T(s+\delta, \tau-\omega, h_2) \le c_4\, \mathcal{H}_T(s,\tau,h_1)\, \mathcal{M}_T^{h_2-h_1}(s,\delta,\tau,\omega) \tag{41}$$

holds for all $s \in R^1$, $\delta > 0$, $\tau, \omega : 0 < \omega < \tau \le T$, and $h_1, h_2 : 0 < h_1 < h_2 < \infty$.

Proof. We introduce the following auxiliary function:

$$\mathcal{X}_h(t) := \int_0^t \left(\int_{\Omega(s)} u^2 \varphi_{s,\delta}(x)\psi(\theta-\omega)\, dx \right)^h d\theta,$$

where the functions $\varphi_{s,\delta}$ and ψ are defined by (30) and (40) accordingly. In view of the Fubini theorem, the relation

$$\mathcal{X}_{h+1}(T) = \int\!\!\int_{Q^T_{T-\tau}(s)} u^2 \varphi_{s,\delta}(x)\psi(t-\omega)\frac{d}{dt}\mathcal{X}_h(t)\, dx\, dt$$

is valid. Let us define the function $\tilde{\mathcal{H}}_T(s,\tau,h)$ by $\tilde{\mathcal{H}}_T(s,\tau,h) := \mathcal{X}_h(T)$ for all $s \in R^1$, $h : 0 < h < \infty$, and $\tau : 0 < \tau \le T$. Substituting the function $\phi(t,x) = \varphi_{s,\delta}(x)\psi(t-\omega)\mathcal{X}_h(t)$ into (4) and using simple transformations, we obtain the following recursive estimate:

$$\tilde{\mathcal{H}}_T(s,\tau,h+1) \le C\, \tilde{\mathcal{H}}_T(s,\tau,h)\, \mathcal{M}_T(s,\delta,\tau,\omega)\ \forall\, h > 0.$$

Iterating this inequality, we find

$$\tilde{\mathcal{H}}_T(s,\tau,h) \le C\, \tilde{\mathcal{H}}_T(s,\tau,h-\ell)\, \mathcal{M}^{\ell}_T(s,\delta,\tau,\omega)\ \forall\, h > \ell,\ \ell \in N,$$

whence, due to the Hölder inequality, it follows the relation:

$$\tilde{\mathcal{H}}_T(s,\tau,h_2) \le C\, \tilde{\mathcal{H}}_T(s,\tau,h_1)\, \mathcal{M}_T^{h_2-h_1}(s,\delta,\tau,\omega)\ \forall\, h_2 > h_1 > 0.$$

It is also easy to see that

$$\mathcal{H}_T(s+\delta,\tau-\omega,h) \le \tilde{\mathcal{H}}_T(s,\tau,h) \le \mathcal{H}_T(s,\tau,h).$$

Hence, taking into account the previous inequality, we obtain the estimate (41).

4. Waiting time phenomenon

In this section, we find the condition on local behaviour of the initial data which results into the infinite WTP for weak solutions of the problem (1), (2).

First we obtain an additional estimate for the functional $\mathcal{E}_T(s)$ from (31).

Lemma 4. *Assume that conditions (3) are satisfied. Then there exists the positive constant c_5 depending on the known parameters only, such that the following estimate*

$$\mathcal{E}_T(s+\delta) \le c_5 \Gamma^{\mu_0}(T)\, \mathcal{P}_T^{1+k_0}(s,\delta) \quad \text{if} \;\; 0<q<p-1 \tag{42}$$

is true for all $s \in R^1$, $\delta > 0$, and $T>0$, where $c_5 = \omega_N^{\frac{\mu_0}{N}}$, ω_N is the volume of the unit ball in R^N, and

$$\mu_0 := \frac{2pq}{2p+N(p-q-1)},\;\; k_0 := \frac{p(p-q-1)}{2p+N(p-q-1)}.$$

Proof. Proof of Lemma 4 is based on the auxiliary Lemma 1. It is of a technical character and is given in Appendix A.

Now we are ready to prove the Theorem 1.

Proof (Proof of Theorem 1). For $q>0$ the finite speed of propagation for the support of the solution follows, for example, from [19] where the equation without absorption was analyzed. Because the absorption can only reduce the speed of propagation the same proof is applicable. Our main interest is the case $0<q<p-1$.

By Lemma 4, for every $s \in R^1$, $\delta>0$ we obtain the following estimate for the function $\mathcal{E}_T(s)$:

$$\mathcal{E}_T(s+\delta) \le c_6(\Gamma^{\mu_0}(T)\delta^{-p(1+k_0)}\mathcal{E}_T^{1+k_0}(s) + \Gamma^{\mu_0}(T)h_0^{1+k_0}(s)) \tag{43}$$

if $0<q<p-1$ and $c_6 = 2^{k_0}c_5$. First, we introduce the notation

$$\delta_T(s) := \left[2\, c_6\, \Gamma^{\mu_0}(T)\, \mathcal{E}_T^{k_0}(s)\right]^{\frac{1}{p(1+k_0)}}.$$

Setting $\delta = \delta_T(s)$ in (43), we have

$$\mathcal{E}_T(s+\delta_T(s)) \le \frac{1}{2}\mathcal{E}_T(s) + c_6\Gamma^{\mu_0}(T)h_0^{1+k_0}(s) \text{ for } 0<q<p-1.$$

Taking both sides of the above inequality to the power $\frac{k_0}{p(1+k_0)} < 1$ and multiplying them by $(2\, c_6\, \Gamma^{\mu_0}(T))^{\frac{1}{p(1+k_0)}}$, we obtain

$$\delta_T(s+\delta_T(s)) \le \varepsilon\, \delta_T(s) + \mathcal{F}_T(s) \;\forall\, s \in R^1, \tag{44}$$

where $0 < \varepsilon = 2^{-\frac{k_0}{p(1+k_0)}} < 1$. Here

$$\mathcal{F}_T(s) := c_7 \Gamma^{\frac{\mu_0}{p}}(T) h_0^{\frac{k_0}{p}}(s) \quad \text{for } 0 < q < p-1,$$

where $c_7 = (2^{\frac{1}{1+k_0}} c_6)^{\frac{1}{p}}$.

We set $s = -2\delta,\ \delta = s' > 0$ in (43) and pass to the limit as $s' \to \infty$. Using the boundedness of functions $\mathcal{E}_T(s)$ and $h_0(s)$, in view of (43), we deduce

$$\mathcal{E}_T(-\infty) \le c_6 \Gamma^{\mu_0}(T) h_0^{1+k_0}(-\infty). \tag{45}$$

Hence, there exists $s_1 \in (-\infty, 0)$ such that

$$\delta_T(s) \le c_8 \mathcal{F}_T(s)\ \forall\, s < s_1 < 0, \tag{46}$$

where the constant $c_8 > 1$.

Below, we show the simple corollary from functional inequality (44) related to the boundedness of the function $\Gamma(t)$. In view of (10), we find from (44) that

$$\delta_T(s + \delta_T(s)) \le \varepsilon\, \delta_T(s)\ \forall\, s \ge 0,\ 0 < \varepsilon < 1. \tag{47}$$

Taking into account property (47), we apply Lemma B.3 of Appendix B to the function $\delta_T(s)$ for $s \ge 0$ and derive

$$\delta_T(s) \equiv 0\ \forall\, s \ge \frac{1}{1-\varepsilon}\, \delta_T(0). \tag{48}$$

Now, from monotonicity of the function $\mathcal{E}_T(s)$ we have

$$\mathcal{E}_T(0) \le \mathcal{E}_T(-\infty).$$

Then from the definition of $\delta_T(s)$ and inequality (45) we deduce

$$\delta_T(0) \le c_9 \Gamma^{\frac{\mu_0}{p}}(T)\, h_0^{\frac{k_0}{p}}(-\infty), \tag{49}$$

where the constant $c_9 = 2^{\frac{1}{p(1+k_0)}} c_6^{\frac{1}{p}}$. The upper estimate of the support boundary

$$\Gamma(T) \le \frac{1}{1-\varepsilon}\, \delta_T(0) \le \frac{c_9}{1-\varepsilon} \Gamma^{\frac{\mu_0}{p}}(T) h_0^{\frac{k_0}{p}}(-\infty).$$

follows from (49). Hence, $\Gamma(T)$ satisfies

$$\Gamma(T) \le \big(\frac{c_9}{1-\varepsilon}\big)^{\frac{p}{p-\mu_0}} h_0^{\frac{k_0}{p-\mu_0}}(-\infty).$$

Then, using the definition (9), and k_0, μ_0, we obtain the following estimate:

$$\Gamma(T) \le c_{10} \Gamma_0, \tag{50}$$

were $c_{10} = \big(\frac{c_9}{1-\varepsilon}\big)^{\frac{p}{p-\mu_0}} > 0$, and

$$\Gamma_0 = \|u_0\|_{L^2(R^N)}^{\frac{k_0}{p-\mu_0}} = \|u_0\|_{L^2(R^N)}^{\frac{p-q-1}{2(p-q)+N(p-q-1)}}. \tag{51}$$

Remark 6. The estimate (50) is also valid for $\Gamma^{(y)}(t)$ in the sense that

$$\Gamma^{(y)}(t) \le c_{10}\Gamma_0 \text{ for all } t > 0 \text{ and } y \in \partial\{\operatorname{supp} u_0\}. \tag{52}$$

Now we proceed with the proof of the Theorem 1. For $0 < q < p-1$ from estimate (50) and the definition of $\mathcal{F}_T(s)$ we deduce

$$\mathcal{F}_T(s) \le c_9(c_{10}\Gamma_0)^{\frac{\mu_0}{p}} \chi^{\frac{k_0}{p}}(-s) \le c_{11}(-s) \tag{53}$$

for all $s < 0$, $0 < \chi < \big(c_{11}\, c_9^{-1}(c_{10}\Gamma_0)^{-\frac{\mu_0}{p}}\big)^{\frac{p}{\mu_0}}$. Next, we apply Lemma B.4 from Appendix B to the functions $\delta_T(s)$. Indeed, the condition (a) from Lemma B.4 is valid for $f(s) := \delta_T(s)$, $g(s) := \mathcal{F}_T(s)$, $\gamma := \varepsilon$, due to the inequality (44). The inequality (53) guarantees the validation of the condition (b) for $d_4 := c_{11}$. The condition (c) of the Lemma B.4 follows from the inequality (46) for $d_5 := c_8 > (1-\varepsilon)^{-1}$. Let us choose the constant c_{11} such that

$$c_{11} < c_8^{-1}(1 - \varepsilon - c_8^{-1}).$$

Then we have

$$\delta_T(s) \equiv 0 \;\forall\, s \ge 0,\ T > 0.$$

Taking into account $c_{11} < (1-\varepsilon)^2$, we derive

$$\kappa_1 := \frac{1}{c_{11}\Gamma_0}\Big(\frac{(1-\varepsilon)^2}{c_9}\Big)^{\frac{p}{\mu_0}} = \frac{\Big(2^{\frac{k_0(p-\mu_0)-p(1+k_0^2)}{p(1+k_0)(2p-\mu_0)}}\big(2^{\frac{k_0}{p(1+k_0)}}-1\big)\Big)^{\frac{p(2p-\mu_0)}{\mu_0(p-\mu_0)}}}{\Big(\omega_N\|u_0\|_{L^2(R^N)}^{\frac{k_0 N}{p}}\Big)^{\frac{p}{N(p-\mu_0)}}}. \tag{54}$$

This completes the proof of Theorem 1.

5. Backward motion phenomenon

In this section, we find the local behaviour asymptotic condition on the initial data which results into the BMP. Let us denote by

$$\mathcal{D}_T(s,\tau) := (|s|\Gamma_0^q)^{\Lambda_1}\mathcal{E}_T^{1+\sigma}(s,\tau) + (|s|\Gamma_0^q)^{\Lambda_2}\mathcal{H}_T^{1+k_0}(s,\tau)$$

an energy functional, where $s \in R^1$, $\tau : 0 < \tau \le T$, and

$$\Lambda_1 := (1-\mu)(1+k_0),\ \Lambda_2 := \varrho(1+\sigma),\ \Lambda := \Lambda_1 + \Lambda_2.$$

Define $g(s)$ as

$$g(s) := (|s|^{\Lambda_1} + |s|^{\Lambda})\,\Gamma_0^{q\Lambda}(H_0^{(0)}(s))^{\beta},\ \beta = (1+k_0)(1+\sigma).$$

Further, we formulate some auxiliary results connected with estimates of the energy functionals $\mathcal{E}_T(s,\tau)$, $\mathcal{H}_T(s,\tau)$, $\mathcal{E}_T(s)$, and $\mathcal{D}_T(s,T)$. The proofs of these lemmas can be found in the Appendix A.

Lemma 5. *Assume that condition (3) is satisfied. Then there exist positive constants c_{12}, c_{13} such that the following estimates*

$$\mathcal{E}_T(s+\delta,\tau-\omega) \le c_{12}\,(|s|\,\varGamma_0^q)^{\varrho}\ \mathcal{M}_T^{1+k_0}(s,\delta,\tau,\omega), \tag{55}$$

$$\mathcal{H}_T(s+\delta,\tau-\omega) \le c_{13}\,(|s|\,\varGamma_0^q)^{1-\mu}\ \mathcal{M}_T^{1+\sigma}(s,\delta,\tau,\omega) \tag{56}$$

are valid for all $s \in R^1$, $\delta > 0$, and $\tau, \omega : 0 < \omega < \tau \le T$, where $\varGamma_0$ is defined by (51), and

$$\varrho := \frac{2p}{2p+N(p-q-1)} = \frac{\mu_0}{q},\ \ \mu := \frac{N(1-q)}{2p+N(p-q-1)},$$

$$\sigma := \frac{p(1-q)}{2p+N(p-q-1)}. \tag{57}$$

Lemma 6. *There exists a positive constant c_{14} such that the following estimate*

$$\mathcal{E}_T(s) \le c_{14}\varGamma_0^{\mu_0}(H_0^{(0)}(s))^{1+k_0} \quad \forall\, s<0,\ T>0 \tag{58}$$

holds.

Lemma 7. *There exists a positive constant c_{15} such that the following estimate*

$$\mathcal{D}_T(s,T) \le c_{15}g(s) \tag{59}$$

is valid for every $s<0$.

Proof (Proof of Theorem 2).

We take both sides of the inequalities (55) and (56) to the powers $1+\sigma$ and $1+k_0$ accordingly, and multiply them by $|s|\varGamma_0^q$ with powers $\varLambda_1$ and $\varLambda_2$ respectively. Summing the obtained inequalities and using (39), we obtain the estimate

$$\mathcal{D}_T(s+\delta,\tau-\omega) \le c_{16}\,(|s|\varGamma_0^q)^{\varLambda}\mathcal{M}_T^{\beta}(s,\delta,\tau,\omega)$$

$$\le c_{17}\left((|s|\varGamma_0^q)^{\varLambda_2}\delta^{-p\beta}\mathcal{D}_T^{1+k_0}(s,\tau) + (|s|\varGamma_0^q)^{\varLambda_1}\omega^{-\beta}\mathcal{D}_T^{1+\sigma}(s,\tau)\right) \tag{60}$$

for all $s \in R^1$, $\delta > 0$, and $\tau, \omega : 0 < \omega < \tau \le T$, where $c_{16} = c_{12}^{1+\sigma} + c_{13}^{1+k_0}$, $c_{17} = 2^{\beta-1}c_{16}$, $\beta = (1+k_0)(1+\sigma)$.

Remark 7. Let $0<q<p-1$. If $s \ge 0$ then, due to (14), $u(t,x)=0$ for all $(t,x) \in R^+ \times \varOmega^{(y)}(s)$. Hence, we can consider the inequality (60) for $s<0$ only.

First, we introduce the notations

$$\delta_T(s,\tau) := \left[2c_{17}\,(|s|\Gamma_0^q)^{\Lambda_2}\mathcal{D}_T^{k_3}(s,\tau)\right]^{\frac{1}{p\beta}},$$

$$\omega_T(s,\tau) := \left[2c_{17}\,(|s|\Gamma_0^q)^{\Lambda_1}\mathcal{D}_T^{\sigma}(s,\tau)\right]^{\frac{1}{\beta}}.$$

Setting $\delta = \delta_T(s,\tau)$ and $\omega = \omega_T(s,\tau)$, we have the following main functional inequality

$$\mathcal{D}_T\big(s+c_{18}\,(|s|\Gamma_0^q)^{\gamma_1}\mathcal{D}_T^{\rho_1}(s,\tau),\ \tau-c_{19}\,(|s|\Gamma_0^q)^{\gamma_2}\mathcal{D}_T^{\rho_2}(s,\tau)\big) \le \frac{1}{2}\mathcal{D}_T(s,\tau) \quad (61)$$

for all $s<0,\ \tau<T$, where $c_{18}=(2c_{17})^{\frac{1}{p\beta}},\ c_{19}=(2c_{17})^{\frac{1}{\beta}}$, and

$$\gamma_1 := \frac{\Lambda_2}{p\beta} = \frac{\varrho}{p(1+k_0)},\ \rho_1 := \frac{k_0}{p\beta};\ \gamma_2 = \frac{\Lambda_1}{\beta} = \frac{1-\mu}{1+\sigma},\ \rho_2 := \frac{\sigma}{\beta}.$$

Let s_0' be an arbitrary negative number such that $s_0' \le s_0 < 0$, where s_0 is from (17). Further, we will consider the estimate (61) for all $s : s_0' < s < 0$.

Next, we apply Lemma B.5 from the Appendix B to $f(s,\tau) := \mathcal{D}_T(s,\tau)$, $k_1 := c_{18}(\Gamma_0^q)^{\gamma_1}$, $k_2 := c_{19}(\Gamma_0^q)^{\gamma_2}$, $\gamma := \frac{1}{2}$, $\alpha_1 := \gamma_1$, $\alpha_2 := \gamma_2$, $\beta_1 := \rho_1$, $\beta_2 := \rho_2$, and $\tau_0' = T$. Due to Lemma B.5, from (61) it follows that for $s_0 \in [s_0',0)$ and $\tau_0 = T$ the following equality

$$\mathcal{D}_T(s,\tau) \equiv 0 \quad (62)$$

holds

$$\forall\,(s,\tau) \in \left\{ \begin{array}{c} s_0 + \frac{c_{18}2^{\rho_1}}{2^{\rho_1}-1}(|s_0|\Gamma_0^q)^{\gamma_1}\mathcal{D}_T^{\rho_1}(s_0,T) \le s < 0, \\ T-\tau \ge \frac{c_{19}2^{\rho_2}}{2^{\rho_2}-1}(|s_0|\Gamma_0^q)^{\gamma_2}\mathcal{D}_T^{\rho_2}(s_0,T) \end{array} \right\}.$$

Using (59), we derive

$$\mathcal{D}_T(s,\tau) \equiv 0\ \forall\,(s,\tau) \in \left\{ \begin{array}{c} s_0 + c_{20}(|s_0|\Gamma_0^q)^{\gamma_1}g^{\rho_1}(s_0) \le s < 0, \\ T-\tau \ge c_{21}(|s_0|\Gamma_0^q)^{\gamma_2}g^{\rho_2}(s_0) \end{array} \right\}, \quad (63)$$

where $c_{20} = \frac{c_{18}(2c_{15})^{\rho_1}}{2^{\rho_1}-1},\ c_{21} = \frac{c_{19}(2c_{15})^{\rho_2}}{2^{\rho_2}-1}$. Further, without loss of generality, we suppose that $-1 < s_0 < 0$. Then, due to (17), we find

$$s_0 + c_{20}(|s_0|\Gamma_0^q)^{\gamma_1}g^{\rho_1}(s_0)$$

$$\le s_0 + c_{20}\Gamma_0^{q(\gamma_1+\rho_1\Lambda)}\big(|s_0|^{\gamma_1+\rho_1\Lambda_1} + |s_0|^{\gamma_1+\rho_1\Lambda}\big)(H_0^{(0)}(s_0))^{\beta\rho_1} \le \mathcal{S}_1(s_0),$$

$$c_{21}(|s_0|\Gamma_0^q)^{\gamma_2}g^{\rho_2}(s_0) \le c_{23}\Gamma_0^{q(\gamma_2+\rho_2\Lambda)}\big(|s_0|^{\gamma_2+\rho_2\Lambda_1} + |s_0|^{\gamma_2+\rho_2\Lambda}\big)(H_0^{(0)}(s_0))^{\beta\rho_2}$$

$$\le \mathcal{S}_2(s_0),$$

where

$$\mathcal{S}_1(s) := \big(1 - c_{22}\Gamma_0^{q(\gamma_1+\rho_1\Lambda)}\chi^{\frac{k_0}{p}}\big)s = \mathcal{C}(\chi)s, \quad c_{22} = 2^{\rho_1}c_{20},$$

$$\mathcal{S}_2(s) := c_{23}\Gamma_0^{q(\gamma_2+\rho_2\Lambda)}|s|^{\gamma_2+\rho_2\Lambda_1}(H_0^{(0)}(s))^{\beta\rho_2}, \quad c_{23} = 2^{\rho_2}c_{21}.$$

Now, for $\mathcal{C}(\chi)$ we impose the additional restriction $\mathcal{C}(\chi) > 0$. Note, $\mathcal{C}(\chi) \to 1$ as $\chi \to 0$. Then χ from (17) has to satisfy the following condition

$$\chi < \tilde{\kappa}_2 := \big(c_{22}\Gamma_0^{q(\gamma_1+\rho_1 \Lambda)}\big)^{-\frac{p}{k_0}}. \tag{64}$$

Choosing in (63) $s \geq \mathcal{S}_1(s_0)$ and $T-\tau \geq \mathcal{S}_2(s_0)$, we deduce

$$\mathcal{D}_T(s,\tau) \equiv 0 \ \forall\,(s,\tau) \in \{s \geq \mathcal{C}(\chi)\, s_0,\ T-\tau \geq \mathcal{S}_2(s_0)\}\,. \tag{65}$$

As $\mathcal{S}_2(\cdot)$ is a monotone function then there exists an inverse function $\mathcal{S}_2^{-1}(\cdot)$. Hence, from $T-\tau \geq \mathcal{S}_2(s_0)$ we find that $s_0 \leq \mathcal{S}_2^{-1}(T-\tau)$. Thus, choosing $s \geq \mathcal{C}(\chi)\mathcal{S}_2^{-1}(T-\tau)$ in (65), we obtain

$$\mathcal{D}_T(s,\tau) \equiv 0 \ \forall\,(s,\tau) \in \big\{s \geq \mathcal{C}(\chi)\mathcal{S}_2^{-1}(T-\tau),\ T-\tau \geq \mathcal{S}_2(s_0)\big\}\,. \tag{66}$$

From (66), taking into account that

$$\beta\rho_2 = \sigma,\ \gamma_2 + \Lambda_1\rho_2 = 1-\mu,\ \gamma_2 + \Lambda\rho_2 = 1-\mu + \frac{\sigma\varrho}{1+k_0},\ \mu_0 = q\varrho,$$

we find the estimate (23) for $\kappa_3 = \mathcal{C}(\chi)$ and $\kappa_4 = (c_{23}\Gamma_0^{q(\gamma_2+\rho_2\Lambda)})^{-1}$. Moreover, χ satisfies condition (21) with $\kappa_2 = \min\{\kappa_1, \tilde{\kappa}_1, \tilde{\kappa}_2\}$, where $\tilde{\kappa}_1$ is from (A.23), $\tilde{\kappa}_2$ is from (64), and κ_1 is defined by (54). Hence

$$\begin{aligned}\kappa_2 &:= \min\{\kappa_1, \Gamma_0^{-\frac{\mu_0}{k_0}}(\max\{2, c_{22}\Gamma_0^{\frac{qk_0(1-\mu)}{p(1+\sigma)}}\})^{-\frac{p}{k_0}}\} = \\ &= \min\{\kappa_1, 2^{-\gamma}\Gamma_0^{-\frac{\mu_0}{k_0}}\} \text{ for } \Gamma_0 \leq \big(2/c_{22}\big)^{\frac{p(1+\sigma)}{qk_0(1-\mu)}}, \\ &= \min\{\kappa_1, c_{22}^{-\gamma}\Gamma_0^{-\frac{\mu_0}{k_0}-\frac{q(1-\mu)}{1+\sigma}}\} \text{ for } \Gamma_0 > \big(2/c_{22}\big)^{\frac{p(1+\sigma)}{qk_0(1-\mu)}}.\end{aligned} \tag{67}$$

This completes the proof.

The proof of Theorem 3 is similar to the one of Theorem 2 right up to (62). The main difference is the estimate of the energy functional $\mathcal{D}_T(s,\tau)$ at the point (s_0, T) under the assumption that the initial data decrease at infinity.

Proof (Proof Theorem 3). Due to (45), (50) and (24), we find

$$\mathcal{E}_T(s) \leq \mathcal{E}_T(-\infty) \leq c_{24}\Gamma_0^{\mu_0} h_0^{1+k_0}(-\infty) \leq c_{24}\Gamma_0^{\mu_0}\overline{H}^{1+k_0} \tag{68}$$

for all $s<0$, $T>0$, where $c_{24} = c_6 c_{10}^{\mu_0}$, μ_0 and k_0 are from (42). Taking $s=-2\delta$, $\delta = s > 0$ in (A.20) and passing to the limit as $s \to +\infty$, in view of (24), we derive

$$\mathcal{H}_T(s) \leq \mathcal{H}_T(-\infty) \leq c_{25}(|s|\Gamma_0^q)^{1-\mu} h_0^{1+\sigma}(-\infty) \leq c_{25}(|s|\Gamma_0^q)^{1-\mu}\overline{H}^{1+\sigma} \tag{69}$$

for all $s<0$, $T>0$, where μ and σ are from (57). By virtue of (68) and (69), we estimate the function $\mathcal{D}_T(s,T)$. As a result, we have

$$\mathcal{D}_T(s,T) \leq c_{26}(|s|^{\Lambda_1} + |s|^{\Lambda})\,\Gamma_0^{q\Lambda}\,\overline{H}^{\beta}$$

for all $s < 0,\ T > 0$, where $c_{26} = \max\{c_{24}^{1+\sigma}, c_{25}^{1+k_0}\}$. Using this estimate in (62), we obtain

$$\mathcal{D}_T(s,\tau) \equiv 0 \, \forall \, (s,\tau) \in \left\{ \begin{matrix} s_0\big(1 - c_{27}|s_0|^{\gamma_1 - 1}\max\{|s_0|^{\Lambda_1\rho_1}, |s_0|^{\Lambda\rho_1}\}\big) \le s < 0, \\ T - \tau \ge c_{28}|s_0|^{\gamma_2}\max\{|s_0|^{\Lambda_1\rho_2}, |s_0|^{\Lambda\rho_2}\} \end{matrix} \right\}, \tag{70}$$

where

$$c_{27} = \Gamma_0^{q(\gamma_1 + \Lambda\rho_1)} \overline{H}^{\beta\rho_1} \frac{c_{18}(2c_{26})^{\rho_1}}{2^{\rho_1} - 1}, \quad c_{28} = \Gamma_0^{q(\gamma_2 + \Lambda\rho_2)} \overline{H}^{\beta\rho_2} \frac{c_{19}(2c_{26})^{\rho_2}}{2^{\rho_2} - 1}.$$

Now we find admissible $s_0 \in (-1, 0)$ from the estimate

$$1 - c_{27}|s_0|^{\gamma_1 - 1}\max\{|s_0|^{\Lambda_1\rho_1}, |s_0|^{\Lambda\rho_1}\} = 1 - c_{27}|s_0|^{\gamma_1 + \Lambda_1\rho_1 - 1} \ge \mathcal{C}(\chi).$$

If $\gamma_1 + \Lambda_1\rho_1 < 1$ then $s_0 \le s_1(\chi)$, and if $\gamma_1 + \Lambda_1\rho_1 > 1$ then $s_0 \ge s_1(\chi)$, where

$$s_1(\chi) := -\Big(\frac{1 - \mathcal{C}(\chi)}{c_{27}}\Big)^{\frac{1}{\gamma_1 + \Lambda_1\rho_1 - 1}}.$$

Let

$$\mathcal{S}_3(s) := c_{28}\max\{|s|^{1-\mu}, |s|^{\gamma_2 + \Lambda\rho_2}\}.$$

Taking into account that

$$\gamma_2 + \Lambda_1\rho_2 = 1 - \mu < \gamma_2 + \Lambda\rho_2, \quad \gamma_1 + \Lambda_1\rho_1 = \frac{1-\mu}{p}\left(\frac{1-\nu}{1+k_0} + \frac{k_0}{1+\sigma}\right),$$

we find

$$\mathcal{D}_T(s,\tau) \equiv 0 \, \forall \, (s,\tau) \in \left\{ \begin{matrix} s \ge \mathcal{C}(\chi)\mathcal{S}_3^{-1}(T-\tau), \\ T - \tau \ge \mathcal{S}_3(s_0) \end{matrix} \right\}, \tag{71}$$

where $\mathcal{C}(\chi)$ is from (66). For the case $\gamma_1 + \Lambda_1\rho_1 < 1$ the statement of the theorem holds for all $T \ge T^* = \mathcal{S}_3(s_1(\chi))$. If $\gamma_1 + \Lambda_1\rho_1 > 1$ then the one holds for all $T > 0$. Thus, the necessary estimate (25) follows from (71). This completes the proof of Theorem 3.

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Appendix A: Proofs of Auxiliary Statements

A.1. Proof of Lemma 4

Applying Lemma B.2 from Appendix B to the function $v = u$ in the domain $Q_T(s+\delta)$ for $\tilde{p} = p > 2,\ r = q+1,\ \tilde{q} = 2,\ b = \frac{N(p-q-1)}{2p+N(p-q-1)}$, and the condition $r \le \tilde{p}$ leads to the restriction $q \le p-1$. As a result, we have

$$\iint_{Q_T(s+\delta)} u^p \, dx\, dt \le d_1 \left(\iint_{Q_T(s+\delta)} |\nabla u|^p \, dx\, dt \right)^b \times$$

$$\left(\iint_{Q_T(s+\delta)} u^{q+1} \, dx\, dt \right)^{1-b} \sup_{t\in[0,T]} \left(\int_{\Omega(s+\delta)} u^2 \, dx \right)^{\frac{p(p-q-1)}{2p+N(p-q-1)}}. \qquad \text{(A.1)}$$

Using the Poincaré inequality

$$\|v\|^q_{L^q(\Omega)} \le (\operatorname{meas} \Omega)^{\frac{q}{N}} \|\nabla v\|^q_{L^q(\Omega)}$$

with $v = u^{\frac{q+1}{q}}$, we deduce

$$\iint_{Q_T(s+\delta)} u^{q+1} \, dx\, dt \le \omega_N^{\frac{q}{N}}\, \Gamma^q(T) \iint_{Q_T(s+\delta)} |\nabla u^{\frac{q+1}{q}}|^q \, dx\, dt, \qquad \text{(A.2)}$$

where ω_N is the volume of the unit ball in R^N. The right side of (A.2) is finite because the solution u is compactly supported (see, e. g. [19]). From (A.1) and (A.2) we obtain (42).

A.2. Proof of Lemma 5

Applying Lemma B.1 from the Appendix B to the function $v = u$ in the domain $\Omega(s+\delta)$ for $a = d = p > 2,\ b = 2,\ i = 0,\ j = 1$, and integrating with respect to time, by Hölder's inequality, we obtain the estimate

$$\mathcal{E}_T(s+\delta, \tau-\omega) \le d_1^p \left(\int_{Q^T_{T-\tau+\omega}(s+\delta)} \left|\nabla u\right|^p dxdt \right)^{\nu} \mathcal{H}_T^{1-\nu}(s+\delta, \tau-\omega, \frac{p}{2}), \qquad \text{(A.3)}$$

where $\nu = \frac{N(p-2)}{2p+N(p-2)}$. To estimate $\mathcal{H}_T^{1-\nu}(s+\delta, \tau-\omega, \frac{p}{2})$ we apply the interpolation inequality from Lemma B.1 (see Appendix B) in $\Omega(s+\delta)$ to the function $v := u$ for $a = 2,\ d = p,\ b = q+1 < 2,\ i = 0,\ j = 1$. Then we derive

$$\int_{\Omega(s+\delta)} u^2 \, dx \le d_1^2 \left(\int_{\Omega(s+\delta)} \left|\nabla u\right|^p dx \right)^{\frac{2\theta}{p}} \left(\int_{\Omega(s+\delta)} u^{q+1} \, dx \right)^{\frac{2(1-\theta)}{q+1}},$$

where

$$\theta := \frac{Np(1-q)}{2(N(p-q-1)+p(q+1))}, \quad q<1.$$

We take both sides of this inequality to the power $\xi > 0$ and integrate them with respect to time from $T-\tau+\omega$ to T. After using Hölder's inequality with the exponents $\frac{p}{2\theta\xi} > 1$ and $\frac{p}{p-2\theta\xi}$, we obtain

$$\mathcal{H}_T(s+\delta,\tau-\omega,\xi) \le d_1^{2\xi}\left(\int\int_{Q^T_{T-\tau+\omega}(s+\delta)} \left|\nabla u\right|^p dx\,dt\right)^{\frac{2\theta\xi}{p}}$$

$$\times\left(\int_{T-\tau+\omega}^{T}\left(\int_{\Omega(s+\delta)} u^{q+1}\,dx\right)^{\frac{2p\xi(1-\theta)}{(q+1)(p-2\theta\xi)}} dt\right)^{1-\frac{2\theta\xi}{p}}. \tag{A.4}$$

Now we choose ξ from the following equality

$$\frac{2p\xi(1-\theta)}{(q+1)(p-2\theta\xi)} = 1.$$

Then

$$0 < \xi = \frac{p(q+1)+N(p-q-1)}{2p+N(p-q-1)} < 1. \tag{A.5}$$

From (A.4) it follows

$$\mathcal{H}_T(s+\delta,\tau-\omega,\xi)$$

$$\le d_1^{2\xi}\left(\int\int_{Q^T_{T-\tau+\omega}(s+\delta)}\left|\nabla u\right|^p dx\,dt\right)^{\mu}\left(\int\int_{Q^T_{T-\tau+\omega}(s+\delta)} u^{q+1}\,dx\,dt\right)^{1-\mu}. \tag{A.6}$$

Setting $h_2 = \frac{p}{2} > 1$, $h_1 = \xi < 1$ in (41) (see Lemma 3), we deduce

$$\mathcal{H}_T(s+\delta,\tau-\omega,\frac{p}{2}) \le c_4\,\mathcal{H}_T(s,\tau,\xi)\mathcal{M}_T^{\frac{p}{2}-\xi}(s,\delta,\tau,\omega), \tag{A.7}$$

where ξ is defined by (A.5), $\mathcal{M}_T(s,\delta,\tau,\omega)$ is from (33).

To estimate the right-hand side of inequality (A.7) we use (A.6) and Lemma 2. Eventually, we obtain

$$\mathcal{H}_T(s+\delta,\tau-\omega,\frac{p}{2}) \le c_{29}\,\mathcal{M}_T^{\mu+\frac{p}{2}-\xi}(s,\delta,\tau,\omega)\left(\int\int_{Q^T_{T-\tau}(s+\delta)} u^{q+1}\,dx\,dt\right)^{1-\mu}, \tag{A.8}$$

where $c_{29} = c_3^{\mu}c_4d_1^{2\xi}$. Taking into account (A.8) and Lemma 2, we estimate the second multiplier in the right-hand side of (A.3). As a consequence, using (A.2) and (50), we have

$$\mathcal{E}_T(s+\delta,\tau-\omega)$$

$$\le d_1^p c_{29}^{1-\nu}\,\mathcal{M}_T^{\nu+\left(\mu+\frac{p}{2}-\xi\right)(1-\nu)}(s,\delta,\tau,\omega)\left(\int\int_{Q^T_{T-\tau}(s+\delta)} u^{q+1}\,dx\,dt\right)^{(1-\mu)(1-\nu)}$$

$$\le c_{30}\,\Gamma_0^{q(1-\mu)(1-\nu)}\mathcal{M}_T^{\nu+\left(\mu+\frac{p}{2}-\xi\right)(1-\nu)}(s,\delta,\tau,\omega)$$
$$\times\left(\int\int_{Q^T_{T-\tau}(s+\delta)}\left|\nabla u^{\frac{q+1}{q}}\right|^q dx\,dt\right)^{(1-\mu)(1-\nu)}. \qquad (A.9)$$

where $c_{30}=d_1^p c_{29}^{1-\nu}(\omega_N^{\frac{q}{N}}c_{10}^q)^{(1-\mu)(1-\nu)}$.

Below we obtain an estimate of integral in the right-hand side of (A.9). For an arbitrary fixed $s_0\in R^1$ and $\delta>0$, we substitute $s=s_i=s_{i-1}+\delta$, $i\in N$ in (39) and sum the obtained inequalities. We arrive at

$$\int\int_{Q^T_{T-\tau}(s_0+\delta)}\left|\nabla u^{\frac{q+1}{q}}\right|^q dx\,dt\le\sum_{k=1}^{+\infty}\int\int_{Q^T_{T-\tau}(s_0+k\delta)}\left|\nabla u^{\frac{q+1}{q}}\right|^q dx\,dt$$
$$\le\frac{c_3}{c_1}\sum_{k=1}^{+\infty}\left(\delta^{-p}\mathcal{E}_T(s_0+k\,\delta,\tau)+\omega^{-1}\mathcal{H}_{T-\tau+\omega}(s+k\,\delta,\omega)\right)$$
$$\le\frac{c_3}{c_1}\sum_{k=0}^{+\infty}\left(\delta^{-p}\mathcal{E}_T(s_0+k\,\delta,\tau)+\omega^{-1}\mathcal{H}_T(s_0+k\,\delta,\tau)\right)$$
$$\le\frac{c_3}{c_1}\int_{s_0}^{+\infty}\left(\delta^{-p}\mathcal{E}_T(z,\tau)+\omega^{-1}\mathcal{H}_T(z,\tau)\right)dz. \qquad (A.10)$$

It follows from the flatness condition (17) that

$$\mathcal{E}_T(0)=\mathcal{H}_T(0)=0\ \forall\,T>0.$$

Therefore, by monotonicity of functions $\mathcal{E}_T(s,\tau)$ and $\mathcal{H}_T(s,\tau)$, it follows from (A.10) that for every $s\in R^1$, $\delta>0$, and $\tau,\,\omega:0<\omega<\tau\le T$ the inequality

$$\int\int_{Q^T_{T-\tau}(s+\delta)}\left|\nabla u^{\frac{q+1}{q}}\right|^q dx\,dt\le\frac{c_3}{c_1}\,|s|\,\mathcal{M}_T(s,\delta,\tau,\omega) \qquad (A.11)$$

holds. Using (A.11) we estimate the right-hand of inequality (A.9), consequently, we derive the estimate (55) for $\tilde c_{12}=c_{30}(c_3/c_1)^{\varrho}$, where

$$\varrho=(1-\mu)(1-\nu),\ k_0=(\frac{p}{2}-\xi)(1-\nu). \qquad (A.12)$$

Now we obtain a similar inequality for $\mathcal{H}_T(s,\tau)$. Setting $h_2=1$, $h_1=\xi$ in (41) (see Lemma 3), we deduce

$$\mathcal{H}_T(s+\delta,\tau-\omega)\le c_4\,\mathcal{H}_T(s,\tau,\xi)\mathcal{M}_T^{1-\xi}(s,\delta,\tau,\omega) \qquad (A.13)$$

for all $s\in R^1$, $\delta>0$, and $\tau,\,\omega:0<\omega<\tau\le T$. Using (A.6), we estimate the right-hand of (A.13). Consequently, we infer

$$\mathcal{H}_T(s+\delta,\tau-\omega)\le d_1^{2\xi}c_3^{\mu}c_4\,\mathcal{M}_T^{1+\mu-\xi}(s,\delta,\tau,\omega)\left(\int\int_{Q^T_{T-\tau}(s+\delta)}u^{q+1}\,dx\,dt\right)^{1-\mu},$$

whence, in view of (A.11), (A.2) and (50), we obtain the inequality (56) for $\tilde c_{13}=d_1^{2\xi}c_3c_4(\omega_N^{\frac{q}{N}}c_{10}^q/c_1)^{1-\mu}$, where $\sigma=1-\xi$. This completes the proof.

A.3. Proof of Lemma 6

From (43), using (50) and (10), we obtain

$$\mathcal{E}_T(s+\delta) \le \tilde{c}_8 \Gamma_0^{\mu_0} \delta^{-p(1+k_0)} \mathcal{E}_T^{1+k_0}(s), \ \forall\ s \in R^1,\ \delta > 0,$$

where $\mu_0 = q\varrho, \tilde{c}_6 = c_6 c_{10}^{\mu_0}$. Setting in the last inequality

$$\delta = \tilde{\delta}_T(s) := [2\tilde{c}_6 \Gamma_0^{\mu_0} \mathcal{E}_T^{k_0}(s)]^{\frac{1}{p(1+k_0)}},$$

analogously to the proof of Theorem 1, we derive

$$\tilde{\delta}_T(s + \tilde{\delta}_T(s)) \le \varepsilon\, \tilde{\delta}_T(s) + c_{31} \Gamma_0^{\frac{\mu_0}{p}} [H_0^{(0)}(s)]^{\frac{k_0}{p}} \quad \forall s \in R^1,$$

where $c_{31} := (2^{\frac{1}{1+k_0}} \tilde{c}_6)^{\frac{1}{p}}$, $0 < \varepsilon = 2^{-\frac{k_0}{p(1+k_0)}} < 1$. From the definition of $\tilde{\delta}_T(s)$ we obtain

$$\mathcal{E}_T(s) = (2\tilde{c}_6)^{-\frac{1}{k_0}} [\tilde{\delta}_T(s)]^{\frac{p(1+k_0)}{k_0}} \Gamma_0^{-\frac{\mu_0}{k_0}}.$$

To estimate $\tilde{\delta}_T(s)$ we apply Lemma B.4 (see Appendix B) to $f(s) = \tilde{\delta}_T(s)$, $g(s) = c_{31} \Gamma_0^{\frac{\mu_0}{p}} [H_0^{(0)}(s)]^{\frac{k_0}{p}}$, $d_4 = c_{31} \chi^{\frac{k_0}{p}} \Gamma_0^{\frac{\mu_0}{p}}$, $d_5 = c_8 > \frac{1}{1-\varepsilon}$, $\gamma = \varepsilon$. The conditions (a)–(d) of Lemma B.4 are satisfied. Now we use the condition (18) of qualified monotonicity for majorant $H_0^{(0)}(s)$ for

$$k = (\varepsilon + c_8^{-1})^{\frac{p}{k_0}} < 1, \quad K = c_8 c_{31} \Gamma_0^{\frac{\mu_0}{p}} > 1. \tag{A.14}$$

Note, that inequalities (A.14) are satisfied due to the choice of the constant c_8 which was an arbitrary up to now. In this case, for the function $H_0^{(0)}(s)$ the following inequality

$$(\varepsilon + c_8^{-1}) H_0^{(0)}(s) \le H_0^{(0)}(s + c_8 c_{31} \Gamma_0^{\frac{\mu_0}{p}} (H_0^{(0)}(s))^{\frac{1}{\gamma}})$$

is true, where $\frac{1}{\gamma} = \frac{k_0}{p} = \frac{p-q-1}{2p+N(p-q-1)}$. Then it is simple to check that condition (e) of Lemma B.4 from the Appendix B is valid, and we obtain

$$\tilde{\delta}_T(s) \le c_8 c_{31} \Gamma_0^{\frac{\mu_0}{p}} [H_0^{(0)}(s)]^{\frac{k_0}{p}} \ \forall\, s < 0,\ T > 0. \tag{A.15}$$

Finally, from the definition of $\tilde{\delta}_T(s)$ and the estimate (A.15), we derive (58) for $\tilde{c}_{14} = \left(\frac{(c_8 c_{31})^{p(1+k_0)}}{2\tilde{c}_6}\right)^{\frac{1}{k_0}}$.

A.4. Proof of Lemma 7

It obviously follows from the definition of $\mathcal{H}_T(s)$ that the estimate

$$\mathcal{H}_T(s) \leq \mathcal{H}_T(s,T,\xi)\left(\sup_{t\in(0,T)}\int\limits_{\Omega(s)} u^2(t)\,dx\right)^{\sigma}, \tag{A.16}$$

holds for every $s \in R^1$ and $T > 0$, where $\sigma = 1-\xi$, ξ is defined by (A.5). Using (A.16) and (A.6), we derive

$$\mathcal{H}_T(s+\delta) \leq d_1^{2\xi}\left(\int\int_{Q_T(s+\delta)} |\nabla u|^p\,dxdt\right)^{\mu}\left(\int\int_{Q_T(s+\delta)} u^{q+1}\,dxdt\right)^{1-\mu}$$
$$\times\left(\sup_{t\in(0,T)}\int_{\Omega(s+\delta)} u^2(t)\,dx\right)^{\sigma},$$

where μ is from Lemma 5. From the definition of $\mathcal{L}_T(s+\delta)$ (see (36)) we deduce

$$\mathcal{H}_T(s+\delta) \leq d_1^{2\xi}\,\mathcal{L}_T^{\mu+\sigma}(s+\delta)\left(\int\int_{Q_T(s+\delta)} u^{q+1}\,dxdt\right)^{1-\mu} \tag{A.17}$$

for every $s \in R^1$, $\delta > 0$, and $T > 0$. By (37) and (A.2) in the right-hand side of (A.17), we obtain

$$\mathcal{H}_T(s+\delta) \leq d_1^{2\xi}c_2^{\mu+\sigma}\,\mathcal{P}_T^{\mu+\sigma}(s,\delta)\left(\omega_N^{\frac{q}{N}}\,\Gamma^q(T)\int\int_{Q_T(s+\delta)} |\nabla u^{\frac{q+1}{q}}|^q\,dxdt\right)^{1-\mu}.$$

In Section 4, we obtained the uniform boundedness from above for $\Gamma(t)$ (see (50)). Thus, we find

$$\mathcal{H}_T(s+\delta) \leq c_{32}\,\Gamma_0^{q(1-\mu)}\mathcal{P}_T^{\mu+\sigma}(s,\delta)\left(\int\int_{Q_T(s+\delta)} |\nabla u^{\frac{q+1}{q}}|^q\,dxdt\right)^{1-\mu} \tag{A.18}$$

for every $s \in R^1$, $\delta > 0$, and $T > 0$, where $c_{32} = d_1^{2\xi}c_2^{\mu+\sigma}(\omega_N c_{10}^N)^{\frac{q(1-\mu)}{N}}$.

To estimate the integral in the right-hand side of the inequality (A.18) we use the same arguments as in the proof of the inequality (A.11). Then we derive

$$\int\int_{Q_T(s+\delta)} |\nabla u^{\frac{q+1}{q}}|^q\,dx\,dt \leq \frac{c_2}{c_1}\,|s|\mathcal{P}_T(s,\delta) \tag{A.19}$$

for every fixed $s \in R^1$, $\delta > 0$, $T > 0$. By (A.18), (A.19), (35), and (17) we have

$$\mathcal{H}_T(s+\delta) \leq c_{33}(|s|\Gamma_0^q)^{1-\mu}\mathcal{P}_T^{1+\sigma}(s,\delta) \leq \tilde{c}_{19}(|s|\Gamma_0^q)^{1-\mu}(\delta^{-p}\mathcal{E}_T(s)+h_0(s))^{1+\sigma}$$

$$\leq c_{33}(|s|\Gamma_0^q)^{1-\mu}(\delta^{-p}\mathcal{E}_T(s)+H_0^{(0)}(s))^{1+\sigma} \tag{A.20}$$

$\forall\, s\in R^1,\ \delta>0,\ T>0$, where $c_{33}=c_{32}(\frac{c_2}{c_1})^{1-\mu}$. In account of (58), we estimate the right-hand side of (A.20). As a result, we have

$$\mathcal{H}_T(s+\delta)\leq c_{33}(|s|\Gamma_0^q)^{1-\mu}\Big(\tilde{c}_{14}\delta^{-p}\Gamma_0^{\mu_0}[H_0^{(0)}(s)]^{1+k_0}+H_0^{(0)}(s)\Big)^{1+\sigma} \tag{A.21}$$

for every $s<0,\ \delta>0,\ T>0$. Setting

$$\delta=\hat{\delta}_T(s):=\Big[\Gamma_0^{\mu_0}[H_0^{(0)}(s)]^{k_0}\Big]^{\frac{1}{p}}$$

in (A.21), we obtain

$$\mathcal{H}_T(s+\hat{\delta}_T(s))\leq c_{34}(|s|\Gamma_0^q)^{1-\mu}(H_0^{(0)}(s))^{1+\sigma}\ \forall\, s<0,\ T>0,$$

where $c_{34}=c_{33}(1+\tilde{c}_{14})^{1+\sigma}$. From the majorant inequality (17) we see

$$\hat{\delta}_T(s)\leq\Gamma_0^{\frac{\mu_0}{p}}\chi^{\frac{k_0}{p}}(-s).$$

Then

$$\mathcal{H}_T\big(s(1-\Gamma_0^{\frac{\mu_0}{p}}\chi^{\frac{k_0}{p}})\big)\leq\mathcal{H}_T\big(s+\hat{\delta}_T(s)\big)\leq c_{34}(|s|\Gamma_0^q)^{1-\mu}(H_0^{(0)}(s))^{1+\sigma} \tag{A.22}$$

$\forall\, s<0,\ T>0$. Let the constant χ from (13) be such that $\Gamma_0^{\frac{\mu_0}{p}}\chi^{\frac{k_0}{p}}<2^{-1}$, i. e.

$$\chi<\tilde{\kappa}_1:=(2\Gamma_0^{\frac{\mu_0}{p}})^{-\frac{p}{k_0}}=(2\Gamma_0^{\frac{\mu_0}{p}})^{-\gamma}. \tag{A.23}$$

Then it follows from (A.22) that

$$\mathcal{H}_T(2^{-1}s)\leq c_{34}(|s|\Gamma_0^q)^{1-\mu}(H_0^{(0)}(s))^{1+\sigma}\ \forall\, s<0,$$

whence, due to the inequality $\mathcal{H}_T(s)\leq\mathcal{H}_T(2^{-1}s)$, we deduce

$$\mathcal{H}_T(s)\leq c_{34}(|s|\Gamma_0^q)^{1-\mu}(H_0^{(0)}(s))^{1+\sigma}\ \forall\, s<0. \tag{A.24}$$

In view of the definition of $\mathcal{D}_T(s,T)$, using the estimates (58) and (A.24), and taking into account that $\mu_0(1+\sigma)=q\Lambda_2$, we infer the inequality (59) for $\tilde{c}_{15}=\tilde{c}_{14}^{1+\sigma}+c_{34}^{1+k_0}$.

Appendix B

Lemma B.1. [18] *Let $\Omega \subset R^N$ be a bounded domain with a piecewise-smooth boundary, $0 < b < a$, $d \geq 1$, and $0 \leq i < j$, $i, j \in N$. Then there exist positive constants d_1 and d_2 ($d_2 = 0$ if Ω is unbounded) depending on Ω, d, j, b, and N only such that the following inequality*

$$\left\|D^i v\right\|_{L^a(\Omega)} \leq d_1 \left\|D^j v\right\|^{\theta}_{L^d(\Omega)} \|v\|^{1-\theta}_{L^b(\Omega)} + d_2 \|v\|_{L^b(\Omega)},$$

is valid for every $v(x) \in W^{j,d}(\Omega) \cap L^b(\Omega)$, where $\theta = \frac{\frac{1}{b} + \frac{i}{N} - \frac{1}{a}}{\frac{1}{b} + \frac{j}{N} - \frac{1}{d}} \in \left[\frac{i}{j}, 1\right)$.

Lemma B.2. [7]. *Let $\Omega \subset R^N$, $\tilde{p} > 1$, $r > 0$, $r \leq \tilde{p}$, $\tilde{q} > 0$, $1 - b = \tilde{p}\tilde{q}\,(\tilde{p}\tilde{q} + N(\tilde{p} - r))^{-1}$. Then for all $v(t,x) \in L^{\tilde{p}}(0,T; W^{1,\tilde{p}}_{loc}(\Omega(s)))$ and $T > 0$ the following inequality*

$$\int_0^T \int_\Omega |v|^{\tilde{p}} \leq d_3 \left(\int_0^T \int_\Omega |\nabla v|^{\tilde{p}}\right)^b \left(\int_0^T \int_\Omega |v|^r\right)^{1-b} \sup_{t\in[0,T]} \left(\int_\Omega |v|^{\tilde{q}}\right)^{\frac{(\tilde{p}-r)(1-b)}{\tilde{q}}}$$

holds, where $0 < d_3 = d_3(N, \tilde{p}, \tilde{q}, r)$ is independent of T.

Lemma B.3. [20]. *Let a nonnegative continuous nonincreasing function $f(s) : [s_0, \infty) \to R^1$ satisfy the following functional relation*

$$f(s + f(s)) \leq \varepsilon\, f(s)\ \forall\, s \geq s_0,\ 0 < \varepsilon < 1.$$

Then $f(s) \equiv 0\ \forall\, s \geq s_0 + (1 - \varepsilon)^{-1} f(s_0)$.

Lemma B.4. [20]. *Let a nonnegative continuous nonincreasing function $f(s)$ satisfy the functional relation*

(*a*) $f(s + f(s)) \leq \gamma\, f(s) + g(s)\ \forall\, s \in R^1,\ 0 < \gamma < 1,$

where $g(s) \geq 0$ is a continuous nonincreasing function satisfying the estimation

(*b*) $g(s) \leq d_4\,(s_0 - s)\ \forall\, s < s_0,\ 0 < d_4 < \infty.$

Also, let the following inequality

(*c*) $f(s) \leq d_5\, g(s)\ \forall\, s < s_1 < s_0,\ d_5 > \frac{1}{1-\gamma}.$

hold for some $s_1 \in (-\infty, s_0)$. If the parameter d_4 from (b) satisfies the restriction

(*d*) $d_4 < d_5^{-1}(1 - \gamma - d_5^{-1})$

then

$$f(s) \equiv 0\ \forall\, s \geq s_0.$$

Moreover, if $g(s)$ satisfies the extra condition

(*e*) $g(s + d_5 g(s)) \geq (\gamma + d_5^{-1}) g(s)\ \forall\, s < s_1$

then

$$f(s) \leq d_5\, g(s)\ \forall\, s < s_0.$$

Lemma B.5. [21]. *Let a nonnegative continuous nonincreasing in both of its arguments function* $f(s,\tau)$ *satisfy the functional relation*

$$f(s + k_1\,|s|^{\alpha_1} f^{\beta_1}(s,\tau), \tau - k_2\,|s|^{\alpha_2} f^{\beta_2}(s,\tau)) \le \gamma\, f(s,\tau)\ \forall\, s > s_0',\ \tau < \tau_0',$$

where $0 < \gamma < 1,\ 0 < k_i < \infty,\ \alpha_i \ge 0,\ \beta_i > 0, i = 1, 2$*. Then for arbitrary* $s_0 \ge s_0',\ \tau_0 \le \tau_0'$*, the following property*

$$f(s,\tau) \equiv 0$$

is valid for every (s,τ) *such that*

$$(s,\tau) \in \big\{ s \ge s_0 + \frac{k_1}{1-\gamma^{\beta_1}} |s_0|^{\alpha_1} f^{\beta_1}(s_0,\tau_0),\ \tau \le \tau_0 - \frac{k_2}{1-\gamma^{\beta_2}} |s_0|^{\alpha_2} f^{\beta_2}(s_0,\tau_0) \big\}.$$

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