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EFFECT OF TIME DELAY OF SUPPORT PROPAGATION IN EQUATIONS OF THIN FILMS

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We prove the existence of the effect of time delay of propagation of the support of "strong" solutions of the Cauchy problem for an equation of thin films and establish exact conditions on the behavior of an initial function near the free boundary that guarantee the appearance of this effect.

1. Introduction

In the present paper, we investigate the start behavior of generalized solutions of the Cauchy problem for the fourth-order degenerate parabolic equation

$$u_t + \operatorname{div}(u^n \nabla \Delta u - u^m \nabla u) + f(x, t, u) = 0, \quad (x, t) \in \mathbb{R}^N \times (0; T), \quad N \in \{1, 2, 3\}, \tag{1.1}$$

where $m \in \mathbb{R}$ and $n \in \mathbb{R}^+$. Equations of this structure appear in the description of the evolution of the height $u(x, t)$ of a thin liquid film propagating over a solid surface. The term of the fourth order simulates the influence of surface tension forces, and the term of the second order simulates the influence of gravity on the motion of the film over the horizontal plane. The lowest term $f(x, t, u)$ corresponds to absorption. Equation (1.1) also appears in Cahn–Hilliard models of the phase separation for binary mixtures, where $u(x, t)$ plays the role of the concentration of one of the components [1], and in models analogous to elastoviscoplastic Norton–Hoff models [2] in the theory of plastic deformations, where $u(x, t)$ describes the density of dislocations. For a detailed survey of mathematical models leading to various degenerate parabolic equations of the fourth and higher orders, see, e.g., [3].

The paper by Bernis and Friedman [4] initiated the mathematical investigation of equations of the type (1.1). In the paper indicated, Bernis and Friedman introduced the notion of a "weak" generalized solution of the Neumann problem for the model equation of thin films

$$u_t + \operatorname{div}(|u|^n \nabla \Delta u) = 0, \tag{1.2}$$

where $N = 1$, established the existence of these solutions for $n \geq 1$, and proved the important property of nonnegativity of these solutions for nonnegative initial functions. In [4–8], it was shown that these solutions have a number of specific qualitative properties depending on the value of the index n. For solutions of Eq. (1.2), for $N = 1$, the property of finiteness of the rate of propagation of perturbations was proved: for $0 < n < 2$, in [5] and [9] (for energy solutions), for $2 \leq n < 3$ in [6, 10], and for $n \geq 4$ in [7]. In [11], this property was established for $N \in \{2, 3\}$ and $1/8 < n < 2$, and a solution of the Cauchy problem with an initial function from H^1 having a compact support was constructed. The paper [2] was the first work where a mathematical model with a structure analogous to the structure of Eq. (1.1) was investigated in the multidimensional case. In [2], the Neumann problem for the equation

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$$u_t + \operatorname{div}\big(m(u)(\nabla\Delta u - \nabla A(u))\big) = g(x, t, u) \tag{1.3}$$

was considered. It was shown there that nonnegative "weak" solutions of Eq. (1.3) exist if the behavior of the coefficient $m(u)$ in the neighborhood of $u = 0$ is the same as that of a power function with exponent $n > 1$, $|A'| \le \text{const} < \infty$, and $g(x, t, u)$ has at most a linear growth, furthermore, there exists a finite constant $R_1 > 0$ such that $g(., ., u) \ge 0$ for $u \le R_1$. It was also also established that these solutions are positive for almost all times for sufficiently large values of n. Equations of the type (1.1) for $N > 1$ were also studied in [1, 12, 13]. In particular, in [12], "strong" generalized solutions of the Neumann and Cauchy problems were constructed for Eq. (1.1) with $f = 0$. The notion of "strong" solution was introduced in [5] for Eq. (1.2) for $N = 1$, where a "strong" solution is understood as a "weak" solution belonging to the class C^1 with respect to x for almost all times $t > 0$. In the multidimensional case, a "strong" solution is understood as a "weak" solution that satisfies a certain integral ("entropy") estimate and additional conditions of regularity. The interest in the study of these solutions is caused by the hypothesis, according to which the class of "strong" solutions is the class of uniqueness of the mentioned problems (see, e.g., [7]). In [12], the existence of nonnegative "strong" solutions of the Neumann problem was established for $m > -1$, $1/8 < n$ if $N \in \{1, 2\}$ and $1/8 < n < 4$ if $N = 3$. It was also established that the rate of propagation of perturbations is infinite for $m < 0$, $0 < m - n + 2 < 1/2$ and finite for $m > 0$, $1/8 < n < 2$, the estimate for this rate was found, and the existence of "strong" solutions of the Cauchy problem with weak initial trace was proved.

In the present paper, we establish the existence of "strong" solutions of the Cauchy problem for Eq. (1.1) and determine exact conditions on the local behavior of an initial function near the boundary of its support that guarantee the appearance of a time delay of propagation of the support of these solutions. For a nonlinear degenerate parabolic equations of the second order with absorption, this and analogous problems of the qualitative theory were investigated by many authors (see, e.g., [14–18]). The phenomenon of time delay of propagation of perturbations, i.e., "inertia," is described in [19] for a one-dimensional equation of a porous medium without absorption. Note that, in the papers indicated above, a "barrier technique" connected with the construction of various sub- and supersolutions was used. By virtue of this, it is inapplicable to equations of the general form including higher-order equations. To study localization properties of generalized energy solutions of certain general parabolic equations of the second order with an absorption term, the method of local energy estimates is also used (see, e.g., [20–24]). In [25, 26], for energy solutions of divergence equations of high order, the method for establishing the time delay of support propagation is proposed. This method can be regarded as a certain adaptation of the method of local energy estimates. It is based on the study of properties of solutions of special functional inequalities (see, e.g., [27]). In [28], using this method, the effect of "inertia" for "strong" nonnegative solutions of Eq. (1.2) was described for $N = 1$ and $0 < n < 3$ as well as for $N \in \{2, 3\}$ and $1/8 < n < 2$.

2. Existence of "Strong" Solutions

We consider the Cauchy problem [in what follows, denoted by (C)]

$$\begin{gathered} u_t + \operatorname{div}(u^n \nabla\Delta u - u^m \nabla u) + f(x, t, u) = 0, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ u(0) = u_0(x) \in H^1(\mathbb{R}^N) \cap L^{m-n+2}(\mathbb{R}^N), \end{gathered} \tag{2.1}$$

where $u_0(x)$ is a nonnegative function with a compact support and the function $f(x, t, z) \in C(\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R})$ and satisfies the condition

$$d_1|z|^{\lambda+1} \le f(x,t,z)z \le d_2|z|^{\lambda+1} \quad \forall\,(x,t,z) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^1,$$

$$0 < d_1 < d_2 < \infty, \quad \lambda \ge 1. \tag{2.2}$$

We understand a solution of problem (C) in the following sense:

Definition 1. *Suppose that* $m > 0$, $n > 0$, *and* $\lambda \ge 1$. *A nonnegative function* $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{loc}}(\mathbb{R}^N)) \cap L^{\lambda+1}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^N)$ *is called a solution of problem* (C) *if:*

(i) $\chi_{\mathrm{P}} u^{n-2}|\nabla u|^3$, $\chi_{\mathrm{P}} u^{n-1}|\nabla u|^2$, *and* $u^n|\nabla u|$ *and* $u^m|\nabla u|$ *belong to the space* $L^1_{\mathrm{loc}}([0;\infty); L^1_{\mathrm{loc}}(\mathbb{R}^N))$, *where* χ_{P} *is the characteristic function of the set* $\mathrm{P} := \{u > 0\}$;

(ii) $\forall \zeta \in C^\infty_c(\mathbb{R}^+ \times [0;\infty))$, *where* C^∞_c *is the space of functions from* C^∞ *having a compact support, and the following equality is true:*

$$-\int_0^\infty \int_{\mathbb{R}^N} u\zeta_t\,dx\,dt - \int_{\mathbb{R}^N} u_0\zeta(x,0)\,dx = \frac{n(n-1)}{2}\iint_{\mathrm{P}} u^{n-2}|\nabla u|^2 \nabla u \nabla\zeta\,dx\,dt$$

$$+\frac{n}{2}\iint_{\mathrm{P}} u^{n-1}|\nabla u|^2 \Delta\zeta\,dx\,dt + n\iint_{\mathrm{P}} u^{n-1}\langle \nabla u, D^2\zeta, \nabla u\rangle\,dx\,dt$$

$$-\int_0^\infty \int_{\mathbb{R}^N} u^m \nabla u \nabla\zeta\,dx\,dt + \int_0^\infty \int_{\mathbb{R}^N} u^n \nabla u \nabla\Delta\zeta\,dx\,dt - \int_0^\infty \int_{\mathbb{R}^N} f\zeta\,dx\,dt;$$

(iii) $u(.,t) \to u_0(.)$ *in* $H^1(\mathbb{R}^N)$ *as* $t \to 0$.

Remark 1. Solutions satisfying Definition 1 are called “weak” entropy solutions (see [5]).

In the multidimensional case, the concept of “weak” solutions is proposed in [1, 2, 12]. In particular, in [12], we construct “weak” solutions with additional regularity and call these solutions “strong” solutions.

The following local theorem on the existence of “strong” solutions of problem (C) is one of the results of the present paper:

Theorem 1. *Suppose that* $N \in \{1, 2, 3\}$, $m > 0$, $n \in (1/8; 2)$ $\forall\,\lambda: 1 \le \lambda$ *for* $N \in \{1, 2\}$ *and* $1 \le \lambda \le 11/4$ *for* $N = 3$ *as well as* $n \in (3\lambda/2 - 4; 2)$ $\forall\,\lambda \in (11/4; 3)$ *for* $N = 3$. *In this case,* $u_0 \in H^1(\mathbb{R}^N) \cap L^{m-n+2}(\mathbb{R}^N)$ *and is a nonnegative function with a compact support. Then, for* $T = T(u_0)$, *the solution* $u(x,t)$ *corresponding to Definition 1 has the following properties:*

(i) $\operatorname{supp} u(t)$ *is compact for almost all* $0 < t < T$;

(ii) $\forall q' \in \left(1; \dfrac{4N}{2N+(N-2)n}\right)$ *for* $N \in \{2, 3\}$, $q' = 2$ *for* $N = 1$, *and the following relation is true:*

$$u_t = -\operatorname{div} J - f(x, t, u) \quad in \quad L^2([0; T); (W_q^1(\mathbb{R}^N))'),$$

where $q = \dfrac{q'}{q'-1}$ *and* $J \in L^2([0; T); L^{q'}(\mathbb{R}^N))$;

(iii) $\forall \alpha \in \left(\max\left\{-1; \dfrac{1}{2} - n\right\}; 2 - n\right) \setminus \{0\}$ *if* $N \in \{1, 2\}$ *and* $\forall \alpha \in \left(\max\left\{-1; \dfrac{1}{2} - n; 2\lambda - 5 - n\right\}; 2 - n\right) \setminus \{0\}$ *if* $N = 3$, *and the following inclusions are true:*

$$u^{m-n+2} \in L^\infty([0; T); L^1(\mathbb{R}^N)), \qquad u^{(\alpha+n+1)/2} \in L^2([0; T); H^2(\mathbb{R}^N)),$$

$$u^{(\alpha+n+1)/4} \in L^4([0; T); W_4^1(\mathbb{R}^N)), \qquad u^{(\alpha+m+1)/2} \in L^2([0; T); H^1(\mathbb{R}^N));$$

(iv) *for all* α *that satisfy (iii), there exist positive constants* C_1, $C_2(\alpha, m, n)$ *such that*

$$\begin{aligned} &\frac{1}{\alpha(\alpha+1)} \int\limits_{\mathbb{R}^N} \zeta^4 u^{\alpha+1}(t)\,dx - \frac{1}{\alpha(\alpha+1)} \int\limits_0^t \int\limits_{\mathbb{R}^N} (\zeta^4)_t u^{\alpha+1}\,dx\,d\tau \\ &\quad + C_1 \int\limits_0^t \int\limits_{\mathbb{R}^N} \zeta^4 \left[\left|\nabla u^{(\alpha+m+1)/2}\right|^2 + \left|\nabla u^{(\alpha+n+1)/4}\right|^4 + \left|D^2 u^{(\alpha+n+1)/2}\right|^2\right] dx\,d\tau \\ &\quad + \frac{1}{\alpha} \int\limits_0^t \int\limits_{\mathbb{R}^N} \zeta^4 u^\alpha f(u)\,dx\,d\tau \\ &\leq \frac{1}{\alpha(\alpha+1)} \int\limits_{\mathbb{R}^N} \zeta^4(0) u_0^{\alpha+1}(x)\,dx + C_2 \int\limits_0^t \int\limits_{\{\zeta(t)>0\}} u^{\alpha+n+1}(|\nabla\zeta|^4 + \zeta^2|\Delta\zeta|^2)\,dx\,d\tau \\ &\quad + C_2 \int\limits_0^t \int\limits_{\{\zeta(t)>0\}} u^{\alpha+m+1}(\zeta^2|\nabla\zeta|^2 + \zeta^3|\Delta\zeta|)\,dx\,d\tau \end{aligned} \tag{2.3}$$

$\forall t \in (0; T)$ *and any nonnegative function* $\zeta \in C^2(\mathbb{R}^N \times [0; t])$.

Proof. Since transformations are rather awkward, we give only the outline of the proof and give main attention to original fragments with respect to [1, 2, 12].

For all $\delta > 0$, $\sigma > 0$, and $\varepsilon > 0$, we consider approximating problems in an arbitrary bounded domain from $\mathbb{R}^N$ with sufficiently smooth boundary for any finite time $T > 0$ [in what follows, denoted by $(P^{\varepsilon\delta\sigma})$] of the form

$$(u_{\varepsilon\delta\sigma})_t + \operatorname{div}\{m_{\delta\sigma}(u_{\varepsilon\delta\sigma})[\nabla\Delta u_{\varepsilon\delta\sigma} - \Psi''_\varepsilon(u_{\varepsilon\delta\sigma})\nabla u_{\varepsilon\delta\sigma}]\} + f(x, t, u_{\varepsilon\delta\sigma}) = 0 \quad \text{in} \quad \Omega \times (0; T),$$

$$\nabla u_{\varepsilon\delta\sigma} \cdot \bar{n} = \nabla\Delta u_{\varepsilon\delta\sigma} \cdot \bar{n} = 0 \quad \text{on} \quad \partial\Omega \times (0; T), \tag{2.4}$$

$$u_{\varepsilon\delta\sigma}(., 0) = u_{0\delta\sigma}(.) := u_0(x) + \delta^{\theta_1} + \sigma^{\theta_2} \quad \text{in} \quad \Omega, \tag{2.4}$$

where $\bar{n}$ is the vector of the outward normal to $\partial\Omega$, $0 < \theta_1 < \frac{1}{s - (\alpha + n + 1)}$, $\theta_2 > 0$, and $0 \le u_0(x) \in H^1(\Omega)$. We also assume that $\| \Psi_0(u_0) \|_{L^1(\Omega)} < \infty$. The coefficients $m_{\delta\sigma}(z)$ and $\Psi''_\varepsilon(z)$ are chosen in the special form

$$m_{\delta\sigma}(z) = \frac{|z|^{n+s}}{\delta|z|^n + |z|^s + \sigma|z|^{n+s}}, \qquad \Psi''_\varepsilon(z) = \frac{|z|^{m-n}}{1 + \varepsilon|z|^{m-n}}.$$

Without loss of generality, we assume that $s > n$. The functions $m_{\delta\sigma}(z)$ and $\Psi''_\varepsilon(z)$ are the following:

$$0 \le m_{\delta\sigma}(z) \le \frac{1}{\sigma}, \qquad 0 \le \Psi''_\varepsilon(z) \le \frac{1}{\varepsilon}. \tag{2.5}$$

In addition, the behavior of $m_{\delta\sigma}(z)$ in the neighborhood of zero is analogous to the behavior of $|z|^s/\delta$. This choice of the coefficients, for sufficiently large value of s, guarantees the existence of a positive solution of problem $(P^{\varepsilon\delta\sigma})$ for almost all $t \in (0; T)$.

Conditionally, we can divide the proof into three main stages.

Stage 1. We construct an auxiliary nondegenerate problem with diffusion coefficient $m_\gamma(z) = m_{\delta\sigma}(z) + \gamma$. We seek its solution by the Faedo–Galerkin method in the form

$$u^N_\gamma(x, t) = \sum_{i=1}^{N} c^N_i(t)\varphi_i(x) \quad \forall t \in [0; T],$$

where, as a basis $\{\varphi_i\}$, we take eigenfunctions of the Neumann problem for the Laplace operator (see [1, 2]), i.e.,

$$-\Delta\varphi_i = \lambda_i\varphi_i \quad \text{in} \quad \Omega,$$

$$\nabla\varphi_i \cdot \bar{n} = 0 \quad \text{on} \quad \partial\Omega \in C^{1,1},$$

where $\{\varphi_i\}$ are orthogonal in $H^1(\Omega)$ and $(\varphi_i, \varphi_j)_{L^2(\Omega)} = \delta_{ij}$.

By virtue of the Peano theorem, there exists a solution $(c_1^N(t), \ldots, c_N^N(t))$ of the Cauchy problem on the interval $(0; t_N)$ for the following system of ordinary differential equations:

$$\frac{d}{dt}(u_\gamma^N, \varphi_j) - \big(m_\gamma(u_\gamma^N)(\nabla\Delta u_\gamma^N - \Psi_\varepsilon''(u_\gamma^N)\nabla u_\gamma^N), \nabla\varphi_j\big) = -\big(f(x, t, u_\gamma^N), \varphi_j\big), \quad j = \overline{1, N},$$

$$u_\gamma^N(x, 0) = \sum_{i=1}^{N} (u_{0\delta\sigma}, \varphi_i)_{L^2(\Omega)}\varphi_i(x). \tag{2.6}$$

For its solvability on a common fixed time interval, it is necessary to determine additional a priory estimates. We scalarly multiply in $L^2(\Omega)$ this equation for u_γ^N by $-\Delta u_\gamma^N + u_\gamma^N$. After simple algebraic transformations, we get

$$\frac{1}{2}\frac{d}{dt}\left\|u_\gamma^N(t)\right\|_{H^1(\Omega)}^2 + c_1\gamma\int_\Omega \left|\nabla\Delta u_\gamma^N\right|^2 dx + d_1\int_\Omega \left|u_\gamma^N\right|^{\lambda+1} dx \le c_2\left\|u_\gamma^N(t)\right\|_{H^1(\Omega)}^2 + \int_\Omega f(.,.,u_\gamma^N)\Delta u_\gamma^N\, dx.$$

We estimate the second term on the right-hand side of the inequality using the Cauchy inequality with "ε" and condition (2.2). As a result, we get

$$\int_\Omega f(.,.,u_\gamma^N)\Delta u_\gamma^N\, dx \le \varepsilon\int_\Omega \left|\Delta u_\gamma^N\right|^2 dx + c(\varepsilon)\int_\Omega f(.,.,u_\gamma^N)\,dx$$

$$\le \tilde{\varepsilon}\int_\Omega \left|\nabla\Delta u_\gamma^N\right|^2 dx + c(\tilde{\varepsilon})\int_\Omega \left|u_\gamma^N\right|^{2\lambda} dx \le \tilde{\varepsilon}\int_\Omega \left|\nabla\Delta u_\gamma^N\right|^2 dx + c(\tilde{\varepsilon})\left\|u_\gamma^N(t)\right\|_{H^1(\Omega)}^{2\lambda},$$

where $\lambda \ge 1$ for $N \in \{1, 2\}$ and $1 \le \lambda \le 3$ for $N = 3$. Here, we used the Poincaré inequality for integral mean taking into account the relation $\int_\Omega \Delta u_\gamma^N\, dx = 0$ and the imbedding $H^1(\Omega) \subset L^{2\lambda}(\Omega)$. Thus, after the choice of $\tilde{\varepsilon}$, the following estimate is true:

$$\frac{1}{2}\frac{d}{dt}\left\|u_\gamma^N(t)\right\|_{H^1(\Omega)}^2 + \tilde{c}_1\gamma\int_\Omega \left|\nabla\Delta u_\gamma^N\right|^2 dx + d_1\int_\Omega \left|u_\gamma^N\right|^{\lambda+1} dx \le c_2\left\|u_\gamma^N(t)\right\|_{H^1(\Omega)}^2 + c_3\left\|u_\gamma^N(t)\right\|_{H^1(\Omega)}^{2\lambda},$$

where $1 \le \lambda$ (in addition, $\lambda \le 3$ for $N = 3$). Integrating the estimate with respect to the time and using the nonlinear Gronwall lemma, we obtain the relation

$$\left\|u_\gamma^N(t)\right\|_{H^1(\Omega)}^2 + \bar{c}_1\gamma\iint_{Q_t} \left|\nabla\Delta u_\gamma^N\right|^2 dx\,d\tau + \bar{d}_1\iint_{Q_t} \left|u_\gamma^N\right|^{\lambda+1} dx\,d\tau \le C(T)$$

$$\forall t\colon 0 \le t \le T = \frac{1}{2c_2(\lambda-1)}\ln\left(1 + \frac{c_2}{c_3}\left\|u_{0\delta\sigma}\right\|_{H^1(\Omega)}^{1-\lambda}\right). \tag{2.7}$$

Let Π_N be the projection of $L^2(\Omega)$ to a linear hull $\{\varphi_1(x), \ldots, \varphi_N(x)\}$. Then, using the Hölder inequality and relations (2.5) and (2.7), for all $\varphi \in L^2((0;T);H^1(\Omega))$, we obtain

$$\left|\iint_{Q_T}(u_\gamma^N)_t\varphi\,dx\,dt\right| = \left|\iint_{Q_T}(u_\gamma^N)_t\Pi_N\varphi\,dx\,dt\right|$$

$$= \left|\iint_{Q_T} m_\gamma(u_\gamma^N)(\nabla\Delta u_\gamma^N - \Psi_\varepsilon''(u_\gamma^N)\nabla u_\gamma^N)\nabla\Pi_N\varphi\,dx\,dt - \iint_{Q_T} f(x,t,u_\gamma^N)\Pi_N\varphi\,dx\,dt\right|$$

$$\le c_1\left|\iint_{Q_T}|\nabla\Pi_N\varphi|^2\,dx\,dt\right|^{1/2} + c_2\iint_{Q_T}\left|u_\gamma^N\right|^\lambda|\Pi_N\varphi|\,dx\,dt$$

$$\le c_1\left|\iint_{Q_T}|\nabla\Pi_N\varphi|^2\,dx\,dt\right|^{1/2}$$

$$+\ c_2\left(\int_0^T\left(\int_\Omega\left|u_\gamma^N\right|^{\lambda+1}dx\right)^{2\lambda/(\lambda+1)}dt\right)^{1/2}\left(\int_0^T\left(\int_\Omega|\Pi_N\varphi|^{\lambda+1}dx\right)^{2/(\lambda+1)}dt\right)^{1/2}$$

$$\le c_1\|\varphi\|_{L^2((0;T);H^1(\Omega))} + c_2\left(\int_0^T\|u_\gamma^N\|_{L^{\lambda+1}(\Omega)}^{2\lambda}dt\right)^{1/2}\|\varphi\|_{L^2((0;T);L^{\lambda+1}(\Omega))}.$$

Using the imbedding $H^1(\Omega)\subset L^{\lambda+1}(\Omega)$ and relation (2.7), we obtain the necessary a priori estimate

$$\left\|\partial_t u_\gamma^N\right\|_{L^2((0;T);(H^1(\Omega))')} \le C$$

$\forall\lambda\ge 1$ for $N\in\{1,2\}$ and $\lambda\in[1;5]$ for $N=3$, where C is independent of N and γ.

The obtained a priori estimates and lemmas on compactness (see, e.g., Corollary 4 in [29, p. 85]) enable one to pass to the limit as $N\to+\infty$ (for details, see [1, 2]) and also to show the validity of the equality

$$\int_0^T\langle(u_\gamma)_t(t),\zeta(t)\rangle_{(H^1)',H^1}\,dx = \iint_{Q_T} m_\gamma(u_\gamma)(\nabla\Delta u_\gamma - \Psi_\varepsilon''(u_\gamma)\nabla u_\gamma)\nabla\zeta\,dx\,dt - \iint_{Q_T} f(x,t,u_\gamma)\zeta\,dx\,dt, \tag{2.8}$$

where $\zeta\in L^2((0;T);H^1(\Omega))$ and $\nabla\zeta\cdot\bar{n}=0$ on $\partial\Omega\times(0;T)$.

To pass to the limit as $\gamma\to 0$, an additional information on the L^2-norm of Δu_γ is necessary. To this end, as a test function in (2.8), we use $\zeta = u_\gamma - \Delta u_\gamma + g_\gamma(u_\gamma)$, where

$$g_\gamma(z) = \int_1^z \frac{d\varsigma}{m_\gamma(\varsigma)} \quad \text{and} \quad G'_\gamma(z) = g_\gamma(z).$$

As a result, after simple transformations, we get

$$\frac{1}{2}\frac{d}{dt}\left\|u_\gamma\right\|^2_{H^1(\Omega)} + \frac{d}{dt}\int_\Omega G_\gamma(u_\gamma)\,dx + c\int_\Omega \left[m_\gamma(u_\gamma)\left|\nabla\Delta u_\gamma\right|^2 + \left|\Delta u_\gamma\right|^2 + \Psi''_\varepsilon(u_\gamma)\left|\nabla u_\gamma\right|^2 + \left|u_\gamma\right|^{\lambda+1}\right]dx$$

$$\leq c\left\|u_\gamma\right\|^2_{H^1(\Omega)} + \int_\Omega f(.,.,u_\gamma)\Delta u_\gamma\,dx - \int_\Omega f(.,.,u_\gamma)g_\gamma(u_\gamma)\,dx := c\left\|u_\gamma\right\|^2_{H^1(\Omega)} + I_1 - I_2.$$

Applying the Cauchy inequality with "ε" to I_1 and using the imbedding $H^1(\Omega) \subset L^{2\lambda}(\Omega)$, we conclude that

$$I_1 \leq \tilde{\varepsilon}\int_\Omega \left|\Delta u_\gamma\right|^2 dx + c(\tilde{\varepsilon})\left\|u_\gamma\right\|^{2\lambda}_{H^1(\Omega)}$$

for all $\lambda \geq 1$ for $N \in \{1, 2\}$ and $1 \leq \lambda \leq 3$ for $N = 3$. Now we estimate the integral $-I_2$. For $u_\gamma \geq 1$ or $u_\gamma \leq 0$, we have

$$-I_2 \leq 0.$$

For $0 < u_\gamma \leq R < 1$, we get

$$0 < -I_2 \leq \begin{cases} c_1 & \text{for} \quad \lambda \geq s-1, \\ c_2 G_{\delta\sigma}(u) & \text{for} \quad 1 \leq \lambda < s-1. \end{cases}$$

If $R < u_\gamma < 1$, then we have

$$0 < -I_2 \leq c_3.$$

The constants c_i, $i = 1, 2, 3$, are independent of γ.

In the investigation of the behavior of $-I_2$ for small values of u_γ, we used the following simple statement:

Lemma 1. *Suppose that*

$$g_\gamma(z) = \int_1^z \frac{d\varsigma}{m_{\delta\sigma}(\varsigma) + \gamma}, \qquad g^{(\alpha)}_{\delta\sigma}(z) = \int_1^z \frac{\varsigma^{\alpha+n-1}}{m_{\delta\sigma}(\varsigma)}\,d\varsigma$$

and

$$G^{(\alpha)}_{\delta\sigma}(z) = \int_1^z g^{(\alpha)}_{\delta\sigma}(\varsigma)\,d\varsigma, \qquad G_\gamma(z) = \int_1^z g_\gamma(\varsigma)\,d\varsigma.$$

Then there exist constants $\tilde{c}_1, \tilde{c}_2 > 0$ *and* $0 < R < 1$ *such that*

$$-zg_\gamma(z) \le \tilde{c}_1 G_\gamma(z) \quad \forall\, 0 < z < R, \quad n > 0,$$

$$-zg^{(\alpha)}_{\delta\sigma}(z) \le \tilde{c}_2 G^{(\alpha)}_{\delta\sigma}(z) \quad \forall\, 0 < z < R, \quad s > \alpha,$$

where $\gamma, \delta, \sigma \ge 0$, *furthermore,* R *is a fixed number depending on known quantities and independent of the parameters* $\gamma, \delta,$ *and* σ.

By virtue of the estimates presented above, applying the nonlinear Gronwall lemma for $1 \le \lambda < s - 1$ and choosing the corresponding $\tilde{\varepsilon}$, we obtain the following main a priori estimate:

$$\frac{1}{2}\|u_\gamma\|^2_{H^1(\Omega)} + \int_\Omega G_\gamma(u_\gamma)\,dx + c\iint_{Q_t}\Big[m_\gamma(u_\gamma)\left|\nabla\Delta u_\gamma\right|^2 + \left|\Delta u_\gamma\right|^2 + \Psi''_\varepsilon(u_\gamma)\left|\nabla u_\gamma\right|^2 + \left|u_\gamma\right|^{\lambda+1}\Big]dx\,d\tau \le C \quad (2.9)$$

$$\forall t: 0 < t < T = T(u_0),$$

where C is independent of γ and ε.

With regard for the uniform boundedness of $(u_\gamma)_t$ in $L^2((0;T);(H^1(\Omega))')$, this estimate enables one to pass to the limit with respect to γ [2] using the corresponding lemmas on compactness [29].

In the same manner as in [2], using the uniform boundedness of the term $\int_\Omega G_\gamma(u_\gamma)\,dx$, we can show the positiveness of a solution of the problem $(P^{\varepsilon\delta\sigma})$ for $\lambda \ge 1$ and sufficiently large values of the index s in the coefficient $m_{\delta\sigma}(z)$. Thus, we obtained an approximation of solutions of problem $(P^{\varepsilon\delta\sigma})$ by positive solutions of a regularized problem.

Stage 2. We pass to the limits with respect to ε, δ, and σ using necessary a priory estimates.

We multiply scalarly in $L^2(\Omega)$ Eq. (2.4) by the function $-\Delta u + \Psi'_\varepsilon(u) + g^{(\alpha)}_{\delta\sigma}(u)$. Then, for $\alpha \in \left(\frac{1}{2} - n; 2 - n\right)$, the following relation is true:

$$\frac{d}{dt}\int_\Omega\left[\frac{1}{2}|\nabla u|^2 + \Psi_\varepsilon(u) + G^{(\alpha)}_{\delta\sigma}(u)\right]dx + \int_\Omega m_{\delta\sigma}(u)\left|\nabla\Delta u - \Psi''_\varepsilon(u)\nabla u\right|^2 dx$$

$$+ c_1\int_\Omega\Big[\left|D^2 u^{(\alpha+n+1)/2}\right|^2 + \left|\nabla u^{(\alpha+n+1)/4}\right|^4\Big]dx + c_2\int_\Omega u^{\alpha+n-1}\Psi''_\varepsilon(u)|\nabla u|^2\,dx$$

$$\le -\int_\Omega f(.,.,u)g^{(\alpha)}_{\delta\sigma}(u)\,dx - \int_\Omega f(.,.,u)\Psi'_\varepsilon(u)\,dx + \int_\Omega f(.,.,u)\Delta u\,dx + \tilde{c}\int_\Omega u^{\alpha+n+1}\,dx$$

$$:= J_1 + J_2 + J_3 + \tilde{c}\int_\Omega u^{\alpha+n+1}\,dx, \quad (2.10)$$

where $c_1 = c_1(\alpha, n)$ and $\tilde{c} = 0$. If Ω is convex, then

$$g^{(\alpha)}_{\delta\sigma}(z) = \int\limits_1^z \frac{\varsigma^{\alpha+n-1}}{m_{\delta\sigma}(\varsigma)}\,d\varsigma \quad \text{and} \quad G^{(\alpha)}_{\delta\sigma} = \int\limits_1^z g^{(\alpha)}_{\delta\sigma}(\varsigma)\,d\varsigma.$$

For a detailed information about the derivation of this inequality for $f = 0$, see [30].

For J_1 and J_2, we use the same reasonings with regard for Lemma 1 as for the integral $-I_2$ at the first stage of the proof of the theorem. We estimate the term J_3 as follows:

$$\begin{aligned} J_3 &\le \tilde{c}_1 \int\limits_\Omega u^{(2\lambda-n-\alpha+1)/2}\left|\Delta u^{(\alpha+n+1)/2}\right| dx + \tilde{c}_2 \int\limits_\Omega u^{\lambda-1}|\nabla u|^2\,dx \\ &\le \tilde{\varepsilon} \int\limits_\Omega \left(\left|D^2 u^{(\alpha+n+1)/2}\right|^2 + \left|\nabla u^{(\alpha+n+1)/4}\right|^4\right) dx + c(\tilde{\varepsilon}) \int\limits_\Omega u^{2\lambda-n-\alpha+1}\,dx. \end{aligned}$$

By virtue of the imbedding $H^1(\Omega) \subset L^{2\lambda-n-\alpha+1}(\Omega)$ for all λ such that $1 \le \lambda$ if $N \in \{1, 2\}$ and $1 \le \lambda \le \frac{n+\alpha+5}{2}$ if $N = 3$, using the nonlinear Gronwall lemma, we obtain the first a priori estimate.

Using the uniform boundedness of the flow $\bar{J}_{\varepsilon\delta\sigma}(u) = m_{\delta\sigma}(u)(\nabla\Delta u - \Psi''_\varepsilon(u)\nabla u)$ in $L^2((0; T); L^{q'}(\Omega))$, where $q' = 2$ for $N = 1$ and $q' \in \left(1; \frac{4N}{2N+(N-2)n}\right)$ for $N \ge 2$, which is proved in the same manner as in [12, 30], we obtain the estimate

$$\begin{aligned} \left|\iint\limits_{Q_T} u_t \varphi\,dx\,dt\right| &= \left|\iint\limits_{Q_T} J_{\varepsilon\delta\sigma}\nabla\varphi\,dx\,dt - \iint\limits_{Q_T} f(x, t, u)\varphi\,dx\,dt\right| \\ &\le \|J_{\varepsilon\delta\sigma}\|_{L^2((0;T);L^{q'}(\Omega))}\|\nabla\varphi\|_{L^2((0;T);L^q(\Omega))} + d_2 \iint\limits_{Q_T} u^\lambda|\varphi|\,dx\,dt \\ &\le c\|\nabla\varphi\|_{L^2((0;T);L^q(\Omega))} + d_2\left(\int\limits_0^T \|u\|^{2\lambda}_{L^{\lambda+1}(\Omega)}\,dt\right)^{1/2} \|\varphi\|_{L^2((0;T);L^{\lambda+1}(\Omega))}, \end{aligned}$$

where $q = \frac{q'}{q'-1} > 2$. By virtue of the uniform boundedness of solutions in $L^\infty((0; T); H^1(\Omega))$ and the imbeddings $H^1(\Omega) \subset L^{\lambda+1}(\Omega)$ and $W^1_q(\Omega) \subset L^{\lambda+1}(\Omega)$, we obtain the inequality

$$\|\partial_t u\|_{L^2((0;T);(W^1_q(\Omega))')} \le C$$

for $\lambda \ge 1$ for $N \in \{1, 2\}$ and $\lambda \in [1; 3]$ for $N = 3$, furthermore, C is independent of ε, δ, and σ.

Using these estimates, we can pass to the limits with respect to ε, δ, and σ following the scheme proposed in [12]. In this paper, an analog of the lemma on compactness proved in Lemma 1.5 in [30, p. 10] is important. In the case under consideration, we can prove this lemma by analogy and formulate it as follows:

Lemma 2. *Suppose that, for* $N \in \{1, 2\}$, $0 < n$, *and* $1 \le \lambda$ $\beta \in \left(\frac{3}{4}; \frac{3}{2}\right)$ *and, for* $N = 3$, $0 < n < 4$, *and* $1 \le \lambda < 7$, $\beta \in \left(\frac{3}{4}; \min\left\{\frac{3}{2}; \frac{6}{\lambda + 1}\right\}\right)$, *and* $\Im$ *is a bounded subset in* $L^{\infty}((0; T); H^1(\Omega))$ *such that the following conditions are true:*

(i) $\forall u \in \Im$ $J(u)$: $u_t = -\operatorname{div} J - f(x, t, u)$ *in* $L^2((0;T);(W_q^1(\Omega))')$, $\{J(u)\}_{u \in \Im}$ *is bounded in* $L^2((0; T); L^{q'}(\Omega))$ $\forall q' < \dfrac{4N}{2N + (N-2)n}$, *and* $\{u^{\lambda+1}\}_{u \in \Im}$ *is bounded in* $L^1(Q_T)$;

(ii) $\{u^{\beta}\}_{u \in \Im}$ *is bounded in* $L^2((0; T); H^2(\Omega))$.

Then $\{u^{\beta}\}_{u \in \Im}$ *is relatively compact in* $L^2((0; T); H^1(\Omega))$.

Stage 3. By analogy with [12, p. 1518] (see Proposition 3.1), using the local entropy estimate (2.3), we show that the above-obtained "strong" solutions of the Neumann problem have a finite rate of propagation. This fact enables us to construct the corresponding "strong" solutions of the Cauchy problem (see Theorem 4.1 in [12, p. 1529]).

Remark 2. If $f(x, t, z) \in C^1(\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^1)$ satisfies condition (2.2) and the conditions

$$d_1'|z|^{\lambda-1} \le f_z(x, t, z) \le d_2'|z|^{\lambda-1} \quad \forall (x, t, z) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^1, \quad 0 < d_1' < d_2' < \infty,$$

$$\sum_{i=1}^{N} f_{x_i}^2(x, t, z) \le d|z|^{2\lambda} \quad \forall (x, t, z) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^1, \quad 0 < d < \infty,$$

then, for $N \in \{1, 2, 3\}$, $m > 0$, $n \in (1/8; 2)$, and $1 \le \lambda$ for $N \in \{1, 2\}$ and $1 \le \lambda \le 5$ for $N = 3$, we can prove a global theorem on the existence of "strong" solutions of problem (C). The scheme of the proof is the same as that presented above. The main difference lies in the estimation of integrals of the form $\int_{\Omega} f(., ., u)\Delta u\, dx$, for which, by virtue of additional assumptions imposed on the function $f(x, t, z)$, the following inequalities are true:

$$\int_{\Omega} f(., ., u)\Delta u\, dx \le -C \int_{\Omega} \left|\nabla u^{(\lambda+1)/2}\right|^2 dx + C(u_0),$$

where the constants C and $C(u_0)$ are positive.

3. Time Delay of Support Propagation

We introduce the following notation:

$$\Omega(s) = \{x = (x_N, x') : x_N > s + a_1|x'|, \; 0 < a_1 < \infty\}, \qquad Q_T(s) = \Omega(s) \times (0; T),$$

$$K_T(s, \delta) = Q_T(s) \backslash Q_T(s+\delta), \quad h_0(s) = \int\limits_{\Omega(s)} u_0^{\alpha+1}(x)\,dx \quad \forall s \in \mathbb{R}^1, \quad \delta > 0.$$

Without loss of generality, we assume that

$$\operatorname{supp} u_0 \cap \{x_N \geq a_2 |x'|\} = \varnothing,$$

where $a_2 \in [0; a_1)$ is a certain fixed number. Thus, we have

$$h_0(s) \equiv 0 \quad \forall s \geq 0. \tag{3.1}$$

In this section, we formulate conditions imposed on the behavior of the initial function $u_0(x)$ in the neighborhood of the boundary of its support that guarantee the appearance of the effect of time delay of propagation of the support of an arbitrary "strong" solution of the Cauchy problem.

Theorem 2. *Suppose that the initial function* $u_0(x)$ *satisfies relation (3.1) and, for certain* $\alpha \in \left(\max\left\{\frac{1}{2} - n; 0\right\}; 2 - n\right)$ *for* $N \in \{1, 2\}$ *and* $\alpha \in \left(\max\left\{\frac{1}{2} - n; 2\lambda - 5 - n; 0\right\}; 2 - n\right)$ *for* $N = 3$, *one of the following conditions of smallness in the neighborhood of the boundary of its support:*

(i) for $\forall \lambda \geq 1$,

$$h_0(s) \leq \begin{cases} k(-s)^{N + \frac{2(\alpha+1)}{m}}, & m \leq \dfrac{n}{2}, \\[2ex] k(-s)^{N + \frac{4(\alpha+1)}{n}}, & m > \dfrac{n}{2}, \end{cases}$$

(ii) for $n + 1 < \lambda < m + 1$,

$$h_0(s) \leq \begin{cases} k(-s)^{N + \frac{2(\alpha+1)}{m-\lambda+1}}, & m \leq \dfrac{n}{2} + \lambda - 1, \\[2ex] k(-s)^{N + \frac{4(\alpha+1)}{n}}, & m > \dfrac{n}{2} + \lambda - 1, \end{cases}$$

(iii) for $m + 1 < \lambda < n + 1$,

$$h_0(s) \leq \begin{cases} k(-s)^{N + \frac{2(\alpha+1)}{m}}, & m \leq \dfrac{n - \lambda + 1}{2}, \\[2ex] k(-s)^{N + \frac{4(\alpha+1)}{n-\lambda+1}}, & m > \dfrac{n - \lambda + 1}{2}, \end{cases}$$

(iv) for $\lambda < 1 + \min\{n; m\}$,

$$h_0(s) \le \begin{cases} k(-s)^{N+\frac{2(\alpha+1)}{m-\lambda+1}}, & m \le \dfrac{n+\lambda-1}{2}, \\ k(-s)^{N+\frac{4(\alpha+1)}{n-\lambda+1}}, & m > \dfrac{n+\lambda-1}{2}, \end{cases}$$

where $0 < -s \le 1$.

Then there exists $T^* = T^*(k) > 0$ *such that, for any "strong" solution of problem* (C),

$$\operatorname{supp}_x u(.,t) \cap \Omega(0) = \varnothing, \quad 0 < t < T^*,$$

in the cases corresponding to conditions (i) – (iii) and

$$\operatorname{supp}_x u(.,t) \cap \Omega(0) = \varnothing \quad \forall t > 0$$

for the case corresponding to condition (iv).

Remark 3. The choice of cones on sufficiently smooth initial functions as a family $\Omega(s)$ enables one to establish an exact condition of slope independent of the dimension of the space N. These exhaustive sets are used in [31] in the investigation of quasilinear multidimensional high-order parabolic equations with a convective term.

Remark 4. The condition of smallness imposed on the behavior of $u_0(x)$ for $m > n/2$ obtained in the first case coincides with the corresponding condition given in [28] for Eq. (1.2) and, for $m < n/2$, in a certain sense, is analogous to the condition of smallness imposed on the behavior of the initial function for the equation of a porous medium $u_t = \operatorname{div}(u^m \nabla u)$ (see, e.g., [19]). The behavior of the initial function near the boundary of its support established in the fourth case, in a certain sense, corresponds to the conditions obtained in [25] for quasilinear parabolic equations of arbitrary order with absorption.

Proof of Theorem 2. The proof is based on the derivation and subsequent analysis of a certain functional inequality connected with energy functions

$$I_T(s) := \iint\limits_{Q_T(s)} u^{\alpha+n+1}\, dx\, dt, \qquad E_T(s) := \iint\limits_{Q_T(s)} u^{\alpha+m+1}\, dx\, dt,$$

where $u(x,t)$ is a "strong" solution of problem (C). This method is proposed in [25] for the study of localization properties and propagation of supports of energy generalized solutions of the Cauchy–Dirichlet problem for divergence high-order parabolic equations with absorption. Before proceeding to the main proof, we show the validity of two auxiliary statements:

Lemma 3. *Suppose that* $u(x,t)$ *is a "strong" solution of problem* (C). *Then, for all* α *that satisfy the condition of Theorem 2, the following inequality is true:*

$$\sup_{(0;T)} \int\limits_{\Omega(s+\delta)} u^{\alpha+1}\,dx + \frac{1}{T} \int\limits_{Q_T(s+\delta)}\!\!\int u^{\alpha+1}\,dx\,dt$$

$$+ \; C \int\limits_{Q_T(s+\delta)}\!\!\int \Big[\big|\nabla u^{(\alpha+m+1)/2}\big|^2 + \big|\nabla u^{(\alpha+n+1)/4}\big|^4 + \big|D^2 u^{(\alpha+n+1)/2}\big|^2 + u^{\lambda+\alpha} \Big] dx\,dt$$

$$\le \; C\left(\delta^{-2} \int\limits_{K_T(s,\delta)}\!\!\int u^{m+\alpha+1}\,dx\,dt + \delta^{-4} \int\limits_{K_T(s,\delta)}\!\!\int u^{n+\alpha+1}\,dx\,dt + h_0(s) \right) \; = \; R_T(s,\delta) \tag{3.2}$$

$$\forall\, s \in \mathbb{R}^1, \quad \delta > 0,$$

where all constants depend only on known parameters of the problem.

Proof. We introduce nonnegative patch functions $\zeta(\tau)$, $\varphi(\tau) \in C^2(\mathbb{R}^1)$ with the following properties:

$$\varphi(\tau) = \begin{cases} 0 & \text{if} \;\; \tau \le 0, \\[2ex] 1 & \text{if} \;\; \tau \ge \dfrac{15}{16}, \end{cases} \qquad \text{in this case,} \quad 0 \le \varphi(\tau) \le 1 \quad \forall\, \tau \in \mathbb{R}^1,$$

$$\zeta(\tau) = \begin{cases} \tau & \text{if} \;\; \tau \ge \dfrac{1}{8}, \\[2ex] \dfrac{1}{16} & \text{if} \;\; \tau < \dfrac{1}{32}, \end{cases} \qquad \text{in this case} \quad \frac{d^2\zeta}{d\tau^2} \ge 0 \quad \forall\, \tau \in \mathbb{R}^1.$$

We define a family of main patch functions as follows:

$$\varphi_{s,\delta}(x) = \varphi\left(-a_1 \zeta\left(\frac{|x'|}{\delta}\right) + \frac{x_N - s}{\delta} \right) \quad \forall\, s \in \mathbb{R}^1, \quad \delta > 0.$$

By virtue of the above-presented properties for the patch functions $\varphi(\tau)$ and $\zeta(\tau)$, functions from the mentioned family satisfy the following relations:

$$\varphi_{s,\delta}(x) = \begin{cases} 0 & \text{if} \;\; x_N \le s + a_1 \delta \zeta\left(\dfrac{|x'|}{\delta}\right), \\[3ex] 1 & \text{if} \;\; x_N \ge s + a_1 \delta \left(\zeta \dfrac{|x'|}{\delta} + \dfrac{15}{16} \right), \end{cases} \qquad 0 \le \varphi_{s,\delta}(x) \le 1 \quad \forall\, x \in \mathbb{R}^N,$$

$$|\nabla \varphi_{s,\delta}| \le \frac{C}{\delta}, \qquad |\Delta \varphi_{s,\delta}| \le \frac{C}{\delta^2}$$

$$\forall\, x\colon \; s + a_1 \delta \zeta\left(\frac{|x'|}{\delta}\right) \le x_N \le s + a_1 \delta \left(\zeta\left(\frac{|x'|}{\delta}\right) + \frac{15}{16} \right).$$

Furthermore, it is easy to verify the validity of the inclusions

$$\Omega(s+\delta) \subset \left\{x\colon x_N \ge s + a_1\delta\left(\zeta\left(\frac{|x'|}{\delta}\right)+\frac{15}{16}\right)\right\} \subset \Omega\left(s+\frac{15}{16}\delta\right) \subset \left\{x\colon x_N \ge s + a_1\delta\zeta\left(\frac{|x'|}{\delta}\right)\right\} \subset \Omega(s),$$

$$K(s,\delta) \supset \left\{x\colon s + a_1\delta\zeta\left(\frac{|x'|}{\delta}\right) \le x_N \le s + a_1\delta\left(\zeta\left(\frac{|x'|}{\delta}\right)+\frac{15}{16}\right)\right\},$$

which imply that $\operatorname{supp}\varphi_{s,\delta}(x) \Subset \Omega\left(s+\frac{15}{16}\delta\right)$.

Setting $\zeta^4(x,t) = \varphi^4_{s,\delta}(x)\exp(-tT^{-1})$ in the local entropy estimate (2.3), after simple transformations, we get inequality (3.2).

Lemma 4. *Suppose that* $u(x,t)$ *is an arbitrary "strong" solution of problem* (C). *We introduce the following energy functions connected with this solution:*

$$I_T(s) := \iint\limits_{Q_T(s)} u^{\alpha+n+1}\,dx\,dt, \qquad E_T(s) := \iint\limits_{Q_T(s)} u^{\alpha+m+1}\,dx\,dt.$$

Then, for all $\lambda \ge 1$, *the following inequalities are true:*

$$I_T(s+\delta) \le C_1 T^{1-\theta_2} R_T^{1+k_2}(s,\delta), \tag{3.3}$$

$$E_T(s+\delta) \le \tilde{C}_1 T^{1-\theta_1} R_T^{1+k_1}(s,\delta), \tag{3.4}$$

where

$$\begin{aligned} \theta_1 &= \frac{mN}{mN+2(\alpha+1)}, \qquad k_1 = \frac{2m}{mN+2(\alpha+1)},\\ \theta_2 &= \frac{nN}{nN+4(\alpha+1)}, \qquad k_2 = \frac{4n}{nN+4(\alpha+1)}, \end{aligned} \tag{3.5}$$

as well as

$$I_T(s+\delta) \le C_2 R_T^{1+l_2}(s,\delta), \quad \text{if} \quad \lambda \le n+1, \tag{3.6}$$

$$E_T(s+\delta) \le \tilde{C}_2 R_T^{1+l_1}(s,\delta), \quad \text{if} \quad \lambda \le m+1, \tag{3.7}$$

where

$$l_1 = \frac{2(m-\lambda+1)}{N(m-\lambda+1)+2(\alpha+1)}, \qquad l_2 = \frac{4(n-\lambda+1)}{N(n-\lambda+1)+4(\alpha+1)}. \tag{3.8}$$

All constants are independent of T.

Proof. Applying the interpolation inequality (4.1) given in Sec. 4 to the function $v(x) = u^{(\alpha+n+1)/4}$ as $\tilde{\delta} \to \infty$ and setting $a = d = 4$, $b = \dfrac{4(\alpha+1)}{n+\alpha+1}$, $i = 1$, we get

$$\int\limits_{\Omega(s+\delta)} u^{n+\alpha+1}\,dx \;\le\; C\left(\int\limits_{\Omega(s+\delta)} \left|\nabla u^{(n+\alpha+1)/4}\right|^4 dx\right)^{\theta_2}\left(\int\limits_{\Omega(s+\delta)} u^{\alpha+1}\,dx\right)^{\frac{(1-\theta_2)(n+\alpha+1)}{\alpha+1}}.$$

Integrating this relation with respect to the time and using the Hölder inequality with exponents θ_2^{-1} and $(1-\theta_2)^{-1}$, we obtain the estimate

$$\iint\limits_{Q_T(s+\delta)} u^{n+\alpha+1}\,dx\,dt \;\le\; C\left(\iint\limits_{Q_T(s+\delta)} \left|\nabla u^{(n+\alpha+1)/4}\right|^4 dx\,dt\right)^{\theta_2}\left(\int\limits_0^T\left(\int\limits_{\Omega(s+\delta)} u^{\alpha+1}\,dx\right)^{1+\frac{n}{\alpha+1}} dt\right)^{1-\theta_2},$$

which, by virtue of Lemma 3, leads to inequality (3.3). In the same manner, we prove inequality (3.4) applying inequality (4.1) to $v(x) = u^{(\alpha+m+1)/2}$ as $\tilde{\delta} \to \infty$ and setting $a = d = 2$, $b = \dfrac{2(\alpha+1)}{m+\alpha+1}$, and $i = 1$. Integrating the obtained inequality with respect to time and using the Hölder inequality, we obtain relation (3.4). We apply inequality (4.2) established in Sec. 4 to the function $v(x) = u^{(\alpha+m+1)/2}$ for $p = 2$, $r = \dfrac{2(\lambda+\alpha)}{m+\alpha+1}$, $q = \dfrac{2(\alpha+1)}{m+\alpha+1}$, and $1 - b = \dfrac{2(\alpha+1)}{N(m-\lambda+1)+2(\alpha+1)}$. The condition $r \le p$ leads to the restriction $\lambda \le m+1$. We perform analogous operations with the function $v(x) = u^{(\alpha+n+1)/4}$ for $p = 4$, $r = \dfrac{4(\lambda+\alpha)}{n+\alpha+1}$, $q = \dfrac{4(\alpha+1)}{n+\alpha+1}$, and $1 - b = \dfrac{4(\alpha+1)}{N(n-\lambda+1)+4(\alpha+1)}$ for $\lambda \le n+1$. As a result, we obtain inequalities (3.6) and (3.7).

We now pass to the proof of Theorem 2.

Assume that condition (i) is satisfied. In this case, the influence of absorption on the propagation of the support of the solution is not taken into account.

We introduce auxiliary functions connected with the energy functions $E_T(s)$ and $I_T(s)$, namely:

$$A_T(s) = E_T^{1+k_2}(s), \qquad B_T(s) = I_T^{1+k_1}(s),$$

where k_i, $i = 1, 2$, are defined in (3.5). For these functions, by virtue of (3.3) and (3.4), the following inequalities are true:

$$A_T(s+\delta) \le cT^{(1-\theta_1)(1+k_2)}R_T^{\beta}(s,\delta),$$

$$B_T(s+\delta) \le cT^{(1-\theta_2)(1+k_1)}R_T^{\beta}(s,\delta) \quad \forall s \in \mathbb{R}^1, \quad \delta > 0,$$

where $\beta := (1+k_1)(1+k_2)$.

We consider the auxiliary function $C_T(s) = T^h A_T(s) + B_T(s)$, where $h = (1-\theta_2)(1+k_1) - (1-\theta_1)(1+k_2)$. Taking into account the above-obtained relations for the function $C_T(s+\delta)$, we get the estimate

$$C_T(s+\delta) \le cT^{(1-\theta_2)(1+k_1)}\left[\delta^{-4}(I_T(s) - I_T(s+\delta)) + \delta^{-2}(E_T(s) - E_T(s+\delta)) + h_0(s)\right]^{\beta}. \qquad (3.9)$$

Using the inequality $(a-b)^{\alpha+1} \le a^{\alpha+1} - b^{\alpha+1}$ $\forall \alpha > 0$, $a > b > 0$, we obtain the inequalities

$$\begin{aligned} C_T(s+\delta) \le\ & cT^{(1-\theta_2)(1+k_1)}\left[\delta^{-4\beta}(B_T(s) - B_T(s+\delta))^{1+k_2}\right. \\ & \left. + \delta^{-2\beta}(A_T(s) - A_T(s+\delta))^{1+k_1} + h_0^{\beta}(s)\right] \\ \le\ & c[\delta^{-4\beta}T^{(1-\theta_2)(1+k_1)}(\Delta C_T(s))^{1+k_2} + \delta^{-2\beta}(\Delta C_T(s))^{1+k_1}T^{(1-\theta_2)(1+k_1)-h(1+k_1)}] \\ & + cT^{(1-\theta_2)(1+k_1)}h_0^{\beta}(s), \end{aligned}$$

where $\Delta C_T(s) = C_T(s) - C_T(s+\delta)$.

Now we choose the parameter δ, which was arbitrary up to now. First, we introduce the notation

$$\delta_T^1(s) := [T^{(1-\theta_2)(1+k_1)-h(1+k_1)}C_T^{k_1}(s)]^{\frac{1}{2\beta}}, \qquad \delta_T^2(s) := [T^{(1-\theta_2)(1+k_1)}C_T^{k_2}(s)]^{\frac{1}{4\beta}},$$

$$S_T^1 := \{s \in \mathbb{R}^+ : \delta_T^1(s) \ge \delta_T^2(s)\}, \qquad S_T^2 := \mathbb{R}^+ \setminus S_T^1, \qquad J_T(s) := \max_{i=1,2}\{\delta_T^i(s)\}.$$

Setting $\delta = \delta_T^1(s)$ in the inequalities derived above, we obtain the estimate

$$C_T(s + \delta_T^1(s)) \le \frac{2c}{2c+1}C_T(s) + cT^{(1-\theta_2)(1+k_1)}h_0^{\beta}(s) \quad \forall s \in S_T^1. \qquad (3.10)$$

Indeed, if $\delta_T^1(s) \ge \delta_T^2(s) > 0$, then we have

$$(\delta_T^1(s))^{-4\beta}C_T^{k_2}(s)T^{(1-\theta_2)(1+k_1)} \le (\delta_T^2(s))^{-4\beta}C_T^{k_2}(s)T^{(1-\theta_2)(1+k_1)} = 1.$$

If $\delta_T^1(s) \ge \delta_T^2(s) = 0$, then $C_T(s) = 0$, which means that $\delta_T^1(s) = 0$ and the statement of the theorem is obvious. By analogy, we perform similar operations for $\delta = \delta_T^2(s)$ and obtain

$$C_T(s + \delta_T^2(s)) \le \frac{2c}{2c+1}C_T(s) + cT^{(1-\theta_2)(1+k_1)}h_0^{\beta}(s) \quad \forall s \in S_T^2. \qquad (3.11)$$

Thus, using relations (3.10) and (3.11), we obtain

$$C_T\big(s + J_T(s)\big) = \begin{cases} C_T(s + \delta_T^1), & s \in S_T^1, \\ C_T(s + \delta_T^2), & s \in S_T^2, \end{cases}$$

which leads to the inequality

$$C_T\big(s + J_T(s)\big) \le \frac{2c}{2c+1}\, C_T(s) + cT^{(1-\theta_2)(1+k_1)} h_0^{\beta}(s) \quad \forall\, s \in \mathbb{R}^1.$$

Raising both sides of this inequality to the power $k_1(2\beta)^{-1}$ and multiplying by $T^{((1-\theta_2)(1+k_1)-h(1+k_1))/(2\beta)}$, we obtain an inequality for $\delta_T^1(s)$. Now we raise both sides of this inequality to the power $k_2(4\beta)^{-1}$ and multiply by $T^{((1-\theta_2)(1+k_1))/(4\beta)}$. As a result, we obtain an inequality for $\delta_T^2(s)$. Combining both these inequalities, we obtain the following main functional relation for the function $J_T(s)$:

$$J_T\big(s + J_T(s)\big) \le \gamma J_T(s) + \bar{c}\, F_T(s) \quad \forall\, s \in \mathbb{R}^1, \quad 0 < \gamma < 1, \tag{3.12}$$

where

$$F_T(s) := \max\big\{T^{(1-\theta_1)/2} h_0^{k_1/2}(s);\, T^{(1-\theta_2)/4} h_0^{k_2/4}(s)\big\}.$$

We set $s = -2\delta$ and $\delta = s$ in relation (3.9) and pass to the limit as $s \to \infty$. Using the boundedness of the functions $I_T(s)$, $E_T(s)$ and $h_0(s)$, we obtain the estimate

$$C_T(-\infty) \le cT^{(1-\theta_2)(1+k_1)} h_0^{\beta}(-\infty).$$

Thus, there exists $-\infty < s_0' < 0$ such that

$$J_T(s) \le \tilde{\tilde{c}}\, F_T(s) \quad \forall\, s < s_0'. \tag{3.13}$$

By virtue of the assumption on the increase in the function $h_0(s)$, we have

$$F_T(s) \le -ks \max\big\{T^{(1-\theta_1)/2};\, T^{(1-\theta_2)/4}\big\} := -as. \tag{3.14}$$

Now we use Lemma 5 formulated in Sec. 4 for the functions $J_T(s)$ and $F_T(s)$ that satisfy relations (3.12)–(3.14). As a result, we get

$$J_T(s) \equiv 0 \quad \forall\, s \ge 0, \quad T^* < T^*(k),$$

which was to be proved.

Assume that condition (ii) is satisfied. The scheme of the proof is the same as for the first case but with regard for the influence of absorption on the motion of a support of the solution. We assume that

$$A_T(s) = E_T^{1+k_2}(s), \quad B_T(s) = I_T^{1+l_1}(s),$$

where k_2 and l_1 are defined in (3.5) and (3.8). By virtue of inequalities (3.3) and (3.7), we have

$$A_T(s+\delta) \le cR_T^{\beta}(s,\delta), \quad B_T(s+\delta) \le cT^{(1-\theta_2)(1+l_1)}R_T^{\beta}(s,\delta),$$

where $\beta := (1+l_1)(1+k_2)$. The function $C_T(s)$ takes the form

$$C_T(s) = A_T(s) + T^{-(1-\theta_2)(1+l_1)}B_T(s).$$

By analogy with the first case, we obtain the functional inequality (3.12), where

$$J_T(s) := \max\{T^{(1-\theta_2)/4}C_T^{k_2/(4\beta)}(s); C_T^{l_1/(2\beta)}(s)\},$$

$$F_T(s) = \max\{T^{(1-\theta_2)/4}h_0^{k_2/4}(s); h_0^{l_1/2}(s)\}.$$

Applying Lemma 5 to the functions $J_T(s)$ and $F_T(s)$, we obtain the necessary statement.

Assume that condition (iii) is satisfied. By analogy with the previous case, we take the influence of absorption into account. We set

$$A_T(s) = E_T^{1+l_2}(s), \quad B_T(s) = I_T^{1+k_1}(s),$$

where k_1 and l_2 are defined in (3.5) and (3.8). Using inequalities (3.4) and (3.6), we get

$$A_T(s+\delta) \le cT^{(1-\theta_1)(1+l_2)}R_T^{\beta}(s,\delta), \quad B_T(s+\delta) \le cR_T^{\beta}(s,\delta),$$

where $\beta := (1+k_1)(1+l_2)$. The function $C_T(s)$ takes the form

$$C_T(s) = T^{-(1-\theta_1)(1+l_2)}A_T(s) + B_T(s).$$

By analogy with the first case, we obtain inequality (3.12), where

$$J_T(s) := \max\{T^{(1-\theta_1)/2}C_T^{k_1/(2\beta)}(s); C_T^{l_2/(4\beta)}(s)\},$$

$$F_T(s) = \max\{T^{(1-\theta_1)/2}h_0^{k_1/2}(s); h_0^{l_2/4}(s)\}.$$

Using Lemma 5, we obtain the necessary result.

Now assume that condition (iv) is satisfied. In this case, the influence of absorption on the support propagation is the most essential. For this reason, we set

$$A_T(s) = E_T^{1+l_2}(s), \quad B_T(s) = I_T^{1+l_1}(s),$$

where l_1 and l_2 are defined by (3.8). Using inequalities (3.4) and (3.6), we get

$$A_T(s+\delta) \le cR_T^{\beta}(s,\delta), \qquad B_T(s+\delta) \le cR_T^{\beta}(s,\delta),$$

where $\beta := (1+l_1)(1+l_2)$. The function $C_T(s)$ can be represented in the form

$$C_T(s) = A_T(s) + B_T(s)$$

and in the corresponding inequality (3.12),

$$J_T(s) := \max\{C_T^{l_1/(2\beta)}(s);\, C_T^{l_2/(4\beta)}(s)\}, \qquad F_T(s) = \max\{h_0^{l_1/2}(s);\, h_0^{l_2/4}(s)\}.$$

Applying Lemma 5, we obtain the required statement.

Thus, Theorem 2 is completely proved.

4. Application

Let $K(s,\tilde{\delta})$ be the domain introduced in the third section. Then it is easy to verify that the interpolation Nirenberg–Gagliardo inequality [32] has the form

$$\|\nu\|_{a,K(s,\tilde{\delta})} \le k_1 \tilde{\delta}^{\frac{-N(a-b)}{ab}} \|\nu\|_{b,K(s,\tilde{\delta})} + k_2 \left\| D^i \nu \right\|^{\theta}_{d,K(s,\tilde{\delta})} \|\nu\|^{1-\theta}_{b,K(s,\tilde{\delta})}, \tag{4.1}$$

where $\nu(x) \in W_d^i(K(s,\tilde{\delta})) \cap L^b(K(s,\tilde{\delta}))$, $N \ge 1$ is the dimension, $\tilde{\delta} > 0$, $b > 0$, $a > 1$, $d > 1$, $\theta \in (0;1)$ and is determined by the relation $\theta = \left(\frac{1}{b} - \frac{1}{a}\right)\left(\frac{1}{b} + \frac{i}{N} - \frac{1}{d}\right)^{-1}$, $0 \le k_1$, and $k_2 < \infty$ are constants independent of $\tilde{\delta}$ and $\nu(x)$.

Let $\Omega(s) \subset \mathbb{R}^N$ be the domain considered in the third section. Then the following interpolation inequality, the proof of which is a simple corollary of the Hölder inequality and (4.1) (see, e.g., [33]), is true:

$$\int_0^T \int_{\Omega(s)} |\nu|^p\,dx\,dt \le \tilde{C} \left(\int_0^T \int_{\Omega(s)} |\nabla \nu|^p\,dx\,dt \right)^{b} \left(\int_0^T \int_{\Omega(s)} |\nu|^r\,dx\,dt \right)^{1-b} \sup_{[0;T]} \left(\int_{\Omega(s)} |\nu|^q\,dx \right)^{(p-r)(1-b)/q}, \tag{4.2}$$

where $\nu \in L^p((0;T); W^1_{p,\mathrm{loc}}(\Omega(s)))$, $T > 0$, $1 - b = pq(pq + N(p-r))^{-1}$, $r \le p$, $q > 0$, and $\tilde{C} > 0$ depends on N, p, q, and r and is independent of T.

Lemma 5 (Lemma 3 in [26]). *Suppose that a certain continuous nonnegative nonincreasing function $f(s)$ satisfies the functional relation*

$$f(s + f(s)) \le \gamma f(s) + g(s) \quad \forall s \in \mathbb{R}^1, \quad 0 < \gamma < 1,$$

where the known nonnegative continuous nonincreasing function $g(s)$ *satisfies the estimate*

$$g(s) \le -as \quad \forall s < 0, \quad 0 < a < \infty.$$

Suppose also that, for certain $-\infty < s_0' < 0$, *the inequality*

$$f(s) \le kg(s) \quad \forall s < s_0' < 0, \quad k > \frac{1}{1-\gamma},$$

is true. In addition, if

$$a < k^{-1}(1-\gamma-k^{-1}),$$

then $f(s) \equiv 0 \ \ \forall s \ge 0.$

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