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On the Thin-Film Equation with Nonlinear Convection in Multidimensional Domains

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Abstract. We investigate the global solvability of the Cauchy problem for a multidimensional thin-film equation with nonlinear convection. Preliminarily, we construct a nonnegative local generalized "strong" solution of the Neumann problem in a bounded domain as the limit of the sequence of solutions of the corresponding regularized boundary-value problems. We establish the finiteness of the speed of support propagation for arbitrary "strong" solutions of the Neumann problem. Using this property, we construct a nonnegative global "strong" solution of the Cauchy problem with an arbitrary finite initial function under optimal conditions on parameters of nonlinearity of the equation.

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1. Introduction and Statement of Main Results

In the present paper, we investigate the degenerate fourth-order parabolic equation

$$u_t + \operatorname{div}\left(|u|^n \nabla\Delta u - |u|^m \nabla u\right) = \overrightarrow{\chi} \cdot \nabla b(u),\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \quad (1.1)$$

where $u = u(t,x), n > 0, m \in \mathbb{R}^1, b(z) \in W^1_{1,loc}(\mathbb{R}^1)$, and $\overrightarrow{\chi} \in \mathbb{R}^N$. Equation (1.1) is a typical representative of a broad class of nonlinear equations of the form

$$u_t + \operatorname{div}\left(f(u)\nabla\Delta u + \nabla A(u)\right) = g(t,x,u,\nabla u) \quad (1.2)$$

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which appear in the simulation of various processes in the theory of elastic deformations and dynamics of fluids and binary alloys [2, 6, 8, 9, 16, 18]. For example, Eq. (1.1) with $b(u) = 0$ describes the evolution of the height $u(t, x)$ a liquid film that propagates over a solid surface under the action of surface tension, viscosity, and (for $m = n$) gravitation. Note that, for $m < 0$, the second-order term in Eq. (1.1) corresponds to the intermolecular Van der Waals forces, whereas for $m > 0$ $(m \neq n)$ it corresponds to internal diffusion forces. The fourth-order term in Eq. (1.1) describes the effect of capillary surface tension. The exponent $n > 0$ characterizes the behavior of a liquid at the line of its contact with a solid surface. For example, the case $n = 3$ corresponds to the no-slip condition, and the case $n \in (0, 3)$ corresponds to the motion of a liquid for slip condition. The case $n = m = 1$ describes the variation in the size of the domain filled with liquid in a half-space Hele–Shaw cell in the lubrication regime [23]. The case $n = 1$ and $m = 0$ corresponds to the principal term (for $u \to 0$) in the Cahn–Hilliard equation for binary alloys with logarithmic free energy:

$$u_t + \operatorname{div}\left[u(1-u)\nabla\left\{\Delta u - \ln\tfrac{u}{1-u}\right\}\right] = 0.$$

The convective term in Eq (1.1) describes the action of various potentials, e.g., gravitation on an inclined plane ($\overrightarrow{\chi} b(u) = (u^n, 0, \ldots, 0)$), thermally induced surface tension ($\overrightarrow{\chi} b(u) = (-u^{n-1}, 0, \ldots, 0)$) [6, 7, 10], etc.

The mathematical investigation of degenerate fourth-order thin-film equations was initiated by Bernis and Friedman in [5], where, in particular, a nonnegative generalized solution of a Neumann problem with an arbitrary nonnegative initial function was constructed for the one-dimensional equation

$$u_t + \operatorname{div}(|u|^n \nabla \Delta u) = 0 \tag{1.3}$$

Later, nonnegative generalized solutions of boundary-value problems for the multidimensional equation (1.3) were constructed, and a number of qualitative properties of the constructed solutions that depend on the parameters n and N (e.g., the finiteness of the speed and waiting time phenomenon of the propagation of supports of solutions, support shrinking, etc.) were described in [1, 3, 4, 11, 13, 19, 20]. In [18], nonnegative solutions were constructed for the Neumann problem for Eq. (1.2) where $g = g(t, x, u)$ is a function of at most linear growth with respect to u, $|A'(u)| \leqslant d_0 f(u)$, and $f(u)$ is a function of power growth in the neighborhood of $u = 0$. For $g = g(t, x, u) \sim |u|^{\lambda-1}u\,(\lambda > 0)$, $A'(u) = -|u|^m$, and $f(u) = |u|^n$, Eq. (1.2) was studied in [25, 26]. In [14], for the multidimensional equation (1.1) with $b(u) = 0$, nonnegative generalized solutions were constructed, conditions on the parameters n and m that guarantee the finiteness or infiniteness of the speed of propagation of perturbations were found, and the exact asymptotics of the motion of the support of a solution were obtained.

In [6, 7, 10], the stability of traveling-wave solutions of the one-dimensional equation (1.1) with $n = m = 3$, $\chi = 1$, and $b(u) = u^3 - u^2$ was studied. In [17], for the one-dimensional equation (1.1) without diffusion term in the case $n \in (0, 3)$, $\lambda > \max\left\{\frac{3}{4}n - 1, \frac{1}{8}\right\}$, and $b(u)$ as in (1.6), a nonnegative local generalized solution and, under the additional condition $\lambda < \frac{9}{2}$, a nonnegative global generalized solution of the Cauchy problem were constructed. It was also proved that the speed of support propagation is finite, and its upper bound for large and small times was obtained.

The main result of the present paper is the construction of a nonnegative global "strong" (entropy) generalized solution of the Cauchy problem

$$(C)\begin{cases} u_t + \operatorname{div}(u^n \nabla\Delta u - u^m \nabla u) = \overrightarrow{\chi} \cdot \nabla b(u),\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, & (1.4)\\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^N),\ u_0 \geqslant 0,\ \operatorname{supp} u_0 \text{ is a compact set}, & (1.5)\\ |b'(z)| \leqslant c\,|z|^{\lambda-1}\ \forall\, z \in \mathbb{R}^1, b(0) = 0, \lambda > 0, c > 0, & (1.6)\end{cases}$$

for $N \leqslant 3$, $m > 0$, and $\frac{1}{8} < n < 2$ and the proof of the finiteness of the speed of propagation of the support of this solution. In a subsequent work, we will describe the evolution of the support of a solution for large and small times and establish exact (in a certain sense) upper bounds for the right and left (with respect to the direction of the vector of convection) fronts of the support of the solution.

Remark 1.1. The method used for the investigation of problem (C) is not connected with the model form of Eq. (1.4) and can easily be adapted to equations with more general structure, e.g., to the case of a more general convective transfer $b = b(t, x, u)$ that does not have a periodic structure with respect to the space variable x.

In the present paper, parallel with problem (C), we consider the Neumann problem for Eq. (1.4), namely,

$$(N)\begin{cases} u_t + \operatorname{div}\left(u^n \nabla\Delta u - u^m \nabla u\right) = \overrightarrow{\chi} \cdot \nabla b(u) \text{ in } Q_T = (0,T) \times \Omega, & (1.7)\\ \nabla u \cdot \vec{n} = \nabla\Delta u \cdot \vec{n} = 0 \text{ on } (0,T) \times \partial\Omega, & (1.8)\\ u(0,x) = u_0(x) \text{ in } \Omega, & (1.9)\end{cases}$$

where $b(z)$ is defined by (1.6), Ω is an arbitrary bounded connected domain in $\mathbb{R}^N$ $(N \leqslant 3)$ with boundary $\partial\Omega$ of the class $C^{1,1}$, and $\vec{n} = \vec{n}(x)$ is the unit vector of the outward normal to $\partial\Omega$ at the point x.

1.1. Notation and Main Assumptions

Sets, Vectors, and Auxiliary Functions: $Q_{t_1}^{t_2} = (t_1, t_2) \times \Omega$; $B(x_0, r) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$; for an $N \times N$ matrix A and vectors $a, b \in \mathbb{R}^N$, we define $\langle a, A, b \rangle := \sum\limits_{i,j=1}^{N} a_i A_{ij} b_j$ and $a \cdot b := \sum\limits_{i=1}^{N} a_i b_i$; χ_A is the characteristic function of a set A; for an arbitrary measurable function $v(t,x)$, we define the positivity set

$$\mathrm{P} := \mathrm{P}(v) = \{v > 0\} = \{(t,x) \in Dom(v) : v(t,x) > 0\}; \tag{1.10}$$

by $\overline{\overline{o}}_\delta(1)$ we denote quantities $f(\delta)$ satisfying the condition $\lim\limits_{\delta \to 0} |f(\delta)| = 0$;

$$\mathcal{M}_\gamma(z) := \begin{cases} z^{\frac{\gamma+1}{2}} \text{ if } \gamma \neq -1, \\ \ln z \text{ if } \gamma = -1. \end{cases} \tag{1.11}$$

Notation for Functional Spaces:
$C^k(\Omega)$ is the space of functions k times continuously differentiable in the domain Ω, and $C_c^k(\Omega) := \{v \in C^k(\Omega) : \operatorname{supp} v \subset \Omega\}$;
$L^p(\Omega)$ is the Banach space of functions measurable in Ω and Lebesgue summable to the power $p \geqslant 1$;
$L^{p-}(\Omega) := \{v \in L^1(\Omega) : v \in L^{p-\varepsilon}(\Omega)$ for arbitrary $0 < \varepsilon < p-1\}$;
$W_p^k(\Omega) := \{v : D^\alpha v \in L^p(\Omega) \, \forall \, |\alpha| \leqslant k\}$ is a Sobolev space;
$H^k(\Omega) := W_2^k(\Omega)$ and $H_*^2(\Omega) := \{v \in H^2(\Omega) : \nabla v \cdot \vec{n} = 0$ a.e. on $\partial\Omega\}$;
X^* is the space dual to a Banach space X;
$\langle w, v \rangle_{X^*, X} = \langle w, v \rangle$ is the coupling of elements of the spaces X^* and X;
$L^p(0,T;X)$ is the space of (classes of) Lebesgue measurable functions $v(t,.)$ taking values in X and such that the function $\|v(t,.)\|_X^p$ is measurable on $(0,T)$;
$X_{loc}(\Omega)$ is the space of functions that belong to the space $X(\Omega')$ for an arbitrary bounded subdomain $\Omega' : \overline{\Omega'} \subset \Omega$;
$X \Subset Y$ means that the space X is imbedded compactly into the space Y.

Operators: $\operatorname{div} \overrightarrow{F} := \sum\limits_{i=1}^{N} \frac{\partial}{\partial x_i} F_i(x)$ is the divergence of a vector field $\overrightarrow{F}$, $D^2 v$ is the tensor of the second derivatives of a scalar function $v \in H^2(\Omega)$, and $D\overrightarrow{F} := \nabla \bullet \overrightarrow{F}$ is the tensor product of the vector ∇ and the vector field $\overrightarrow{F}$.

Measures: $\mathfrak{L}^N$ is an N-dimensional Lebesgue measure and $\mathcal{H}^{N-1}$ is an $(N-1)$-dimensional Hausdorff measure.

Types of Constants: C and C_i (c *and* c_i) are positive constants dependent on all known parameters of problem (N) (independent of the initial function u_0 and the domain Ω); $C(\Omega)$ and $C_i(\Omega)$ ($c(u_0)$ and $c_i(u_0)$) are positive constants dependent on the known parameters of problem (N) and the domain Ω (the function u_0) and independent of u_0 (the domain Ω); d and d_i are positive constants that depend neither on the parameters of problem (N), nor on u_0, nor on Ω.

Integration: In the case where the domain of integration is obvious from the context, the corresponding differentials are omitted.

We introduce the set $\Delta_{n,\lambda} := \left(\max\left\{\frac{1}{2} - n, -\lambda\right\}, \min\left\{\frac{n+1}{3}, 2-n\right\}\right) \setminus \{0, -1\}$ and denote

$$\tilde{\Delta}_{n,\lambda} := \Delta_{n,\lambda} \text{ for } N < 3,\ \tilde{\Delta}_{n,\lambda} := \Delta_{n,\lambda} \cap (-2, +\infty) \text{ for } N = 3. \quad (1.12)$$

As initial functions in problem (N), we will use functions $u_0(x)$ such that

$$0 \leqslant u_0(x) \in H^1(\Omega) \cap \left\{ v : \int\limits_{\Omega} |v|^{\alpha+1} < \infty \right\} \cap \left\{ v : \int\limits_{\Omega} \Psi_0(v) < \infty \right\} \quad (1.13)$$

for certain $\alpha \in \tilde{\Delta}_{n,\lambda}$. Here,

$$\Psi_0(z) := \begin{cases} \frac{z^{m-n+2}}{(m-n+1)(m-n+2)} + \frac{R^{m-n+1}}{m-n+2} - \frac{R^{m-n+1}}{m-n+1} z, m-n+2 \neq 0, 1, \\ -\ln z + z - 1,\ m-n+2 = 0, \\ z \ln z - z + 1,\ m-n+2 = 1 \end{cases} \quad (1.14)$$

where $R = 0$ if $m - n + 1 > 0$ and $R = 1$ if $m - n + 1 < 0$.

1.2. Formulation of Main Results

First, we prove the solvability of problem (N).

Definition 1.1. *Let $m > -1$, $n > 0$, and $\lambda > 0$. Following the concept of generalized solution from* [13, 14]*, we say that a nonnegative function $u(t, x) \in L^\infty(0, T;\ H^1(\Omega))$ is a solution of problem (N) if the following conditions are satisfied:*

(i) $\nabla b(u) \in L^2(0, T; (W^1_q(\Omega))^*)$ $\left(q = \frac{q'}{q'-1}\right)$ for any $q' \in \Delta_1 := \left(1, \frac{4N}{2N+n(N-2)}\right)$ $(\Delta_1 := \{2\}$ if $N = 1)$ and there exists a vector function $\overrightarrow{J} \in L^2(0, T; L^{q'}(\Omega; \mathbb{R}^N))$ $\forall\, q' \in \Delta_1$ such that $u_t = -\mathrm{div}\, \overrightarrow{J} + \overrightarrow{\chi} \cdot \nabla b(u)$ in $L^2(0, T; (W^1_q(\Omega))^*)$;

(ii) $\chi_{\mathrm{P}} u^{n-2} |\nabla u|^3$, $\chi_{\mathrm{P}} u^{n-1} |\nabla u|^2$, $u^n |\nabla u|$, u^{m+1}, and $b(u)$ belong to the space $L^1(0, T; L^1(\Omega))$, where $P := P(u)$ is defined in (1.10)*;*

(iii) $u(t,x)$ satisfies the boundary conditions (1.8) *and is a solution of Eq.* (1.7) *in the following sense:*

$$\int_0^T \langle u_t(t),\zeta(t)\rangle\,dt - \int_0^T \langle \overrightarrow{\chi}\cdot\nabla b(u),\zeta\rangle\,dt = \tfrac{n(n-1)}{2}\iint\limits_{\mathrm{P}} u^{n-2}|\nabla u|^2\nabla u\,\nabla\zeta$$
$$+\tfrac{n}{2}\iint\limits_{\mathrm{P}} u^{n-1}|\nabla u|^2\Delta\zeta + n\iint\limits_{\mathrm{P}} u^{n-1}\left\langle\nabla u, D^2\zeta,\nabla u\right\rangle$$
$$+\iint\limits_{Q_T} u^n\nabla u\nabla\Delta\zeta + \tfrac{1}{m+1}\iint\limits_{Q_T} u^{m+1}\Delta\zeta =: \mathcal{U}^{(T)}(u,\zeta) \quad (1.15)$$

$\forall\,\zeta\in C^3(\overline{Q}_T)\colon \nabla\zeta\cdot\vec{n}=0$ *on* $(0,T)\times\partial\Omega$*;*

(iv) $u(t,x)$ satisfies the initial condition (1.9) *in the sense that* $u(t,.)\underset{t\to 0}{\rightrightarrows} u_0(.)$ *in* $L^2(\Omega)$.

Theorem 1.1. *Suppose that $N\leqslant 3$ and the parameters n, m, and λ satisfy the relations*

$$\begin{aligned}&m>-1,\ n>\tfrac{1}{8},\ \lambda>\max\left\{\tfrac{1}{8},\tfrac{3n-1}{7},n-2,n-m-1\right\};\\ &n<4 \text{ and } \lambda<\min\left\{\tfrac{4n+7}{3},4\right\} \text{ if } N=3.\end{aligned} \quad (1.16)$$

Also assume that a parameter $\alpha\in\tilde{\Delta}_{n,\lambda}$ ($\tilde{\Delta}_{n,\lambda}$ is defined by (1.12)) *satisfies the relations*

$$\lambda>\max\left\{\tfrac{3(\alpha+n)-1}{4},(n-m-1)_+\right\};\ \lambda<\alpha+n+2 \text{ for } N=3 \quad (1.17)$$

(*it is easy to verify that, for any n, m, and λ from* (1.16)*, the set of these α is nonempty) and the initial function $u_0\geqslant 0$ satisfies condition* (1.13) *for this α. Then there exists a solution $u(t,x)$ of problem (N) in the domain $(0,T_{loc})\times\Omega$ ($T_{loc}>0$ depends on the known parameters of problem (N)) with the following properties:*

(i) $\Psi_0(u)\in L^\infty(0,T_{loc};L^1(\Omega))$ and $u(t,x)>0$ a.e. in Ω for almost all $t\in[0,T_{loc}]$ if $m-n+2\leqslant 0$;

(ii) if $\alpha+m+1>0$, then $\mathcal{M}_{\alpha+m}(u)\in L^2(0,T_{loc};H^1(\Omega))$, where $\mathcal{M}_\gamma(z)$ is defined by (1.11)*;*

(iii) if $\alpha+m+1\leqslant 0$, then $\mathcal{M}_{\alpha+m}(u(t,x))\in L^2_{loc}(\Omega)$ and $u(t,x)>0$ a.e. in Ω for almost all $t\in(0,T_{loc})$, $\nabla\mathcal{M}_{\alpha+m}(u)=u^{\frac{\alpha+m-1}{2}}\nabla u\in L^2(Q_{T_{loc}})$, and, furthermore, $L^2_{loc}(\Omega)$ can be replaced by $L^2(\Omega)$ if the domain Ω is convex;

(iv) $u^{m+1}\in L^1(0,T_{loc};W^1_1(\Omega))$ if $m>0$, $u^{\frac{\alpha+n+1}{4}}\in L^4(0,T_{loc};W^1_4(\Omega))$, $u^{\frac{\alpha+n+1}{2}}\in L^2(0,T_{loc};H^2(\Omega))$, and $u^{\lambda+\alpha}\in L^1(Q_{T_{loc}})$ (it is obvious that the inequality $\lambda+\alpha>0$ follows from (1.16) *and* (1.12) *);*

(v) there exist positive constants c_1 and $C_2(\Omega)$ such that, for almost all $T \leqslant T_{loc}$, the following inequality is true:

$$c_1^{-1}\iiint\limits_{Q_T}\Big\{\big|D^2u^{\frac{\alpha+n+1}{2}}\big|^2+\big|\nabla u^{\frac{\alpha+n+1}{4}}\big|^4+\big|\nabla\mathcal{M}_{\alpha+m}(u)\big|^2\Big\}$$
$$+\tfrac{1}{\alpha(\alpha+1)}\int\limits_{\Omega}u^{\alpha+1}(T,x)\,dx\leqslant\tfrac{1}{\alpha(\alpha+1)}\int\limits_{\Omega}u_0^{\alpha+1}(x)dx$$
$$+\int\limits_0^T\int\limits_{\partial\Omega}\overrightarrow{\chi}\cdot\vec{n}\,\mathcal{B}^{(\alpha)}(u)+C_2(\Omega)\iint\limits_{Q_T}u^{\alpha+n+1}\tag{1.18}$$

where $\mathcal{M}_\gamma(z)$ is defined by relation (1.11), $\mathcal{B}^{(\alpha)}(z):=\frac{1}{\alpha}\int\limits_0^z b'(\tau)\tau^\alpha\,d\tau$, *and* $C_2(\Omega)=0$ *if the domain Ω is convex;*

(vi) for almost all $0\leqslant t_1<t_2\leqslant T_{loc}$, the following local version of inequality (1.18) *is true if $\alpha\in\tilde{\Delta}_{n,\lambda}\cap(-(m+1),+\infty)$:*

$$\tfrac{1}{\alpha(\alpha+1)}\int\limits_{\Omega}\zeta^4u^{\alpha+1}(t_2,x)\,dx-\tfrac{1}{\alpha(\alpha+1)}\iint\limits_{Q_{t_1}^{t_2}}(\zeta^4)_tu^{\alpha+1}$$
$$+c_3^{-1}\iint\limits_{Q_{t_1}^{t_2}}\zeta^4\left\{\big|\nabla u^{\frac{\alpha+m+1}{2}}\big|^2+\big|\nabla u^{\frac{\alpha+n+1}{4}}\big|^4+\big|D^2u^{\frac{\alpha+n+1}{2}}\big|^2\right\}$$
$$\leqslant\tfrac{1}{\alpha(\alpha+1)}\int\limits_{\Omega}\zeta^4u^{\alpha+1}(t_1,x)dx+c_3\int\limits_{t_1}^{t_2}\int\limits_{\{\zeta(t)>0\}}u^{\alpha+m+1}(\zeta^2\,|\nabla\zeta|^2+\zeta^3\,|\Delta\zeta|)$$
$$+c_3\int\limits_{t_1}^{t_2}\int\limits_{\{\zeta(t)>0\}}u^{\alpha+n+1}(|\nabla\zeta|^4+\zeta^2\,|\Delta\zeta|^2)-\int\limits_{t_1}^{t_2}\int\limits_{\{\zeta(t)>0\}}\overrightarrow{\chi}\mathcal{B}^{(\alpha)}(u)\nabla\zeta^4\tag{1.19}$$

where $\zeta\in C^2(\overline{Q}_{t_1}^{t_2})$ is an arbitrary nonnegative cut-off function such that $\operatorname{supp}\zeta\subset[t_1,t_2]\times\Omega$, *and, furthermore, $t_1=0$ and $\zeta_t(t,x)\leqslant 0$ if $\alpha+1<0$;*

(vii) for almost all $0\leqslant t_1<t_2\leqslant T_{loc}$ ($t_1=0$ if $m-n+2\leqslant 0$), the following inequality is true:

$$\tilde{\mathcal{E}}_\Omega(u(t_2))\leqslant\tilde{\mathcal{E}}_\Omega(u(t_1))+\int\limits_{t_1}^{t_2}\int\limits_{\partial\Omega}\overrightarrow{\chi}\cdot\vec{n}\,\tilde{\Psi}_0(u)-\iint\limits_{\mathrm{P}}\overrightarrow{\chi}\,b'(u)\nabla u\Delta u,\tag{1.20}$$

where $\tilde{\mathcal{E}}_\Omega(u(t)) := \int\limits_\Omega [\frac{1}{2}|\nabla u|^2 + \Psi_0(u)]dx$, $\tilde{\Psi}_0(z) := \int\limits_0^z b'(\tau)\Psi_0'(\tau)d\tau$, $\Psi_0(z)$ *is defined by relation* (1.14) *and* $P := P(u)$ *is defined by* (1.10).

Remark 1.2. If $n < 3$ and $0 < m - n + 2$ ($\leqslant 6$ for $N = 3$), then $\tilde{\Delta}_{n,\lambda} \cap (-1, +\infty) \neq \emptyset$ and an arbitrary nonnegative function $u_0(x) \in H^1(\Omega)$ is admissible in Theorem 1.1. If $n \geqslant 3$ or $m - n + 2 \leqslant 0$, then condition (1.13) leads to the requirement of the integrability of the function $u_0(x)$ to a certain negative power, for which, in particular, it is necessary that $\operatorname{mes}\{\Omega \backslash \operatorname{supp} u_0\} = 0$.

Corollary 1.1. *Suppose that* $\frac{1}{8} < n < 2$, $m - n + 1 > 0$, *and*

$$\max\left\{1, \tfrac{3n}{4}, n - m + \tfrac{1}{2} + \left(\tfrac{1-2n}{4}\right)_+\right\} < \lambda \left(< \min\left\{\tfrac{4n+7}{3}, 4\right\},\ N = 3\right). \tag{1.21}$$

Then $u(t,x) \equiv 0$ *is a unique solution in the sense of Definition* 1.1 *of problem* (N) *with* $u_0(x) = 0$.

The proof of Corollary 1.1 is given in Appendix A.

Under additional restrictions on the parameters of nonlinearity of Eq. (1.7), one can establish the finiteness of the speed of propagation of the support of the solution constructed in Theorem 1.1.

Theorem 1.2. *Suppose that* $N \in \{1, 2, 3\}$,

$$\begin{aligned} &m > 0,\ \tfrac{1}{8} < n < 2,\quad 1 < \lambda < \mu + 1 + \max\{n + \mu, m\} \text{ for } N < 3, \\ &1 < \lambda < \min\left\{\tfrac{4n+7}{3}, 4\right\} \text{ for } N = 3, \text{ and } \mu := \tfrac{2}{N}\min\left\{\tfrac{n+4}{3}, 3 - n\right\} \end{aligned} \tag{1.22}$$

(*it is easy to verify that, under these assumptions, the interval* $\tilde{\Delta}_{n,\lambda} \cap (0, +\infty)$ *is nonempty*), *and* $0 \leqslant u_0(x)$ *is an arbitrary function from* $H^1(\Omega) \cap L^{m-n+2}(\Omega)$ *such that* $\operatorname{supp} u_0(x) \subset B(0, \tilde{R}_0)$, *where* $\tilde{R}_0 > 0$ *and* $R_0 > 0 : B(0, \tilde{R}_0 + 3R_0) \Subset \Omega$. *Then there exist an increasing function* $\Gamma(t) \in C[0, T_{loc}]$, $\Gamma(0) = 0$, *and* $\tilde{T} := \tilde{T}(R_0) > 0$ *such that the solution* $u(t,x)$ *constructed in Theorem* 1.1 *can be represented in the form* $u(t,x) = u_1(t,x) + u_2(t,x)$, *where* u_1 *is a solution of problem* (N) *with the aforementioned initial function* $u_0(x)$ *and* u_2 *is the solution of the homogeneous* (*with* $u_0(x) \equiv 0$) *problem* (N), *and, furthermore,*

$$\begin{aligned} &\operatorname{supp} u_1(t,.) \subset B(0, \tilde{R}_0 + C\,\Gamma(t)) \Subset B(0, \tilde{R}_0 + R_0)\ \forall\, t \leqslant T_1, \\ &\operatorname{supp} u_2(t,.) \Subset \Omega \backslash B(0, \tilde{R}_0 + 2R_0)\ \forall\, t \leqslant T_1 := \min\{\tilde{T}, T_{loc}\}, \end{aligned}$$

where T_{loc} *is defined in Theorem* 1.1. *Moreover, if the parameters* λ, m, *and* n *also satisfy conditions* (1.21), *then* $u_2(t,x) \equiv 0$.

Using Theorems 1.1 and 1.2, one can construct a global solution of the Cauchy problem (C).

Definition 1.2. *Let $N \leqslant 3$, $m > -1$, $n > 0$, and $\lambda > 0$. We say that a nonnegative function $u(t,x) \in L^{\infty}_{loc}(\mathbb{R}^+; H^1_{loc}(\mathbb{R}^N))$ is a solution of problem (C) if the following conditions are satisfied:*

(i) $\chi_{\mathrm{P}} u^{n-2}|\nabla u|^3$, $\chi_{\mathrm{P}} u^{n-1}|\nabla u|^2$, $u^n|\nabla u|$, u^{m+1}, and $b(u)$ belong to the space $L^1_{loc}([0,\infty); L^1_{loc}(\mathbb{R}^N))$, where $P = P(u)$ is defined by (1.10);

(ii) for any function $\zeta \in C^3_c([0,\infty) \times \mathbb{R}^N)$, the following equality is true:

$$-\int\limits_0^\infty \int\limits_{\mathbb{R}^N} u\zeta_t - \int\limits_{\mathbb{R}^N} u_0 \zeta(0,x) + \int\limits_0^\infty \int\limits_{\mathbb{R}^N} \overrightarrow{\chi} \cdot \nabla\zeta\, b(u) = \mathcal{U}^{(\infty)}(u,\zeta)$$

where $\mathcal{U}^{(T)}(u,\zeta)$ is defined in (1.15).

Theorem 1.3. *Suppose that $N \in \{1,2,3\}$, $m > 0$, $\frac{1}{8} < n < 2$,*

$$\begin{aligned} &\max\left\{1, \tfrac{3n-1}{4}\right\} < \lambda < \tfrac{5N+8}{4N} + \min\left\{n, \tfrac{5}{4}\right\} \ \text{if } N < 3, \\ &\max\left\{1, \tfrac{3n-1}{4}\right\} < \lambda < 2 + \min\left\{n, \tfrac{5}{4}\right\} \ \text{if } N = 3, \end{aligned} \tag{1.23}$$

and $u_0(x) \in H^1(\mathbb{R}^N) \cap L^{m-n+2}(\mathbb{R}^N)$ is a nonnegative function with $\operatorname{supp} u_0 \subset B(0,R_0), R_0 < +\infty$. *Then there exists a global solution $u(t,x)$ of problem (C) with the following properties:*

(i) $\operatorname{supp} u(t,.)$ *is compact for any $t > 0$ and there exists an increasing function $\Gamma(t) \in C[0,+\infty)$, $\Gamma(0) = 0$, such that* $\operatorname{supp} u(t,.) \subset B(0, R_0 + c(u_0)\Gamma(t))\ \forall\, t > 0$;

(ii) for any $\alpha \in \tilde{\Delta}_{n,\lambda} \cap (0,+\infty)$ satisfying the conditions

$$\begin{aligned} &\max\left\{\tfrac{3(\alpha+n)-1}{4}, 1\right\} < \lambda \leqslant \tfrac{N+2}{N} + \tfrac{3(\alpha+n)}{4} \ \text{for } N < 3, \\ &\max\left\{\tfrac{3(\alpha+n)-1}{4}, 1\right\} < \lambda \leqslant \tfrac{3(\alpha+n)+7}{4} \ \text{for } N = 3 \end{aligned} \tag{1.24}$$

(it is easy to verify that, under the assumptions of the theorem, the set of these α is nonempty), the following inclusions are true:

$$\begin{aligned} &u^{m-n+2} \in L^{\infty}_{loc}([0,\infty); L^1(\mathbb{R}^N)), u^{\frac{\alpha+n+1}{4}} \in L^4_{loc}([0,\infty); W^1_4(\mathbb{R}^N)), \\ &u^{\frac{\alpha+n+1}{2}} \in L^2_{loc}([0,\infty); H^2(\mathbb{R}^N)), u^{\frac{\alpha+m+1}{2}} \in L^2_{loc}([0,\infty); H^1(\mathbb{R}^N)); \end{aligned}$$

(iii) for almost all $0 \leqslant t_1 < t_2$, inequality (1.19) with $\Omega = \mathbb{R}^N$ and an arbitrary nonnegative function $\zeta \in C^2([t_1,t_2] \times \mathbb{R}^N)$ is true;

(iv) $u(t,.) \underset{t \to 0}{\rightrightarrows} u_0(.)$ in $L^2(\mathbb{R}^N)$.

Remark 1.3. In the case $N = 1$, following the procedure described in [3, 5], one can construct a generalized solution $u(t,x) \in C([0,\infty)\times\mathbb{R}^1)\cap L^\infty(\mathbb{R}^+;H^1(\mathbb{R}^1))$ of problem (C) with compact $\operatorname{supp} u(t,.)$ $\forall\, t \geqslant 0$ for $m > 0$, $0 < n < 2$, and $\max\{1, \frac{3n-1}{4}\} < \lambda < \frac{9}{2}$ such that $u \in C^{4,1}(\mathrm{P})$, $u^{\frac{n}{2}}(u_{xxx} - u^{m-n}u_x) \in L^2(\mathrm{P})$, $\mathcal{B}(u) := \int\limits_0^u b'(\tau)\,d\tau \in L^1(\mathbb{R}^+;L^1(\mathbb{R}^1))$, and

$$\int\limits_0^\infty\int\limits_{\mathbb{R}^1} u\zeta_t + \iint\limits_{\mathrm{P}} u^n(u_{xxx} - u^{m-n}u_x)\zeta_x - \int\limits_0^\infty\int\limits_{\mathbb{R}^1}\mathcal{B}(u)\zeta_x = 0$$

where $\mathrm{P} = \mathrm{P}(u)$ is defined by (1.10) and $\zeta \in C_c^1((0,\infty)\times\mathbb{R}^1)$ is an arbitrary cut-off function. Furthermore, using the approach related to periodic approximations that was proposed in [17], one can construct a generalized solution of problem (C) for $m > -1$, $n \in (0,2)$, and $\lambda \in (\max\{\frac{3n}{4} - 1, \frac{1}{8}\}, \frac{9}{2})$ without the assumption that the initial function is finite.

2. Proof of Theorem 1.1. A Priori Estimates

2.1. Regularized Neumann Problem

For arbitrary $\delta > 0$, $\sigma > 0$, $\varepsilon > 0$, and $\gamma > 0$, we introduce the functions

$$\begin{aligned}
&m_\gamma(z) := m_{\delta\sigma}(z) + \gamma := \tfrac{|z|^{n+s}}{|z|^s+\delta|z|^n+\sigma|z|^{n+s}} + \gamma > 0,\\
&b'_{\delta\sigma}(z) := \tfrac{b'(z)}{1+\delta(|z|^{\lambda-1}+|z|^{\lambda-s-1})+\sigma(|z|^{\lambda-1}+|z|^{\lambda+\beta-1})},\\
&\Psi''_\varepsilon(z) := \tfrac{|z|^{m-n}}{1+\varepsilon|z|^{m-n}},\ u_{0\delta\sigma}(x) := u_0(x) + \delta^{\theta_1} + \sigma^{\theta_2}
\end{aligned}$$

where

$$0 < \theta_1 < \tfrac{2}{2s-3},\ \theta_2 > 0,\ \beta > \max\{2, m-n+1\}, \tag{2.1}$$

$$s > \max\{n,4\} \text{ if } N < 3 \quad\text{and}\quad s > \max\{n,8\} \text{ if } N = 3. \tag{2.2}$$

Consider the four-parameter family of boundary-value Neumann problems

$$(N_{\gamma\varepsilon\delta\sigma})\quad \begin{cases} u_t + \operatorname{div}\left\{m_\gamma(u)\left[\nabla\Delta u - \Psi''_\varepsilon(u)\nabla u\right]\right\} = \overrightarrow{\chi}\cdot b'_{\delta\sigma}(u)\nabla u \text{ in } Q_T,\\ \nabla u\cdot\vec{n} = \nabla\Delta u\cdot\vec{n} = 0 \text{ on } (0,T)\times\partial\Omega,\\ u(0,x) = u_{0\delta\sigma}(x) \text{ in } \Omega.\end{cases}$$

By analogy with [16, 18], for arbitrary values of the parameters $\gamma > 0$, $\varepsilon > 0$, $\delta > 0$, and $\sigma > 0$, using the Faedo–Galerkin method we construct a global regular solution $u_{\gamma\varepsilon\delta\sigma}$ of the regularized problem $(N_{\gamma\varepsilon\delta\sigma})$, which naturally does not possess the property of nonnegativity. Using the method proposed in [16, 18], we can also pass to the limit as $\gamma \to 0$ and obtain a global solution $u_{\varepsilon\delta\sigma}$ of problem $(N_{\varepsilon\delta\sigma})$ (i.e., problem $(N_{\gamma\varepsilon\delta\sigma})$ with $\gamma = 0$), which, by virtue of conditions (2.1) and (2.2) for the regularization parameters s and β, is strictly positive for almost all $t \geqslant 0$. Passing to the limit as $\varepsilon \to 0$ in Theorem A.1 (see Appendix A), we construct a global nonnegative solution $u_{\delta\sigma}(t,x)$ of problem $(N_{\delta\sigma})$ (a special case of problem $(N_{\varepsilon\delta\sigma})$ with $\varepsilon = 0$). This theorem is proved by using the procedure proposed in [14] (in the present paper, the proof is omitted). Now, the main part of the proof of Theorem 1.1 consists in the passage to the limit as $\delta \to 0$ and $\sigma \to 0$. By this passage, we obtain new a priori estimates for the solutions $u_{\delta\sigma}$ independent of $\delta > 0$ and $\sigma > 0$.

2.2. Main Local a Priori Estimate of $u_{\delta\sigma}$ Uniform in δ and σ

First, consider the following two simple inequalities:

$$\iint\limits_{Q_T} u^{\nu} \leqslant \hat{\gamma} B_0 + c \int\limits_0^T \Bigl(\int\limits_{\Omega} u^6 \Bigr)^{\frac{\nu+3(\alpha+n+1)}{3(\alpha+n-\nu+9)}} + c_1 \int\limits_0^T \Bigl(\int\limits_{\Omega} u^6 \Bigr)^{\frac{\nu}{6}} \leqslant \hat{\gamma} B_0$$
$$+ c R_1(h_1) + c_1 R_1(\tfrac{\nu}{2}) \quad \forall\, \hat{\gamma} > 0,\ B_0 := \iint\limits_{Q_T} \bigl| \nabla u^{\frac{\alpha+n+1}{4}} \bigr|^4 \qquad (2.3)$$

where $R_1(s) := \int\limits_0^T \|u\|^{2s}_{H^1(\Omega)} dt,\ h_1 = \frac{\nu+3(\alpha+n+1)}{\alpha+n-\nu+9}$, $6 < \nu < \alpha + n + 9$, and $N = 3$, and

$$\iint\limits_{Q_T} u^{\nu} \leqslant \hat{\gamma} B_0 + c R_2(h_2) + c_1 R_2(\tfrac{\nu}{m-n+2}) \quad \forall\, \hat{\gamma} > 0,\ B_0 \text{ is defined in (2.3)} \qquad (2.4)$$

where $R_2(s) := \int\limits_0^T \|u\|^{s(m-n+2)}_{L^{m-n+2}(\Omega)} dt$, $h_2 = \frac{\nu+3(\alpha+n+1)}{3(\alpha+n+1-\nu)+4(m-n+2)}$, $1 < m - n + 2 < \nu < \alpha + n + 1 + \frac{4}{3}(m-n+2)$, and $N = 3$. To obtain these inequalities, we apply the Nirenberg–Gagliardo interpolation inequality (Lemma B.5) to the function $v = u^{\frac{\alpha+n+1}{4}}$ for $a = \frac{4\nu}{\alpha+n+1}$, $d = 4$, $i = 0$, $j = 1$, and $b = \frac{24}{\alpha+n+1}$ in the case of (2.3) and for $a = \frac{4\nu}{\alpha+n+1}$, $d = 4$, $i = 0$, $j = 1$ and $b = \frac{4(m-n+2)}{\alpha+n+1}$ in the case of (2.4), integrate the results obtained with respect to time, and use the Young inequality.

Now consider inequalities (A.2) and (A.4) from Theorem A.1. We successively estimate the terms on the right-hand sides of these inequalities (note that all further calculations are carried out for $N = 3$; in the case

$N < 3$, the required estimates for the right-hand sides follow directly from the imbeddings $H^1(\Omega) \subset L^\infty(\Omega)$ for $N = 1$ and $H^1(\Omega) \subset L^q(\Omega)\ \forall q < \infty$ for $N = 2$). Using the Cauchy inequality, we get

$$B_1 := \iint\limits_{Q_T} m_{\delta\sigma}(u_{\delta\sigma})\big|\nabla u_{\delta\sigma}\big|^2 \leqslant c \iint\limits_{Q_T} u_{\delta\sigma}^{\frac{n-\alpha+3}{2}} |\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}|^2$$
$$\leqslant \gamma B_0 + cB_1^{(1)} \quad \forall \gamma > 0,\ B_1^{(1)} := \iint\limits_{Q_T} u_{\delta\sigma}^{n-\alpha+3},\ B_0 \text{ from (2.3).} \qquad (2.5)$$

For $0 < n - \alpha + 3 \leqslant 6$, the term $B_1^{(1)}$ is estimated with the use of the imbedding $H^1(\Omega) \subset L^q(\Omega)\ \forall q \leqslant 6$: $B_1^{(1)} \leqslant C(\Omega)R_1(\frac{n-\alpha+3}{2})$. For $0 < n-\alpha+3 \leqslant m-n+2$, $m-n+1 > 0$, we estimate this term using the imbedding $L^{m-n+2}(\Omega) \subset L^q(\Omega)\ \forall\, q \leqslant m-n+2$: $B_1^{(1)} \leqslant C(\Omega)R_2(\frac{n-\alpha+3}{m-n+2})$. For $6 < n-\alpha+3 < \alpha+n+9$ ($m-n+2 < n-\alpha+3 < \alpha+n+1+\frac{4}{3}(m-n+2)$, $m-n+1 > 0$), to estimate $B_1^{(1)}$ we use inequality (2.3) ((2.4)): $B_1^{(1)} \leqslant \tilde\gamma B_0 + c(\tilde\gamma)\{R_1(s_1) + R_1(s_2)\}$ ($\leqslant \tilde\gamma B_0 + c(\tilde\gamma)\{R_2(s_3) + R_2(s_4)\}$), where the quantities $R_i(s)$ are defined in (2.3) and (2.4), $s_1 = \frac{2(2n+\alpha+3)}{\alpha+3}$, $s_2 = \frac{n-\alpha+3}{2}$, $s_3 = \frac{n-\alpha+3}{m-n+2}$, and $s_4 = \frac{2n+\alpha+3}{3(\alpha-1)+2(m-n+2)}$. Substituting the estimates obtained into inequality (2.5), we get

$$B_1 \leqslant \gamma_1 B_0 + C(\Omega)\Bigg(\sum_{i=1,2} R_1(s_i) + \sum_{i=3,4} R_2(s_i)\Bigg)\ \forall \gamma_1 > 0 \qquad (2.6)$$

if the parameter α satisfies one of the relations

$$-3 < \alpha < n+3, \qquad (2.7)$$

$$-\tfrac{2}{3}(m - n + \tfrac{1}{2}) < \alpha < n+3,\ m-n+1 > 0. \qquad (2.8)$$

Moreover, the sum $\sum\limits_{i=1,2} R_1(s_i)$ is present only in case (2.7) (furthermore, $R_1(s_1)$ is absent if $n-3 \leqslant \alpha < n+3$), and the sum $\sum\limits_{i=3,4} R_2(s_i)$ is present only in case (2.8) (furthermore, $R_2(s_4)$ is absent if $2n-m+1 \leqslant \alpha < n+3$).

Remark 2.1. In what follows, estimating the terms B_i on the right-hand sides of relations (A.2) and (A.4), we omit simple calculations similar to those presented between estimates (2.5) and (2.6).

Using inequalities (2.3) and (2.4) again, we get

$$|B_2| := \left| \iint\limits_{Q_T} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) u_{\delta\sigma} \nabla u_{\delta\sigma} \right| \leqslant c \iint\limits_{Q_T} u_{\delta\sigma}^{\frac{4\lambda-\alpha-n+3}{4}} \left| \nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}} \right|$$

$$\leqslant \gamma_2 B_0 + C(\Omega) \Bigl(\sum_{i=5,6} R_1(s_i) + \sum_{i=7,8} R_2(s_i) \Bigr) \quad \forall\, \gamma_2 > 0 \quad (2.9)$$

if the parameters α and λ satisfy one of the relations

$$\left(\tfrac{\alpha+n-3}{4}\right)_+ < \lambda < \alpha + n + 6, \qquad (2.10)$$

$$\left(\tfrac{\alpha+n-3}{4}\right)_+ < \lambda < \alpha + m + 2,\ m - n + 1 > 0. \qquad (2.11)$$

Here, B_0 and $R_i(s)$ are defined in (2.3) and (2.4), $s_5 = \frac{\lambda+2(\alpha+n)+3}{\alpha+n-\lambda+6}$, $s_6 = \frac{4\lambda-\alpha-n+3}{6}$, $s_7 = \frac{4\lambda-\alpha-n+3}{m-n+2}$, and $s_8 = \frac{\lambda+2(\alpha+n)+3}{3(\alpha+m-\lambda+2)}$. The sum $\sum\limits_{i=5,6} R_1(s_i)$ is present in relation (2.9) only in case (2.10), and the sum $\sum\limits_{i=7,8} R_2(s_i)$ is present only in case (2.11).

Taking into account relations (1.6) and (1.14) and Remark 2.1, in the case $m - n + 2 \neq 1$ we obtain

$$|B_3| := \left| \iint\limits_{Q_T} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) \Psi'_0(u_{\delta\sigma}) \nabla u_{\delta\sigma} \right| \leqslant R^{\frac{4(m-n+1)}{3}} C(\Omega) \sum_{i=13,14} R_1(s_i)$$

$$+ \gamma_3 B_0 + C(\Omega) \Bigl(\sum_{i=9,10} R_1(s_i) + \sum_{i=11,12} R_2(s_i) \Bigr) \quad \forall\, \gamma_3 > 0, \quad (2.12)$$

if one of the following conditions is satisfied:

$$n - m + \tfrac{n+\alpha-3}{4} < \lambda < 2n - m + \alpha + 6, \qquad (2.13)$$

$$n - m + \tfrac{n+\alpha-3}{4} < \lambda < \alpha + n + 2,\ m - n + 1 > 0, \qquad (2.14)$$

$$\tfrac{\alpha+n+1}{4} < \lambda < \alpha + n + 7,\ m - n + 1 < 0. \qquad (2.15)$$

Here, B_0 and $R_i(s)$ are defined in (2.3) and (2.4), $s_9 = \frac{n+m+\lambda+2\alpha+3}{2n-m-\lambda+\alpha+6}$, $s_{10} = \frac{4(\lambda+m)-5n-\alpha+3}{6}$, $s_{11} = \frac{4(\lambda+m)-5n-\alpha+3}{m-n+2}$, $s_{12} = \frac{n+m+\lambda+2\alpha+3}{3(\alpha+n+2-\lambda)}$, $s_{13} = \frac{\lambda+2(\alpha+n+1)}{n+\alpha-\lambda+7}$, and $s_{14} = \frac{4\lambda-n-\alpha-1}{6}$. Furthermore, the sum $\sum\limits_{i=9,10} R_1(s_i)$ is present in relation (2.12) only in case (2.13), the sum $\sum\limits_{i=11,12} R_2(s_i)$ is present only in case (2.14), and the sum $\sum\limits_{i=13,14} R_1(s_i)$ is present only in case (2.15). Integrating B_3 by parts and using the imbedding theorem in addition to (2.12) we obtain the estimate

$$|B_3| \leqslant C(\Omega)\left(R_1(s_{15}) + R^{m-n+1}R_1(s_{16})\right), \; m-n+2 \neq 1 \qquad (2.16)$$

(R is defined in (1.14), $s_{15} = \frac{\lambda+m-n+1}{2}$, and $s_{16} = \frac{\lambda}{2}$) which is valid for

$$(n-m-1)_+ < \lambda \leqslant \min\{n-m+3, 4\}. \qquad (2.17)$$

For $m-n+2=1$, using the inequality $|\ln z| \leqslant d(\xi)(z^{-\xi}+z^{\xi})\ \forall\, \xi>0$ and reasoning as in the derivation of inequalities (2.12) and (2.16), we get

$$|B_3| \leqslant \gamma_3 B_0 + C(\Omega)\sum_{i=17}^{20} R_1(s_i) \quad \forall\, \gamma_3>0,\ B_0 \text{ is defined in (2.3)},$$

$$|B_3| \leqslant C(\Omega)\Bigg(\sum_{i=21,22} R_1(s_i) + R_1(s_{16})\Bigg)$$

for $\xi + \frac{\alpha+n+1}{4} < \lambda < \alpha+n+7-\xi$ and $\xi \leqslant \lambda < 4-\xi$, respectively. Here, the quantities $R_i(s)$ are defined in (2.3) and (2.4), $s_{17} = \frac{\lambda+\xi+2(\alpha+n+1)}{n+\alpha-\lambda-\xi+7}$, $s_{18} = \frac{4\lambda-n-\alpha-1+4\xi}{6}$, $s_{19} = \frac{\lambda-\xi+2(\alpha+n+1)}{n+\alpha-\lambda+\xi+7}$, $s_{20} = \frac{4\lambda-n-\alpha-1-4\xi}{6}$, $s_{21} = \frac{\lambda+\xi}{2}$, and $s_{22} = \frac{\lambda-\xi}{2}$.

Remark 2.2. Combining conditions (2.13)–(2.15), and (2.17) and taking into account that, for $m-n+2=1$, the parameter ξ in the condition for λ can be taken arbitrarily small, we conclude that, for any $\lambda \in ((n-m-1)_+, \alpha+n+2+(n-m+4)_+)$, at least one of the estimates for $|B_3|$ presented above is true.

After elementary calculations, we obtain

$$\left|b'_{\delta\sigma}(z) g^{(\alpha)}_{\delta\sigma}(z)\right| \leqslant \frac{c\,z^{\lambda-1}}{1+\delta(z^{\lambda-1}+z^{\lambda-1-s})+\sigma(z^{\lambda-1}+z^{\lambda-1+\beta})}\Big|\frac{1}{\alpha}z^{\alpha} + \frac{\delta}{\alpha+n-s}z^{\alpha+n-s} +$$

$$+\frac{\sigma}{\alpha+n}z^{\alpha+n}\Big| \leqslant \begin{cases} \frac{c\lambda}{|\alpha|}z^{\alpha+\lambda-1} + \frac{c\lambda}{s-\alpha-n}z^{\alpha+n} + \frac{c\lambda}{\alpha+n}z^{\alpha+n} & \forall z \leqslant 1, \\ \frac{c\lambda}{|\alpha|}z^{\alpha+\lambda-1} + \frac{c\lambda}{s-\alpha-n}z^{\alpha+n-s} + \frac{c\lambda}{\alpha+n}z^{\alpha+n-\beta} & \forall z > 1. \end{cases}$$

Therefore, $\left|b'_{\delta\sigma}(z) g^{(\alpha)}_{\delta\sigma}(z)\right| \leqslant \frac{2\lambda c}{|\alpha|}z^{\alpha+\lambda-1} + \frac{2\lambda s c}{(s-\alpha-n)(\alpha+n)}\ \forall\, z>0$ if $s>\alpha+n$ and $\beta \geqslant \alpha+n$. By virtue of conditions (2.1) and (2.2) and Remark 2.1, we get

$$|B_4| := \Bigl| \iint\limits_{Q_T} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) g^{(\alpha)}_{\delta\sigma}(u_{\delta\sigma}) \nabla u_{\delta\sigma} \Bigr|$$
$$\leqslant c \iint\limits_{Q_T} u_{\delta\sigma}^{\frac{4\lambda+3\alpha-n-1}{4}} \bigl|\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}\bigr| + c \iint\limits_{Q_T} |\nabla u_{\delta\sigma}| \leqslant C(\Omega) R_1(\tfrac{1}{2}) + \gamma_4 B_0$$
$$+ C(\Omega)\Bigl(\sum_{i=23,24} R_1(s_i) + \sum_{i=25,26} R_2(s_i) \Bigr) \quad \forall\, \gamma_4 > 0; \quad (2.18)$$

furthermore, the sum $\sum\limits_{i=23,24} R_1(s_i)$ is present only if $\frac{n-3\alpha+1}{4} < \lambda < n+7$, and the sum $\sum\limits_{i=25,26} R_2(s_i)$ is present only if $\frac{n-3\alpha+1}{4} < \lambda < m+3$ and $m-n+1>0$. Integrating B_4 by parts and using relation (1.6) and the imbedding $H^1(\Omega) \subset L^4(\partial\Omega)$, in addition to (2.18) we get

$$|B_4| \leqslant c \int\limits_0^T \int\limits_{\partial\Omega} \{u_{\delta\sigma}^{\lambda+\alpha} + u_{\delta\sigma}\} \leqslant C(\Omega)\bigl(R_1(s_{27}) + R_1(\tfrac{1}{2})\bigr)$$

for $0 < \lambda + \alpha < 4$. Here, B_0 and $R_i(s)$ are defined in (2.3) and (2.4), $s_{23} = \frac{2n+\lambda+3\alpha+2}{n-\lambda+7}$, $s_{24} = \frac{4\lambda+3\alpha-n-1}{6}$, $s_{25} = \frac{4\lambda+3\alpha-n-1}{m-n+2}$, $s_{26} = \frac{\lambda+2n+3\alpha+2}{3(m-\lambda+3)}$, and $s_{27} = \frac{\lambda+\alpha}{2}$.

Remark 2.3. At least one of the estimates for $|B_4|$ presented above is true for any $\lambda \in (\min\{-\alpha, \frac{n-3\alpha+1}{4}\}, \max\{4-\alpha, n+7+(m-n-4)_+\})$.

Using the identity $u_{x_ix_j} = \gamma^{-1} u^{1-\gamma}(u^\gamma)_{x_ix_j} - (\gamma-1)u^{-1}u_{x_i}u_{x_j}$, where $\gamma = \frac{\alpha+n+1}{2}$, we rewrite $B_5 := \iint\limits_{Q_T} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) \Delta u_{\delta\sigma} \nabla u_{\delta\sigma}$ in an equivalent form and then estimate it as follows:

$$|B_5| \leqslant c \iint\limits_{Q_T} u_{\delta\sigma}^{\Lambda} \bigl|\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}\bigr|^3 + c \iint\limits_{Q_T} u_{\delta\sigma}^{\Lambda} \bigl|\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}\bigr| \bigl|\Delta u_{\delta\sigma}^{\frac{\alpha+n+1}{2}}\bigr|,$$

where $\Lambda := \frac{4\lambda-3(\alpha+n)+1}{4}$. Applying the Young inequality with exponents 4 and $\frac{4}{3}$ to the first term and with exponents 4, 4, and 2 to the second term, we get

$$|B_5| \leqslant \gamma_5 \iint\limits_{Q_T} \bigl|D^2 u_{\delta\sigma}^{\frac{\alpha+n+1}{2}}\bigr|^2 + \gamma_5 B_0$$
$$+ C(\Omega)\Bigl(\sum_{i=28,29} R_1(s_i) + \sum_{i=30,31} R_2(s_i) \Bigr) \quad \forall\, \gamma_5 > 0; \quad (2.19)$$

furthermore, the sum $\sum\limits_{i=28,29} R_1(s_i)$ is present only if $\frac{3(\alpha+n)-1}{4} < \lambda < \alpha+n+2$, and the sum $\sum\limits_{i=30,31} R_2(s_i)$ is present only if $\frac{3(\alpha+n)-1}{4} < \lambda < \alpha+n+\frac{1}{3}(m-n+2)$ and $m-n+1>0$. Here, B_0 and $R_i(s)$ are defined in (2.3) and (2.4), $s_{28} = \frac{\lambda+1}{n-\lambda+\alpha+2}$, $s_{29} = \frac{4\lambda-3(\alpha+n)+1}{2}$, $s_{30} = \frac{4\lambda-3(\alpha+n)+1}{m-n+2}$, and $s_{31} = \frac{2(\lambda+1)}{3(n-\lambda+\alpha)+m-n+2}$.

Remark 2.4. Combining the restrictions on λ obtained in the derivation of estimates for $|B_5|$, we conclude that relation (2.19) holds for $\lambda \in \big(\frac{3(\alpha+n)-1}{4}, \alpha+n+2+(\frac{m-n-4}{3})_+\big)$.

Summing up inequalities (A.2) and (A.4), taking into account estimates (2.6), (2.9), (2.12), (2.16), (2.18), and (2.19), and choosing sufficiently small γ_i, $i=\overline{1,5}$, we get

$$\begin{gathered}\mathcal{E}_\Omega(u_{\delta\sigma}) + \int\limits_\Omega G^{(\alpha)}_{\delta\sigma}(u_{\delta\sigma})dx + c_1^{-1}\iint\limits_{Q_T}\Big\{\big|D^2 u_{\delta\sigma}^{\frac{\alpha+n+1}{2}}\big|^2 + \big|\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}\big|^4\Big\} \\ + c_1^{-1}\iint\limits_{Q_T} u_{\delta\sigma}^{\alpha+m-1}|\nabla u_{\delta\sigma}|^2 \leqslant \mathcal{E}_\Omega(u_{0\delta\sigma}) + \int\limits_\Omega G^{(\alpha)}_{\delta\sigma}(u_{0\delta\sigma}(x))dx \\ + C(\Omega)\Big(R_1(\tfrac{1}{2}) + \sum_{i=1}^{31} R_1(s_i) + \sum_{i=1}^{31} R_2(s_i)\Big) + C_2(\Omega)R_1(s_{32}) \leqslant \mathcal{E}_\Omega(u_{0\delta\sigma}) \\ + \int\limits_\Omega G^{(\alpha)}_{\delta\sigma}(u_{0\delta\sigma}(x))dx + C_0(\Omega)\int\limits_0^T \big\{\mathcal{E}_\Omega^{s_{\min}}(u_{\delta\sigma}(\tau)) + \mathcal{E}_\Omega^{s_{\max}}(u_{\delta\sigma}(\tau))\big\}\,d\tau\end{gathered} \tag{2.20}$$

where $\mathcal{E}_\Omega(u_{\delta\sigma})$ is defined in assertion (h) of Theorem A.1, $G^{(\alpha)}_{\delta\sigma}(z)$ is defined in assertion (j) of Theorem A.1, $s_{32} = \frac{\alpha+n+1}{2}$, $C_2(\Omega)=0$ if Ω is convex, $s_{\min} = \min\{s_i, 2^{-1}\}$, and $s_{\max} = \max\{s_i, 2^{-1}\}$ $(i=\overline{1,32})$. Combining Remarks 2.2–2.4 and conditions (2.7), (2.8), (2.10), and (2.11), we obtain the following final restrictions on λ and α sufficient for the validity of (2.20):

$$\left.\begin{array}{l}\max\Big\{\frac{3(\alpha+n)-1}{4}, (n-m-1)_+\Big\} < \lambda (< \alpha+n+2 \text{ if } N=3), \\ \alpha \in \Delta_\lambda := (\max\{\frac{1}{2}-n, -\lambda\}, 2-n)\backslash\{0,-1\} \text{ for } N<3, \\ \alpha \in \Delta_\lambda \cap (\min\{-3, -\frac{2}{3}(m-n+\frac{1}{2})\}, +\infty) \text{ for } N=3.\end{array}\right\} \tag{2.21}$$

Taking into account the definition of the function $G^{(\alpha)}_{\delta\sigma}(z)$ (see assertion (j) in Theorem A.1) and condition (2.1), by analogy with [13] we get

$$\mathcal{E}_\Omega(u_{0\delta\sigma}(x)) + \int\limits_\Omega G^{(\alpha)}_{\delta\sigma}(u_{0\delta\sigma}(x)) \leqslant \tfrac{1}{2}\left\|u_{0\delta\sigma}\right\|^2_{H^1(\Omega)}$$
$$+ \int\limits_\Omega \Psi_0(u_{0\delta\sigma}) + \left|\tfrac{1}{\alpha(\alpha+1)}\right| \int\limits_\Omega u^{\alpha+1}_{0\delta\sigma} + \overline{\overline{o}}_{\delta\sigma}(1) \leqslant c_1(u_0) \quad (2.22)$$

where the constant $c_1(u_0) := c_1(\mathcal{E}_\Omega(u_0), \|u_0\|_{L^{\alpha+1}(\Omega)})$ is independent of δ and σ, and $\Psi_0(z)$ is defined by relation (1.14). Applying Lemma B.9 to inequality (2.20) and taking inequality (2.22) into account, we obtain the following local a priori estimate for almost all $T \leqslant \hat{T} := \frac{T_0}{2}$:

$$\mathcal{E}_\Omega(u_{\delta\sigma}(T)) + \int\limits_\Omega G^{(\alpha)}_{\delta\sigma}(u_{\delta\sigma}(T))\,dx$$
$$+ c_1^{-1}\iint\limits_{Q_T}\Big\{\big|D^2 u_{\delta\sigma}^{\frac{\alpha+n+1}{2}}\big|^2 + \big|\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}\big|^4 + u_{\delta\sigma}^{\alpha+m-1}|\nabla u_{\delta\sigma}|^2\Big\} \leqslant C_4 \quad (2.23)$$

where $C_4 = 2^{\frac{1}{s_{\max}-1}}(c_1(u_0) + C_0(\Omega))$, $s_{\max} > 1$, and

$$T_0 := \tfrac{1}{C_0(\Omega)(s_{\max}-1)}(c_1(u_0) + C_0(\Omega))^{1-s_{\max}}. \quad (2.24)$$

Here, $C_0(\Omega)$ is defined by (2.20), and $c_1(u_0)$ is defined by (2.22). Estimate (2.23) is the main a priori estimate obtained in this section. It follows from (2.23) that uniformly in $\delta > 0$ and $\sigma > 0$:

$$\{u_{\delta\sigma}\}_{\delta,\sigma>0} \text{ are bounded in } \ L^\infty(0,\hat{T}; H^1(\Omega)), \quad (2.25)$$

$$\{\Psi_0(u_{\delta\sigma})\}_{\delta,\sigma>0} \text{ are bounded in } \ L^\infty(0,\hat{T}; L^1(\Omega)), \quad (2.26)$$

$$\{u_{\delta\sigma}^{\frac{\alpha+m-1}{2}}|\nabla u_{\delta\sigma}|\}_{\delta,\sigma>0} \text{ are bounded in } \ L^2(Q_{\hat{T}}), \quad (2.27)$$

$$\int\limits_\Omega G^{(\alpha)}_{\delta\sigma}(u_{\delta\sigma}(t))dx \text{ is bounded } \ \forall\, t \in [0,\hat{T}], \quad (2.28)$$

$$\{u_{\delta\sigma}^{\frac{\alpha+n+1}{2}}\}_{\delta,\sigma>0} \text{ are bounded in } \ L^2(0,\hat{T}; H^2(\Omega)), \quad (2.29)$$

$$\{u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}\}_{\delta,\sigma>0} \text{ are bounded in } L^4(0,\hat{T}; W^1_4(\Omega)). \quad (2.30)$$

By virtue of (2.25), the following sequence is uniformly bounded:

$$\{u_{\delta\sigma}\}_{\delta,\sigma>0} \text{ in } L^\infty(0,\hat{T}; L^q(\Omega))\ \forall\, q < \infty,\ N < 3 \text{ and } \forall\, q < 6,\ N = 3. \quad (2.31)$$

Remark 2.5. Reasoning as in the derivation of inequalities (2.3) and (2.4) and using estimate (2.23), we establish the uniform boundedness of the sequence

$$\{u_{\delta\sigma}\}_{\delta,\sigma>0} \text{ in } L^{\alpha+n+9}(Q_{\hat{T}}) \text{ and in } L^r(Q_{\hat{T}}) \text{ if } m-n+1>0, \qquad (2.32)$$

for $N=3$ where $r=\alpha+n+1+\frac{4}{3}(m-n+2)$.

2.3. Additional a Priori Estimates Uniform in δ and σ

If $N<3$ or $N=3$ and $\alpha+m\leqslant 11\,(\alpha+m>11)$, then relation (2.31) ((2.26)) yields

$$\int_\Omega u_{\delta\sigma}^{\frac{\alpha+m+1}{2}}(t,x)dx \leqslant C \text{ for almost all } t\leqslant \hat{T} \text{ if } \alpha+m+1>0. \qquad (2.33)$$

In view of (2.27), this implies that the sequence

$$\{u_{\delta\sigma}^{\frac{\alpha+m+1}{2}}\}_{\delta,\sigma>0} \text{ is bounded in } L^2(0,\hat{T};H^1(\Omega)),\ \alpha+m+1>0. \qquad (2.34)$$

By virtue of the interpolation Lemma B.3, relations (2.34) and (2.31) yield

$$\iint_{Q_{\hat{T}}} u_{\delta\sigma}^{\alpha+m+5} \leqslant C \text{ if } \alpha+m+1>0 \text{ and } N=3. \qquad (2.35)$$

Following [14], one can show that, for $N<3$ or $N=3,\ 0<m<3$, or $N=3,\ m\geqslant 3,\ \alpha>-2$, there exists $\eta>0$ such that

$$\{|\nabla H_{\delta\sigma}(u_{\delta\sigma})|\}_{\delta,\sigma>0} \text{ is bounded in } L^{1+\eta}(Q_{\hat{T}}) \qquad (2.36)$$

where $H_{\delta\sigma}(z)$ is defined in assertion (c) of Theorem A.1.

In the remaining part of this section, we establish optimal conditions for the parameter λ that guarantee the uniform boundedness (with respect to δ and σ) of the sequence

$$\{\partial_t u_{\delta\sigma}\}_{\delta,\sigma>0} \text{ in the space } L^2(0,\hat{T};(W_q^1(\Omega))^*),\ q=\tfrac{q'}{q'-1} \qquad (2.37)$$

where q' is an arbitrary number from the interval Δ_1 from Definition 1.1. By virtue of assertion (b) of Theorem A.1, for arbitrary function $\varphi\in L^2(0,\hat{T};H^1(\Omega))$ we obtain

$$\int_0^{\hat{T}} \langle \partial_t u_{\delta\sigma},\varphi\rangle = \iint_{Q_{\hat{T}}} \Big(\overrightarrow{J}_{\delta\sigma}\cdot\nabla\varphi + \overrightarrow{\chi}\cdot\nabla\mathcal{B}_{\delta\sigma}(u_{\delta\sigma})\varphi\Big) =: \mathrm{I}+\mathrm{II} \qquad (2.38)$$

where $\mathcal{B}_{\delta\sigma}(z) := \int\limits_0^z b'_{\delta\sigma}(\tau)\,d\tau$. By virtue of estimates (2.6), (2.9), (2.12), (2.16), and (2.19) and a priori estimate (2.23), relation (A.1) yields

$$\int\limits_0^{\hat{T}} \|\overrightarrow{J}_{\delta\sigma}\|^2_{L^{q'}(\Omega)} \leqslant C \sup_{t\in(0,\hat{T})} \left\|u^n_{\delta\sigma}(t,.)\right\|_{L^{\frac{q'}{2-q'}}(\Omega)} \leqslant C_1. \tag{2.39}$$

The last inequality follows from property (2.25) and the imbedding $H^1(\Omega) \subset L^{\frac{nq'}{2-q'}}(\Omega)$, which is true because $\frac{nq'}{2-q'} < \frac{2N}{N-2} \Leftrightarrow q' < \frac{4N}{2N+n(N-2)}$. Relation (2.39) obviously yields

$$|\mathrm{I}| \leqslant C_2\, \|\varphi\|_{L^2(0,\hat{T};W^1_q(\Omega))}, \quad q = \tfrac{q'}{q'-1}. \tag{2.40}$$

Let us find an analogous estimate for the term II in (2.38), which can be represented in the form

$$\mathrm{II} = \int\limits_0^{\hat{T}}\int\limits_{\partial\Omega} \overrightarrow{\chi}\cdot\vec{n}\,\mathcal{B}_{\delta\sigma}(u_{\delta\sigma})\varphi - \iint\limits_{Q_{\hat{T}}} \overrightarrow{\chi}\cdot\mathcal{B}_{\delta\sigma}(u_{\delta\sigma})\nabla\varphi =: \mathrm{II}_1 - \mathrm{II}_2. \tag{2.41}$$

By virtue of (1.6), we have

$$|\mathrm{II}_2| \leqslant c\|u^\lambda_{\delta\sigma}\|_{L^2(0,\hat{T};L^{q'}(\Omega))}\|\nabla\varphi\|_{L^2(0,\hat{T};L^q(\Omega))}. \tag{2.42}$$

Let us analyze the character of the summability of the solutions $u_{\delta\sigma}$ defined by properties (2.25)–(2.30) for $N=3$. Using relations (2.29) and (2.31) ((2.26)), we establish the uniform boundedness of the family

$$\{v_{\delta\sigma}\} := \{u^\mu_{\delta\sigma}\} \text{ in } L^2(0,\hat{T};W^1_6(\Omega)),\ \mu = \tfrac{\alpha+n+1}{2} \in \left(\tfrac{3}{4},\tfrac{3}{2}\right), \tag{2.43}$$

and

$$\{v_{\delta\sigma}\} \text{ in } L^\infty(0,\hat{T};L^{\frac{6}{\mu}}(\Omega))\big(\text{in } L^\infty(0,\hat{T};L^{\frac{m-n+2}{\mu}}(\Omega)), m-n+1>0\big). \tag{2.44}$$

Let us find conditions for λ that guarantee the following estimate uniform in δ and σ:

$$\|u^\lambda_{\delta\sigma}\|_{L^2(0,\hat{T};L^{q'}(\Omega))} \equiv \int\limits_0^{\hat{T}}\Bigg(\int\limits_\Omega v_{\delta\sigma}^{\frac{\lambda q'}{\mu}}\Bigg)^{\frac{2}{q'}} dt \leqslant C\ \forall\, q' \in \Delta_1. \tag{2.45}$$

By virtue of the Nirenberg–Gagliardo interpolation inequality (Lemma B.5), we have

$$\Bigg(\int\limits_\Omega v^a\Bigg)^{\frac{1}{a}} \leqslant d_1\Bigg(\int\limits_\Omega |\nabla v|^6\Bigg)^{\frac{\theta_i}{6}}\Bigg(\int\limits_\Omega v^{b_i}\Bigg)^{\frac{1-\theta_i}{b_i}} + d_2\Bigg(\int\limits_\Omega v^{b_i}\Bigg)^{\frac{1}{b_i}},\ i=1,2, \tag{2.46}$$

where $a = \frac{\lambda q'}{\mu}$, $b_1 = \frac{6}{\mu}$, $b_2 = \frac{m-n+2}{\mu}$, $\theta_1 = \frac{\mu(\lambda q'-6)}{\lambda q'(1+\mu)}$, $\theta_2 = \frac{6\mu(\lambda q'-m+n-2)}{\lambda q'(6\mu+m-n+2)}$, and $N = 3$. Raising inequality (2.46) to the power $\frac{2\lambda}{\mu}$ and integrating the result with respect to t, we get

$$\int_0^{\hat{T}} \Big(\int_\Omega v^{\frac{\lambda q'}{\mu}} \Big)^{\frac{2}{q'}} \leqslant c \int_0^{\hat{T}} \Big(\int_\Omega |\nabla v|^6 \Big)^{\frac{\lambda \theta_i}{3\mu}} \Big(\int_\Omega v^{b_i} \Big)^{\frac{2\lambda(1-\theta_i)}{\mu b_i}} + c \int_0^{\hat{T}} \Big(\int_\Omega v^{b_i} \Big)^{\frac{2\lambda}{\mu b_i}},$$

$i = 1, 2$. Applying the inequality obtained to the function $v = v_{\delta\sigma}$ and using properties (2.43) and (2.44), we get estimate (2.45), provided that one of the following inequalities is true: $\frac{\lambda \theta_i}{\mu} \leqslant 1 \Leftrightarrow \lambda \leqslant \frac{\mu}{\theta_i}$ $\forall\, q' \in \Delta_1$, $i = 1, 2$. It is easy to verify that these relations are equivalent to the following restrictions imposed on the parameter λ:

$$\lambda \leqslant \mu + \max\left\{1, \tfrac{m-n+2}{6}\right\} \left(4 + \tfrac{n}{2}\right) \text{ for } N = 3 \ (\mu \text{ from (2.43)}). \qquad (2.47)$$

Let us estimate the term II_1 in (2.41). It is obvious that

$$|\mathrm{II}_1| \leqslant c \int_0^{\hat{T}} \|\varphi\|_{L^d(\partial\Omega)} \|u_{\delta\sigma}\|^{\lambda}_{L^{\frac{\lambda d}{d-1}}(\partial\Omega)} dt \quad \forall\, d > 1. \qquad (2.48)$$

For $N < 3$, by virtue of (2.25) and the imbedding $W^1_p(\Omega) \subset L^r(\partial\Omega)\ \forall\, r < \infty, \forall\, p \geqslant 2$, this yields

$$|\mathrm{II}_1| \leqslant C_3 \, \|\varphi\|_{L^2(0,\hat{T}; W^1_q(\Omega))} \quad \forall\, \lambda > 0. \qquad (2.49)$$

For $N = 3$, we have the imbedding $W^1_q(\Omega) \subset L^{d_q}(\partial\Omega)$, where $d_q = \frac{2q}{3-q}$ if $q < 3$ and d_q is an arbitrarily large number if $q \geqslant 3$. To obtain an estimate of the form (2.49), by virtue of (2.48) it suffices to establish the uniform estimate

$$\mathrm{II}_1^{(1)} := \int_0^{\hat{T}} \Big(\int_{\partial\Omega} |v_{\delta\sigma}|^{\frac{\lambda d_q}{\mu(d_q-1)}} d\mathcal{H}^{N-1} \Big)^{\frac{2(d_q-1)}{d_q}} dt \leqslant C, \ \mu \text{ is defined in (2.43).} \qquad (2.50)$$

We apply the interpolation inequality from Lemma B.4 to the function $v_{\delta\sigma}$ with $a = \frac{\lambda d_q}{\mu(d_q-1)}$, $d = 6$, $b = b_i$ from (2.46) ($i = 1, 2$), and $N = 3$, raise the result obtained to the power $\frac{2\lambda}{\mu}$, and integrate it with respect to t. As a result, we get

$$|\mathrm{II}_1^{(1)}| \leqslant c \int_0^{\hat{T}} \Big(\int_\Omega |\nabla v_{\delta\sigma}|^6 \Big)^{\frac{\lambda\theta_i}{3\mu}} \Big(\int_\Omega |v_{\delta\sigma}|^{b_i} \Big)^{\frac{2\lambda(1-\theta_i)}{\mu b_i}} + c \int_0^{\hat{T}} \Big(\int_\Omega |v_{\delta\sigma}|^{b_i} \Big)^{\frac{2\lambda}{\mu b_i}}, i = 1, 2$$

where $\theta_1 = \frac{\mu[\lambda d_q - 4(d_q-1)]}{\lambda d_q(1+\mu)}$ and $\theta_2 = \frac{2\mu[3\lambda d_q - 2(d_q-1)(m-n+2)]}{\lambda d_q[6\mu+m-n+2]}$. By virtue of properties (2.43) and (2.44), this yields estimate (2.50) if $\lambda \min\{\theta_1, \theta_2\} \leqslant \mu$. Simple calculations show that this relation is equivalent to the restriction $\lambda \leqslant \mu + \max\bigl\{1, \frac{m-n+2}{6}\bigr\}\bigl(1 + \frac{4(d_q-1)}{d_q}\bigr)$ $\forall\, q' = \frac{q}{q-1} \in \Delta_1$, which, in turn, by virtue of the definitions of μ and d_q, is equivalent to the relation

$$\lambda \leqslant \tfrac{\alpha+n+1}{2} + \max\left\{1, \tfrac{m-n+2}{6}\right\}\left(4 + \tfrac{\min\{n,2\}}{2}\right) \text{ for } N = 3. \tag{2.51}$$

Therefore, if conditions (2.47) and (2.51) are satisfied, then estimates (2.40), (2.42), (2.45), and (2.49) are valid, and, hence, relation (2.37) is true.

3. Proof of Theorem 1.1. Passages to the Limit

First, we pass to the limit as $\delta \to 0$.

3.1. Compactness of the Sequence $u_{\delta\sigma}$

By virtue of Lemma B.1, it follows from (2.25) and (2.37) that there exist an element u_σ and a subsequence $\delta = \delta_k \to 0$ such that

$$u_{\delta\sigma} \underset{\delta\to 0}{\Rightarrow} u_\sigma \text{ in } L^2(Q_{\hat{T}}) \text{ and almost everywhere in } Q_{\hat{T}}. \tag{3.1}$$

In addition, by virtue of Lemma B.2, we have

$$u_{\delta\sigma} \underset{\delta\to 0}{\Rightarrow} u_\sigma \text{ in } C(0,\hat{T}; L^2(\Omega)). \tag{3.2}$$

By virtue of Lemma B.6, it follows from (2.29), (2.39), and (2.43) that

$$u_{\delta\sigma}^{\frac{\alpha+n+1}{2}} \underset{\delta\to 0}{\Rightarrow} u_\sigma^{\frac{\alpha+n+1}{2}} \text{ in } L^2(0,\hat{T}; H^1(\Omega)). \tag{3.3}$$

Using (2.29) and (3.3), we also establish that

$$u_{\delta\sigma}^{\frac{\alpha+n+1}{2}} \underset{\delta\to 0}{\rightharpoonup} u_\sigma^{\frac{\alpha+n+1}{2}} \text{ in } L^2(0,\hat{T}; H^2(\Omega)). \tag{3.4}$$

By virtue of relations (2.30), (2.34), and (3.1) and Lemma B.7, we have

$$u_{\delta\sigma}^{\frac{\alpha+n+1}{4}} \underset{\delta\to 0}{\rightharpoonup} u_{\delta\sigma}^{\frac{\alpha+n+1}{4}} \text{ in } L^4(0,\hat{T}; W_4^1(\Omega)), \tag{3.5}$$

$$u_{\delta\sigma}^{\frac{\alpha+m+1}{2}} \underset{\delta\to 0}{\rightharpoonup} u_\sigma^{\frac{\alpha+m+1}{2}} \text{ in } L^2(0,\hat{T}; H^1(\Omega)) \text{ if } \alpha+m+1 > 0. \tag{3.6}$$

Using relations (2.25), (2.39), (2.37), and (3.1), we get

$$u_{\delta\sigma} \underset{\delta\to 0}{\overset{*}{\rightharpoonup}} u_\sigma \text{ in } L^\infty(0,\hat{T};H^1(\Omega)), \tag{3.7}$$

$$\overrightarrow{J}_{\delta\sigma} \underset{\delta\to 0}{\rightharpoonup} \overrightarrow{J}_\sigma \text{ in } L^2(0,\hat{T};L^{q'}(\Omega))\ \forall\, q' \in \Delta_1, \Delta_1 \text{ from Def. 1.1}, \tag{3.8}$$

$$\partial_t u_{\delta\sigma} \underset{\delta\to 0}{\rightharpoonup} \partial_t u_\sigma \text{ in } L^2(0,\hat{T};(W^1_q(\Omega))^*), \tag{3.9}$$

where $q = \frac{q'}{q'-1}$ and q' is the same as in (3.8). By virtue of the Vitali theorem (see [13]), relations (3.1) and (3.5) yield

$$\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}} \underset{\delta\to 0}{\Rightarrow} \nabla u_\sigma^{\frac{\alpha+n+1}{4}} \text{ in } L^{4-}(\{u_\sigma > 0\}) \text{ and a.e. on } \{u_\sigma > 0\}. \tag{3.10}$$

Using relations (3.3), (3.5), and (3.10) (see. [13]), we get $\nabla u_{\delta\sigma} \underset{\delta\to 0}{\Rightarrow} \nabla u_\sigma$ in $L^2(Q_{\hat{T}})$. With regard for (3.1) and (3.11), this yields

$$u_{\delta\sigma} \underset{\delta\to 0}{\Rightarrow} u_\sigma \text{ in } L^2(0,\hat{T};H^1(\Omega)) \text{ and a.e. in } Q_{\hat{T}}. \tag{3.11}$$

Remark 3.1. If $m > 0$, then, using relations (2.36) and (3.11) and the Vitali theorem, we establish the convergence

$$m_{\delta\sigma}(u_{\delta\sigma})u_{\delta\sigma}^{m-n}\nabla u_{\delta\sigma} \underset{\delta\to 0}{\Rightarrow} m_\sigma(u_\sigma)u_\sigma^{m-n}\nabla u_\sigma \text{ in } L^1(Q_{\hat{T}}).$$

3.2. Passage to the Limit in Cases (b) and (c) of Theorem A.1

By analogy with [13, 14], we prove that the limit flow $\overrightarrow{J}_\sigma$ formally obtained in (3.8) satisfies the equality $\overrightarrow{J}_\sigma = m_\sigma(u_\sigma)[\nabla\Delta u_\sigma - u_\sigma^{m-n}\nabla u_\sigma]$ in the sense of the limit ($\delta = 0$) equality (c). This leads to the restriction $\alpha < \frac{n+1}{3}$ contained in the definition of the interval $\tilde{\Delta}_{n,\lambda}$ in (1.12). Thus, it remains to pass to the limit as $\delta \to 0$ in the second term on the right-hand side of equality (b), or, which is equivalent, in term II in (2.38). We again integrate II by parts, represent this term in the form (2.41), and pass to the limit as $\delta \to 0$ in II_1 and II_2. Since $\mathcal{B}_{\delta\sigma}(z) \underset{\delta\to 0}{\rightrightarrows} \mathcal{B}_\sigma(z) \leqslant c\, z^\lambda$, $z \in [0,\infty)$ $\mathcal{B}_{\delta\sigma}(z)$ is defined in (2.38), relation (3.11) implies that $\mathcal{B}_{\delta\sigma}(u_{\delta\sigma}) \underset{\delta\to 0}{\to} \mathcal{B}_\sigma(u_\sigma)$ a.e. in $Q_{\hat{T}}$ and a.e. in $(0,\hat{T}) \times \partial\Omega$. Using (2.25) and estimate (2.50) with $d_q = 2$, we establish that the majorizing sequence $\{u^\lambda_{\delta\sigma}\}$ is uniformly bounded with respect to δ and σ in $L^{1+\tilde{\sigma}}(Q_{\hat{T}})$ and $L^{1+\tilde{\sigma}}((0,\hat{T}) \times \partial\Omega)$ ($\tilde{\sigma} > 0$ is sufficiently small) if $0 < \lambda$ ($< 4, N = 3$), whence, taking into account that $u^\lambda_{\delta\sigma} \underset{\delta\to 0}{\to} u^\lambda_\sigma$ a.e. in $Q_{\hat{T}}$ and $(0,\hat{T}) \times \partial\Omega$, by virtue of the Vitali theorem we get $u^\lambda_{\delta\sigma} \underset{\delta\to 0}{\Rightarrow} u^\lambda_\sigma$ in $L^1(Q_{\hat{T}})$ and

$L^1((0,\hat{T})\times\partial\Omega)$. Using the generalized Lebesgue lemma (see Appendix B), we establish that $\mathcal{B}_{\delta\sigma}(u_{\delta\sigma}) \underset{\delta\to 0}{\rightrightarrows} \mathcal{B}_\sigma(u_\sigma)$ in $L^1(Q_{\hat{T}})$ and $L^1((0,\hat{T})\times\partial\Omega)$ if $0<\lambda$ $(<4, N=3)$. This yields the validity of the passage to the limit in II_1 and II_2. Consequently,

$$\mathrm{II} \underset{\delta\to 0}{\to} \int\limits_0^{\hat{T}} \langle \overrightarrow{\chi}\cdot\nabla\mathcal{B}_\sigma(u_\sigma),\zeta\rangle\,dt$$

$$= \int\limits_0^{\hat{T}}\int\limits_{\partial\Omega} \overrightarrow{\chi}\cdot\vec{n}\,\mathcal{B}_\sigma(u_\sigma)\zeta - \iint\limits_{Q_{\hat{T}}} \overrightarrow{\chi}\mathcal{B}_\sigma(u_\sigma)\nabla\zeta. \quad (3.12)$$

3.3. Positivity of the Solution u_σ for $\alpha+m+1\leqslant 0$

Lemma 3.1. *Under the conditions of Theorem* 1.1*, the interval of solvability* $T_{loc}>0$ *can be chosen so that the inequality* $\int\limits_\Omega u_{\delta\sigma}(t)\,dx > c(u_0) > 0$ *holds uniformly in* $\delta>0$ *and* $\sigma>0$ *for all* $t<T_{loc}$*.*

Proof. Taking $\zeta\equiv 1$ as a test function in identity (b) of Theorem A.1, we get $\int\limits_\Omega u_{\delta\sigma}(t)dx = \int\limits_\Omega u_{0\delta\sigma}(x)dx + \int\limits_0^t\int\limits_{\partial\Omega} \mathcal{B}_{\delta\sigma}(u_{\delta\sigma})\overrightarrow{\chi}\cdot\vec{n}\,d\mathcal{H}^{N-1}dt := D_1+D_2$, where $\mathcal{B}_{\delta\sigma}$ is defined in (2.38). By virtue of relations (1.6) and (2.25) and the imbedding $H^1(\Omega)\subset L^\lambda(\partial\Omega)$ $\forall\,\lambda\leqslant 4$, for $N=3$, we get $|D_2|\leqslant C_1 t$ $\forall t\leqslant\hat{T}$, where $\hat{T}$ is defined in (2.23) and C_1 is independent of δ, σ, and t. Taking into account that $u_{0\delta\sigma}(x)>u_0(x)$, we establish that $\int\limits_\Omega u_{\delta\sigma}(t)\,dx > \int\limits_\Omega u_0(x)\,dx - C_1 t > 0$, which implies that Lemma 3.1 with

$$T_{loc} = \min\{\hat{T}, (2C_1)^{-1}\|u_0\|_{L^1(\Omega)}\} \quad (3.13)$$

is true. □

By analogy with [14] (Lemma 2.1), using Lemma 3.1 we prove the important property of positivity for $\alpha+m+1\leqslant 0$.

Lemma 3.2. *If* $\alpha+m+1\leqslant 0$*, then* $u_\sigma(t,x)>0$ *almost everywhere in* Ω *and* $\mathcal{M}_{\alpha+m}(u_\sigma)\in L^2_{loc}(\Omega)$ *for almost all* $0<t<T_{loc}$*, where* T_{loc} *is defined by* (3.13) *and* $\mathcal{M}_\gamma(z)$ *is defined by* (1.11)*; furthermore,* $L^2_{loc}(\Omega)$ *can be replaced by* $L^2(\Omega)$ *if the domain* Ω *is convex.*

3.4. Derivation of the Entropy Inequality (1.18)

Inequality (1.18) is obtained by passing to the limit as $\delta\to 0$ and $\sigma\to 0$ in (A.4). As a consequence of Lemma 3.2 and convergence (3.11), for $\alpha+m+1\leqslant 0$ we get

$$u_{\delta\sigma}^{\frac{\alpha+m-1}{2}} \nabla u_{\delta\sigma} \underset{\delta\to 0}{\rightharpoonup} u_{\sigma}^{\frac{\alpha+m-1}{2}} \nabla u_\sigma = \nabla \mathcal{M}_{\alpha+m}(u_\sigma) \text{ in } L^2_{loc}(Q_{T_{loc}}) \tag{3.14}$$

where T_{loc} is defined by (3.13). By virtue of the Fatou lemma, a priori estimate (2.23) implies that $\Psi_0(u_\sigma) \in L^\infty(0, T_{loc}; L^1(\Omega))$ ($\Psi_0(z)$ is defined by (1.14)). Furthermore, if $m - n + 2 \leqslant 0$, then

$$u_\sigma(t,x) > 0 \text{ a.e. in } \Omega \text{ for almost all } 0 \leqslant t \leqslant T_{loc}. \tag{3.15}$$

Using relations (2.25), (2.26), and (2.35), we establish that the sequence $\{\Psi_0(u_{\delta\sigma})\}$ is uniformly bounded in $L^{1+\eta}(Q_{T_{loc}})$ for a certain $\eta > 0$ if $m - n + 2 > 0$. By virtue of the Vitali theorem, this yields

$$\Psi_0(u_{\delta\sigma}) \underset{\delta\to 0}{\rightrightarrows} \Psi_0(u_\sigma) \text{ in } L^1(Q_{T_{loc}}) \text{ if } m - n + 2 > 0. \tag{3.16}$$

On the left-hand side of inequality (A.4), the passage to the limit is a simple consequence of the Fatou lemma, convergence (3.14), the results on compactness obtained in Sec. 3.1, and the inequality $\|\lim f_\delta\|_{L^p(X)} \leqslant \lim \|f_\delta\|_{L^p(X)}$, which is true for an arbitrary sequence $\{f_\delta\}$ weakly compact in $L^p(X)$. Convergence (3.3) yields $\iint\limits_{Q_{T_{loc}}} u_{\delta\sigma}^{\alpha+n+1} \underset{\delta\to 0}{\to} \iint\limits_{Q_{T_{loc}}} u_{\sigma}^{\alpha+n+1}$. Using the boundedness $\int\limits_\Omega G_{\delta\sigma}^{(\alpha)}(u_{0\delta\sigma})\,dx \leqslant c(u_0)$ ($c(u_0)$ is independent of δ and σ), convergence (3.11), and the Lebesgue theorem, we get $\int\limits_\Omega G_{\delta\sigma}^{(\alpha)}(u_{0\delta\sigma})\,dx \underset{\delta\to 0}{\to} \int\limits_\Omega G_{\sigma}^{(\alpha)}(u_{0\sigma})\,dx$. We pass now to the limit as $\delta \to 0$ in the third term on the right-hand side of inequality (A.4), or, which is equivalent, in the integral B_4 in (2.18). Integrating by parts, we rewrite B_4 in the form

$$B_4 = \int\limits_0^{T_{loc}} \int\limits_{\partial\Omega} \overrightarrow{\chi} \cdot \vec{n}\, \mathcal{B}_{\delta\sigma}^{(\alpha)}(u_{\delta\sigma}) \text{ where } \mathcal{B}_{\delta\sigma}^{(\alpha)}(z) := \int\limits_0^z b'_{\delta\sigma}(\tau) g_{\delta\sigma}^{(\alpha)}(\tau)\,d\tau. \tag{3.17}$$

Since $\mathcal{B}_{\delta\sigma}^{(\alpha)}(z) \underset{\delta\to 0}{\rightrightarrows} \mathcal{B}_{\sigma}^{(\alpha)}(z) \leqslant c(z^{\lambda+\alpha} + z)$, $z \in [0,\infty)$ (here, $\alpha > -\lambda$), relation (3.11) implies that $\mathcal{B}_{\delta\sigma}^{(\alpha)}(u_{\delta\sigma}) \underset{\delta\to 0}{\to} \mathcal{B}_{\sigma}^{(\alpha)}(u_\sigma)$ almost everywhere in $(0, T_{loc}) \times \partial\Omega$. Using (2.25) and the imbedding $H^1(\Omega) \subset L^r(\partial\Omega)\ \forall\, r \leqslant 4$, for $N = 3$, we establish the uniform boundedness (in δ and σ) of the majorizing sequence

$$\{u_{\delta\sigma}^{\lambda+\alpha} + u_{\delta\sigma}\} \text{ in } L^{1+\tilde\sigma}((0,T_{loc}) \times \partial\Omega)\ (\tilde\sigma > 0 \text{ is sufficiently small}) \tag{3.18}$$

if $-\alpha < \lambda$ ($< \min\{4, 4-\alpha\}, N = 3$). For $N = 3$ and $0 < \alpha < \lambda < 4$, the boundedness of (3.18) follows from estimate (2.50) with $d_q = 2$. Thus, relation (3.18) is true for $-\alpha < \lambda$ ($< 4, N = 3$). By virtue of (3.18), the convergence $u_{\delta\sigma}^{\lambda+\alpha} + u_{\delta\sigma} \underset{\delta\to 0}{\to} u_{\sigma}^{\lambda+\alpha} + u_{\sigma}$ a.e. in $(0, T_{loc}) \times \partial\Omega$, and the Vitali theorem, we get $u_{\delta\sigma}^{\lambda+\alpha} + u_{\delta\sigma} \underset{\delta\to 0}{\Rightarrow} u_{\sigma}^{\lambda+\alpha} + u_{\sigma}$ in $L^1((0, T_{loc}) \times \partial\Omega)$. Using the generalized Lebesgue lemma (see Appendix B), we establish that $\mathcal{B}_{\delta\sigma}^{(\alpha)}(u_{\delta\sigma}) \underset{\delta\to 0}{\Rightarrow} \mathcal{B}_{\sigma}^{(\alpha)}(u_{\sigma})$ in $L^1((0, T_{loc}) \times \partial\Omega)$ if $-\alpha < \lambda$ ($< 4, N = 3$). Therefore,

$$B_4 \underset{\delta\to 0}{\to} \int\limits_0^{T_{loc}} \int\limits_{\partial\Omega} \overrightarrow{\chi} \cdot \vec{n}\, \mathcal{B}_{\sigma}^{(\alpha)}(u_{\sigma}) \text{ if inequalities (1.17) are true,} \tag{3.19}$$

where B_4 is defined by relation (2.18) and $\mathcal{B}_{\sigma}^{(\alpha)}(z)$ coincides with $\mathcal{B}_{\delta\sigma}^{(\alpha)}(z)$ in (3.17) for $\delta = 0$. The convergences obtained prove inequality (1.18).

3.5. Derivation of the Energy Inequality (1.20)

We rewrite inequality (A.3) in an equivalent form:

$$\begin{aligned}\tilde{\mathcal{E}}_{\Omega}(u_{\delta\sigma}(t)) \leqslant \tilde{\mathcal{E}}_{\Omega}(u_{0\delta\sigma}(x)) - \tfrac{\alpha+n-1}{2} \iint\limits_{Q_{T_{loc}}} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) u_{\delta\sigma}^{-1} |\nabla u_{\delta\sigma}|^2 \nabla u_{\delta\sigma} \\ + \tfrac{2}{\alpha+n+1} \iint\limits_{Q_{T_{loc}}} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) u_{\delta\sigma}^{\frac{1-\alpha-n}{2}} \Delta u_{\delta\sigma}^{\frac{\alpha+n+1}{2}} \nabla u_{\delta\sigma} \\ + \iint\limits_{Q_{T_{loc}}} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) \Psi'_0(u_{\delta\sigma}) \nabla u_{\delta\sigma} =: \tilde{\mathcal{E}}_{\Omega}(u_{0\delta\sigma}(x)) + I_1 + I_2 + I_3.\end{aligned} \tag{3.20}$$

By virtue of relations (3.11), (3.15), and (3.16) and the Fatou lemma, we can pass to the limit as $\delta \to 0$ on the left-hand side of inequality (3.20) and in the term $\tilde{\mathcal{E}}_{\Omega}(u_{0\delta\sigma}(x))$. Then we pass to the limit as $\delta \to 0$ in I_1. We rewrite I_1 in the form

$$I_1 = -\tfrac{\alpha+n-1}{2} \Big(\tfrac{4}{\alpha+n+1}\Big)^3 \iint\limits_{Q_{T_{loc}}} R_{\delta\sigma}^{(1)}(u_{\delta\sigma}),$$

where $R_{\delta\sigma}^{(1)}(u_{\delta\sigma}) := \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) u_{\delta\sigma}^{\frac{5-3(\alpha+n)}{4}} |\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}|^2 \nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}$. Taking (1.6) into account, we get

$$|R_{\delta\sigma}^{(1)}(u_{\delta\sigma})| \leqslant \tilde{R}_{\delta\sigma}^{(1)}(u_{\delta\sigma}) := c\, u_{\delta\sigma}^{\frac{4\lambda-3(\alpha+n)+1}{4}} |\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}|^3,$$

where $\frac{3(\alpha+n)-1}{4} < \lambda\,(< \alpha+n+2,\ N=3)$. Relations (2.32) and (2.30) imply that the majorizing sequence $\{\tilde{R}^{(1)}_{\delta\sigma}(u_{\delta\sigma})\}$ is uniformly bounded with respect to δ and σ in $L^{1+\tilde{\sigma}}(Q_{T_{loc}})$ (for sufficiently small $\tilde{\sigma} > 0$). By virtue of relations (3.10) and (3.11), Lemma B.7, and the Vitali theorem, we conclude that $\tilde{R}^{(1)}_{\delta\sigma}(u_{\delta\sigma}) \underset{\delta\to 0}{\Rightarrow} \tilde{R}^{(1)}_{\sigma}(u_\sigma)$ in $L^1(\{u_\sigma > 0\})$ and, hence, by virtue of the generalized Lebesgue lemma, $R^{(1)}_{\delta\sigma}(u_{\delta\sigma}) \underset{\delta\to 0}{\Rightarrow} R^{(1)}_{\sigma}(u_\sigma)$ in $L^1(\{u_\sigma > 0\})$. Further, we show that $\tilde{R}^{(1)}_{\delta\sigma}(u_{\delta\sigma}) \underset{\delta\to 0}{\to} 0$ (and, hence, $R^{(1)}_{\delta\sigma}(u_{\delta\sigma}) \underset{\delta\to 0}{\to} 0$) on $\{u_\sigma = 0\}$. Indeed, by virtue of the Egorov theorem, it follows from (3.11) that, for any $\gamma > 0$, there exists a set $S_\gamma \subset \{u_\sigma = 0\}$ such that $\mathfrak{L}^{N+1}(S_\gamma) \leqslant \gamma$ and $u_{\delta\sigma} \underset{\delta\to 0}{\rightrightarrows} u_\sigma$ on $\{u_\sigma = 0\}\backslash S_\gamma$. Using the Hölder inequality, we get

$$\iint\limits_{\{u_\sigma=0\}} \tilde{R}^{(1)}_{\delta\sigma}(u_{\delta\sigma}) \leqslant \bar{\bar{o}}_{\delta\sigma}(1)\Bigg(\iint\limits_{\{u_\sigma=0\}\backslash S_\gamma} |\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}|^4\Bigg)^{\frac{3}{4}} + c\Bigg(\iint\limits_{S_\gamma} u_{\delta\sigma}^{4\lambda-3(\alpha+n)+1}\Bigg)^{\frac{1}{4}}\Bigg(\iint\limits_{S_\gamma} |\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}|^4\Bigg)^{\frac{3}{4}}.$$

By virtue of the Vitali theorem, it follows from (2.30) that the last term on the right-hand side tends to zero as $\mathfrak{L}^{N+1}(S_\gamma) \underset{\gamma\to 0}{\to} 0$. Thus, by virtue of the generalized Lebesgue theorem, we have

$$I_1 \underset{\delta\to 0}{\to} -\tfrac{\alpha+n-1}{2} \iint\limits_{\{u_\sigma>0\}} \overrightarrow{\chi}\, b'_\sigma(u_\sigma) u_\sigma^{-1} |\nabla u_\sigma|^2 \nabla u_\sigma \tag{3.21}$$

provided that relations (1.17) are true. Then we pass to the limit as $\delta \to 0$ in I_3. Integrating by parts, we rewrite I_3 in the following form:

$$I_3 = \int\limits_0^{T_{loc}} \int\limits_{\partial\Omega} \overrightarrow{\chi} \cdot \vec{n}\, \tilde{\Psi}_{\delta\sigma}(u_{\delta\sigma}) \text{ where } \tilde{\Psi}_{\delta\sigma}(z) := \int\limits_0^z b'_{\delta\sigma}(\tau)\Psi'_0(\tau)\,d\tau,$$

$|\tilde{\Psi}_{\delta\sigma}(z)| \leqslant c(z^{\lambda+m-n+1} + R^{m-n+1}z^\lambda)$ and R is defined in (1.14). By analogy with the proof of relation (3.19), we get

$$I_3 \underset{\delta\to 0}{\to} \int\limits_0^{T_{loc}} \int\limits_{\partial\Omega} \overrightarrow{\chi} \cdot \vec{n}\, \tilde{\Psi}_\sigma(u_\sigma) \text{ if inequalities (1.17) are true,} \tag{3.22}$$

where $\tilde{\Psi}_\sigma(z)$ coincides with $\tilde{\Psi}_{\delta\sigma}(z)$ for $\delta = 0$. It remains to pass to the limit in the term I_2. We rewrite I_2 in the form

$$I_2 = \tfrac{8}{(\alpha+n+1)^2} \iint\limits_{Q_{T_{loc}}} R^{(2)}_{\delta\sigma}(u_{\delta\sigma}) \Delta u_{\delta\sigma}^{\frac{\alpha+n+1}{2}},$$

where $R^{(2)}_{\delta\sigma}(u_{\delta\sigma}) := \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) u_{\delta\sigma}^{\frac{5-3(\alpha+n)}{4}} \nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}$. Taking (1.6) into account, we get

$$|R^{(2)}_{\delta\sigma}(u_{\delta\sigma})| \leqslant \tilde{R}^{(2)}_{\delta\sigma}(u_{\delta\sigma}) := c\, u_{\delta\sigma}^{\frac{4\lambda-3(\alpha+n)+1}{4}} |\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}|,$$

where $\frac{3(\alpha+n)-1}{4} < \lambda\, (< \alpha+n+2,\ N=3)$. Relations (2.32) and (2.30) imply that the majorizing sequence $\{\tilde{R}^{(2)}_{\delta\sigma}(u_{\delta\sigma})\}$ is uniformly bounded with respect to δ and σ in $L^{2+\tilde{\sigma}}(Q_{T_{loc}})$ (for $0 < \tilde{\sigma} < 2$ if $N < 3$ and for $0 < \tilde{\sigma} < \frac{4(\alpha+n-\lambda+2)}{2\lambda-\alpha-n+5}$ if $N = 3$). By virtue of relations (3.10) and (3.11), Lemma B.7, and the Vitali theorem, we conclude that $\tilde{R}^{(2)}_{\delta\sigma}(u_{\delta\sigma}) \underset{\delta\to 0}{\rightrightarrows} \tilde{R}^{(2)}_{\sigma}(u_\sigma)$ in $L^2(\{u_\sigma > 0\})$. By virtue of the generalized Lebesgue lemma, this yields

$$R^{(2)}_{\delta\sigma}(u_{\delta\sigma}) \underset{\delta\to 0}{\rightrightarrows} \overrightarrow{\chi}\, b'_\sigma(u_\sigma) u_\sigma^{\frac{1-\alpha-n}{2}} \nabla u_\sigma \text{ in } L^2(\{u_\sigma > 0\}). \tag{3.23}$$

Convergence (3.4) implies, in particular, that

$$\Delta u_{\delta\sigma}^{\frac{\alpha+n+1}{2}} \underset{\delta\to 0}{\rightharpoonup} \Delta u_\sigma^{\frac{\alpha+n+1}{2}} \text{ in } L^2(Q_{T_{loc}}). \tag{3.24}$$

Thus, using (3.23) and (3.24), we establish the convergence of the term I_2 on the set $\{u_\sigma > 0\}$. Let us show that $I_2 \underset{\delta\to 0}{\to} 0$ on $\{u_\sigma = 0\}$. Indeed, by virtue of the Egorov theorem, relation (3.11) implies that, for any $\gamma > 0$, there exists a set $S_\gamma \subset \{u_\sigma = 0\}$ such that $\mathfrak{L}^{N+1}(S_\gamma) \leqslant \gamma$ and $u_{\delta\sigma} \underset{\delta\to 0}{\rightrightarrows} u_\sigma$ on $\{u_\sigma = 0\} \backslash S_\gamma$. Using the Hölder inequality, we get

$$\begin{aligned} &\iint\limits_{\{u_\sigma = 0\}} \tilde{R}^{(2)}_{\delta\sigma}(u_{\delta\sigma}) |\Delta u_{\delta\sigma}^{\frac{\alpha+n+1}{2}}| \\ &\quad \leqslant \bar{\bar{o}}_{\delta\sigma}(1) \Bigg(\iint\limits_{\{u_\sigma=0\}\backslash S_\gamma} |\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}|^4 \Bigg)^{\frac{1}{4}} \Bigg(\iint\limits_{\{u_\sigma=0\}\backslash S_\gamma} |\Delta u_{\delta\sigma}^{\frac{\alpha+n+1}{2}}|^2 \Bigg)^{\frac{1}{2}} \\ &\quad + c \Bigg(\iint\limits_{S_\gamma} u_{\delta\sigma}^{4\lambda-3(\alpha+n)+1} \Bigg)^{\frac{1}{4}} \Bigg(\iint\limits_{S_\gamma} |\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}|^4 \Bigg)^{\frac{1}{4}} \Bigg(\iint\limits_{S_\gamma} |\Delta u_{\delta\sigma}^{\frac{\alpha+n+1}{2}}|^2 \Bigg)^{\frac{1}{2}}. \end{aligned}$$

Since, for $\lambda < \alpha + n + 2$ and $N = 3$, the imbedding $L^{\alpha+n+9}(Q_{T_{loc}}) \subset L^{4\lambda-3(\alpha+n)+1}(Q_{T_{loc}})$ is true, by virtue of relations (2.32), (2.25), (2.29), and (2.30) and the Vitali theorem the last term on the right-hand side tends to zero as $\mathfrak{L}^{N+1}(S_\gamma) \underset{\gamma\to 0}{\to} 0$. Thus, by virtue of the generalized Lebesgue lemma, we have

$$I_2 \underset{\delta\to 0}{\to} \tfrac{2}{\alpha+n+1} \iint\limits_{\{u_\sigma>0\}} \overrightarrow{\chi}\, b'_\sigma(u_\sigma) u_\sigma^{\frac{1-\alpha-n}{2}} \Delta u_\sigma^{\frac{\alpha+n+1}{2}} \nabla u_\sigma, \tag{3.25}$$

provided that inequalities (1.17) are true. Taking into account (3.20), (3.21), and (3.23)–(3.25), we get

$$\tilde{\mathcal{I}}_{\delta\sigma}(u_{\delta\sigma}) \underset{\delta\to 0}{\to} \int\limits_0^{T_{loc}} \int\limits_{\partial\Omega} \overrightarrow{\chi} \cdot \vec{n}\, \tilde{\Psi}_\sigma(u_\sigma) - \iint\limits_{\{u_\sigma>0\}} \overrightarrow{\chi}\, b'_\sigma(u_\sigma) \nabla u_\sigma \Delta u_\sigma,$$

provided that inequalities (1.17) are true. Here, $\tilde{\mathcal{I}}_{\delta\sigma}(z)$ is defined in assertion (h) of Theorem A.1 and $\tilde{\Psi}_\sigma(z) := \int\limits_0^z b'_\sigma(\tau)\Psi'_0(\tau)d\tau$. Inequality (1.20) is proved.

Remark 3.2. To obtain the local entropy estimate (1.19), we pass to the limit as $\delta \to 0$ and $\sigma \to 0$ in the corresponding local version of the entropy inequality (A.4). Since this analysis is analogous to the analysis carried out in Sec. 3.4, we omit it here.

Remark 3.3. We also omit the passage to the limit as $\sigma \to 0$, which is based on the same a priori estimates and can be carried out, with slight modifications, by analogy with the passage to the limit as $\delta \to 0$ carried out in Secs. 3.1–3.5.

3.6. Derivation of Restrictions on n, m, and λ

Combining all intermediate restrictions on the parameters of problem (N) used in Secs. 3.1–3.5 (namely, relations (2.21), (2.48), and (2.51), the condition $\alpha > -2$ for $N = 3$ and $m > -1$, and the condition $\alpha < \frac{n+1}{3}$), one can easily verify that they hold simultaneously if the sole condition (1.17) is satisfied. Thus, Theorem 1.1 is proved.

4. Proof of Theorem 1.2. Finite Speed of Propagation

We introduce the following notation: $\Omega(s) := B(x_1, s)$, $Q_T(s) = (0,T) \times \Omega(s)$, and $K(s) := \int\limits_{\Omega(s)} u_0^{\alpha+1}(x)\, dx\ \forall\, s > 0$, where $x_1 \in \partial B(0, \tilde{R}_0 +$

$\frac{3}{2}R_0)$ is an arbitrary point and $R_0 > 0$ is such that $B(0, \tilde{R}_0 + 3R_0) \Subset \Omega$. Assuming that $\operatorname{supp} u_0(x) \subset B(0, \tilde{R}_0)$, we obtain $K(s) \equiv 0\ \forall s \in (0, \frac{3}{2}R_0)$. For arbitrary $s > 0$ and $\delta > 0$, we introduce a function $\varphi(x) \in C^2(\Omega)$ such that $\varphi(x) = 0$ if $x \in \Omega \backslash \Omega(s)$, $\varphi(x) = 1$ if $x \in \Omega(s-\delta)$, $0 \leqslant \varphi(x) \leqslant 1\ \forall x \in \Omega$, $|\nabla\varphi| \leqslant \frac{d}{\delta}$, and $|\Delta\varphi| \leqslant \frac{d}{\delta^2}$. In the local entropy estimate (1.19), we set $\zeta^4(x,t) = \varphi^4(x)\exp(-\frac{t}{T})\ \forall T \leqslant T_{loc}$. We fix $\alpha \in \tilde{\Delta}_{n,\lambda} \cap (0, +\infty)$, which is possible because $n < 2$. After simple transformations, we get

$$\sup_{t\in(0,T)} \int\limits_{\Omega(s-\delta)} u^{\alpha+1}(t) + T^{-1} \iint\limits_{Q_T(s-\delta)} u^{\alpha+1}$$
$$+ c_3^{-1} \iint\limits_{Q_T(s-\delta)} \Big\{ \big|\nabla u^{\frac{\alpha+m+1}{2}}\big|^2 + \big|\nabla u^{\frac{\alpha+n+1}{4}}\big|^4 + \big|D^2 u^{\frac{\alpha+n+1}{2}}\big|^2 \Big\} \leqslant K(s)$$
$$+ c\Bigg(\delta^{-2} \iint\limits_{Q_T(s)} u^{\alpha+m+1} + \delta^{-4} \iint\limits_{Q_T(s)} u^{\alpha+n+1} + \delta^{-1} \iint\limits_{Q_T(s)} u^{\alpha+\lambda} \Bigg) =: R_T(s,\delta) \tag{4.1}$$

$\forall s > 0,\ \delta > 0$, and $\forall T \leqslant T_{loc}$, where T_{loc} is defined by (3.13).

We introduce the following energy functions related to the solution $u(t,x)$:

$$G_T^{(1)}(s) := \iint\limits_{Q_T(s)} u^{\alpha+m+1}, \quad G_T^{(2)}(s) := \iint\limits_{Q_T(s)} u^{\alpha+n+1}, \quad G_T^{(3)}(s) := \iint\limits_{Q_T(s)} u^{\alpha+\lambda}. \tag{4.2}$$

We apply the interpolation inequality from Lemma B.5 to the functions $v_1 := u^{\frac{\alpha+m+1}{2}}$ for $a = d = 2$, $b = \frac{2(\alpha+1)}{\alpha+m+1}$, $i = 0$, $j = 1$, and $\theta_1 = \frac{mN}{mN+2(\alpha+1)}$; $v_2 := u^{\frac{\alpha+n+1}{4}}$ for $a = d = 4$, $b = \frac{4(\alpha+1)}{\alpha+n+1}$, $i = 0$, $j = 1$, and $\theta_2 = \frac{nN}{nN+4(\alpha+1)}$; v_1 for $a = \frac{2(\alpha+\lambda)}{\alpha+m+1}$, $d = 2$, $b = \frac{2(\alpha+1)}{\alpha+m+1}$, $i = 0$, $j = 1$, $\tilde{\theta}_3 = \frac{N(\lambda-1)(\alpha+m+1)}{(\alpha+\lambda)(mN+2(\alpha+1))}$, and $\lambda > 1$, and v_2 for $a = \frac{4(\alpha+\lambda)}{\alpha+n+1}$, $d = 4$, $b = \frac{4(\alpha+1)}{\alpha+n+1}$, $i = 0$, $j = 1$, $\tilde{\theta}_4 = \frac{N(\lambda-1)(\alpha+n+1)}{(\alpha+\lambda)(nN+4(\alpha+1))}$, and $\lambda > 1$. Then we integrate the inequalities obtained with respect to t. Taking (4.1) into account, we obtain the following relations for the energy functions from (4.2):

$$G_T^{(1)}(s-\delta) \leqslant cT^{1-\theta_1} R_T^{1+k_1}(s,\delta) + cTR_T^{\frac{\alpha+m+1}{\alpha+1}}(s,\delta), \tag{4.3}$$

$$G_T^{(2)}(s-\delta) \leqslant cT^{1-\theta_2} R_T^{1+k_2}(s,\delta) + cTR_T^{\frac{\alpha+n+1}{\alpha+1}}(s,\delta), \tag{4.4}$$

$$G_T^{(3)}(s-\delta) \leqslant c \int_0^T \Bigg(\int_{\Omega(s-\delta)} |\nabla v_i|^{2i} \Bigg)^{\frac{\tilde{\theta}_{i+2}(\alpha+\lambda)}{\alpha+n+1}} \Bigg(\int_{\Omega(s-\delta)} u^{\alpha+1} \Bigg)^{\frac{(1-\tilde{\theta}_{i+2})(\alpha+\lambda)}{\alpha+1}}$$

$$+ c \int_0^T \Bigg(\int_{\Omega(s-\delta)} u^{\alpha+1} \Bigg)^{\frac{\alpha+\lambda}{\alpha+1}} \leqslant cT^{1-\theta_{i+2}} R_T^{1+k_{i+2}}(s,\delta) + cT R_T^{\frac{\alpha+\lambda}{\alpha+1}}(s,\delta) \quad (4.5)$$

where $i = 1,2$, $R_T(s,\delta)$ is defined by (4.1), $k_1 = \frac{2m}{mN+2(\alpha+1)}$, $k_2 = \frac{4n}{nN+4(\alpha+1)}$, $\theta_3 = \frac{N(\lambda-1)}{mN+2(\alpha+1)}$ and $k_3 = \frac{2(\lambda-1)}{mN+2(\alpha+1)}$ if $1 < \lambda \leqslant m+1+\frac{2(\alpha+1)}{N}$, and $\theta_4 = \frac{N(\lambda-1)}{nN+4(\alpha+1)}$ and $k_4 = \frac{4(\lambda-1)}{nN+4(\alpha+1)}$ if $1 < \lambda \leqslant n+1+\frac{4(\alpha+1)}{N}$.

We apply Lemma B.8 with $s_1 = \frac{3}{2}s_0$ to the system of inequalities (4.3)–(4.5). For the function $G_T(s) := f_1(T) \sum_{i=1}^{4} \Big(G_T^{(i)}(s)\Big)^{\frac{\beta}{1+k_i}}$, where $f_1(T) \in C[0,T_{loc}]$ and $f_1(0)=0$, constructed in the lemma indicated, we get

$$G_T(s) \equiv 0 \quad \forall\, 0 < s \leqslant \tfrac{3}{2}R_0 - f_2(T) \sum_{i=1}^{4} \Big(G_T(\tfrac{3}{2}R_0)\Big)^{\frac{k_i}{\alpha_i \beta}}. \quad (4.6)$$

Here, the function $f_2(T) \in C[0,T_{loc}]$, $f_2(0) = 0$, is also constructed in Lemma B.8, $\beta = (1+k_1)(1+k_2)(1+k_3)(1+k_4)$, $\alpha_1 = 2$, $\alpha_2 = 4$, and $\alpha_3 = \alpha_4 = 1$.

Further, we obtain an estimate for $G_T(\frac{3}{2}R_0)$. By virtue of Theorem 1.1 ($u \in L^\infty(0,T_{loc}; H^1(\Omega))$, $u^{\alpha+m+1} \in L^2(0,T_{loc};\ H^1(\Omega))$, and $u^{\lambda+\alpha} \in L^1(Q_{T_{loc}})$), the right-hand side of inequality (4.1) (i.e., $R_T(s,\delta)$) is bounded for all $T \leqslant T_{loc}$. Taking this into account and setting $s = 2R_0$ and $\delta = \frac{R_0}{2}$ in inequalities (4.3)–(4.5), we get

$$G_T(\tfrac{3}{2}R_0) := f_1(T) \sum_{i=1}^{4} \Big(G_T^{(i)}(\tfrac{3}{2}R_0)\Big)^{\frac{\beta}{1+k_i}} \leqslant C_5\, f_3(T) \quad (4.7)$$

where $f_3(T) \in C[0,T_{loc}]$ and $f_3(0) = 0$. Therefore, relation (4.6) is true for any s satisfying conditions

$$0 < s \leqslant \tfrac{3}{2}R_0 - C_6\Gamma(T),\ \Gamma(T) := f_2(T) \sum_{i=1}^{4} (f_3(T))^{\frac{k_i}{\alpha_i\beta}}. \quad (4.8)$$

It is easy to verify that the function $\Gamma(T)$ has all properties mentioned in Theorem 1.2. Relation (4.8) yields

$$G_T\left(\tfrac{R_0}{2}\right) = 0 \quad \forall\, T \leqslant \tilde{T} := \Gamma^{-1}(\tfrac{R_0}{C_6}). \quad (4.9)$$

Since the point $x_1 \in \partial B(0, \tilde{R}_0 + \frac{3}{2}R_0)$ is arbitrary, the constructed solution $u(t,x)$ possesses the following property:

$$u(t,x) \equiv 0 \quad \forall\,(t,x) \in (0,T_1) \times \{B(0,\tilde{R}_0 + 2R_0) \backslash B(0,\tilde{R}_0 + R_0)\}, \tag{4.10}$$

where $T_1 := \min\{\tilde{T}, T_{loc}\}$. This enables us to represent the constructed solution $u(t,x)$ in the form $u = u_1 + u_2$, where $u_1 = u$ in $(0,T_1) \times B(0,\tilde{R}_0 + R_0)$, $u_1 = 0$ in $(0,T_1) \times \{\Omega \backslash B(0,\tilde{R}_0 + R_0)\}$, $u_2 = u$ in $(0,T_1) \times \{\Omega \backslash B(0,\tilde{R}_0 + 2R_0)\}$, and $u_2 = 0$ in $(0,T_1) \times B(0,\tilde{R}_0 + 2R_0)$. By virtue of property (4.10), the supports of the functions u_1 and u_2 do not intersect. Therefore, it is easy to verify that u_1 is a solution of problem (N) and u_2 is a solution of problem (N) with $u_0(x) \equiv 0$.

It is easy to verify that, for any n, m, and λ from (1.22), there exists $\alpha \in \tilde{\Delta}_{n,\lambda} \cap (0,+\infty)$ such that

$$\begin{gathered} m > 0, \ \tfrac{1}{8} < n < 2, \\ 1 < \lambda < \Phi \text{ if } N < 3, \ \Phi := \max\{n + 1 + \tfrac{4(\alpha+1)}{3}, m + 1 + \tfrac{2(\alpha+1)}{3}\}, \\ 1 < \lambda < \min\{\alpha + n + 2, \Phi\} = \alpha + n + 2 \text{ if } N = 3, \end{gathered}$$

which is sufficient for the functional system (4.3)–(4.5) to be true. Theorem 1.2 is proved.

5. Proof of Theorem 1.3. Cauchy Problem

We construct a solution $U(t,x)$ of the Cauchy problem (C) on the basis of solutions of problem (N). Assume that $\operatorname{supp} u_0 \Subset B(0,R_0)$, $R_0 > 0$, $\Omega := \Omega_1 = B(0,4R_0)$, and $u(t,x)$ is a solution of problem (N) from Theorem 1.1 in the domain $(0,T^{(1)}_{loc}) \times \Omega_1$ $(T_{loc} := T^{(1)}_{loc})$. By virtue of Theorem 1.2 with $\tilde{R}_0 = R_0$, we have $u(t,x) = u_1(t,x) + u_2(t,x)$ in the domain $(0,T_1) \times \Omega_1$, where $\operatorname{supp} u_1(t,.) \Subset B(0,2R_0)$ and $\operatorname{supp} u_2(t,.) \Subset \Omega_1 \backslash B(0,3R_0)$. We set

$$U(t,x) := \begin{cases} u_1(t,x) \quad \forall\,(t,x) \in (0,T_1) \times B(0,2R_0), \\ 0 \quad \forall\,(t,x) \in (0,T_1) \times \{\mathbb{R}^N \backslash B(0,2R_0)\}. \end{cases}$$

It is obvious that the function $U(t,x)$ is a solution of problem (C) in the domain $(0,T_1) \times \Omega_1$ in the sense of Definition 1.2. We write the refined entropy inequality (1.18) and energy inequality (1.20) for the solution

$U(t,x)$, taking into account that $\operatorname{supp} U(t,.) \Subset B(0,2R_0) \Subset \Omega$:

$$\sup_{t\in(0,T_1)} \int\limits_{\Omega} U^{\alpha+1}(t,x) + c_1^{-1} \iint\limits_{Q_t} \Big\{ \big|D^2 U^{\frac{\alpha+n+1}{2}}\big|^2 + \big|\nabla U^{\frac{\alpha+n+1}{4}}\big|^4 \Big\}$$
$$+ c_1^{-1} \iint\limits_{Q_t} \big|\nabla U^{\frac{\alpha+m+1}{2}}\big|^2 \leqslant \int\limits_{\Omega} u_0^{\alpha+1}(x) \quad \forall\, t \leqslant T_1, \quad (5.1)$$

$$\tilde{\mathcal{E}}_{\Omega}(U(t)) \leqslant \tilde{\mathcal{E}}_{\Omega}(u_0(x)) - \iint\limits_{\mathrm{P}} \overrightarrow{\chi}\, b'(U)\nabla U \Delta U \leqslant \tilde{\mathcal{E}}_{\Omega}(u_0(x))$$
$$+ c\int\limits_0^t \Big(\int\limits_{\Omega} \big|\nabla U^{\frac{\alpha+n+1}{4}}\big|^4\Big)^{\frac14} \Big(\int\limits_{\Omega} \big|D^2 U^{\frac{\alpha+n+1}{2}}\big|^2\Big)^{\frac12} \Big(\int\limits_{\Omega} U^{\xi}\Big)^{\frac14}$$
$$+ c\int\limits_0^t \Big(\int\limits_{\Omega} \big|\nabla U^{\frac{\alpha+n+1}{4}}\big|^4\Big)^{\frac34} \Big(\int\limits_{\Omega} U^{\xi}\Big)^{\frac14}, \ \xi := 4\lambda - 3(\alpha+n) + 1, \quad (5.2)$$

where $\tilde{\mathcal{E}}_{\Omega}(u(t))$ is defined in (1.20). Applying Lemma B.5 to the function $v(x) = U$ for $a = \xi > 1$ $\big(\Rightarrow \lambda > \frac{3(\alpha+n)}{4}\big)$, $b = 1$, $d = 2$, $i = 0$, and $j = 1$, we get

$$\int\limits_{\Omega} U^{\xi} \leqslant c\Big(\int\limits_{\Omega} |\nabla U|^2\Big)^{\frac{N(\xi-1)}{N+2}} \Big(\int\limits_{\Omega} U\Big)^{1+\frac{(2-N)(\xi-1)}{N+2}} + c\Big(\int\limits_{\Omega} U\Big)^{\xi} \quad (5.3)$$

for $\lambda > \frac{3(\alpha+n)}{4}$ if $N < 3$ and for $\frac{3(\alpha+n)}{4} < \lambda \leqslant \frac{3(\alpha+n)+5}{4}$ if $N = 3$. If $N = 3$ and $\frac{3(\alpha+n)-1}{4} < \lambda \leqslant \frac{3(\alpha+n)}{4} \Leftrightarrow 0 < \xi \leqslant 1$, then $\int\limits_{\Omega} U^{\xi} \leqslant |\Omega|^{3(\alpha+n)-4\lambda}\big(\int\limits_{\Omega} U\big)^{\xi}$. In the case where $N = 3$ and $\lambda > \frac{3(\alpha+n)+5}{4}$, applying Lemma B.5 to the function $v(x) = U^{\frac{\alpha+n+1}{4}}$ with $a = \frac{\xi}{\alpha+n+1}$, $b = \frac{24}{\alpha+n+1}$, $d = 4$, $i = 0$, and $j = 1$, we get

$$\int\limits_{\Omega} U^{\xi} \leqslant c\Big(\int\limits_{\Omega} \big|\nabla U^{\frac{\alpha+n+1}{4}}\big|^4\Big)^{\nu} \Big(\int\limits_{\Omega} U^6\Big)^{\frac{4(\lambda+1)}{3(\alpha+n+3)}} + c\Big(\int\limits_{\Omega} U^6\Big)^{\frac{\xi}{6}}, \quad (5.4)$$

where $\nu := \frac{4\lambda - 3(\alpha+n)-5}{\alpha+n+3}$.

Taking into account inequalities (5.1) and (5.3) and the property of the conservation of mass $\big(\int\limits_{\Omega} U(t,x)dx = \int\limits_{\Omega} u_0(x)dx\big)$, which follows from the finiteness of the support of $u(t,.)$, we deduce from (5.2) that, for any $t \leqslant T_1$,

$$\tilde{\mathcal{E}}_\Omega(U(t)) \leqslant A_1(\tilde{\mathcal{E}}_\Omega(u_0),T_1) + c_2(u_0)\int\limits_0^t g(\tau)\Big(\tilde{\mathcal{E}}_\Omega(U(\tau))\Big)^{\frac{N(\xi-1)}{4(N+2)}}\,d\tau \qquad (5.5)$$

for $\lambda > \frac{3(\alpha+n)}{4}$ if $N<3$ and for $\frac{3(\alpha+n)}{4} < \lambda \leqslant \frac{3(\alpha+n)+5}{4}$ if $N=3$. Here, $A_1(\tilde{\mathcal{E}}_\Omega(u_0),T_1) := \tilde{\mathcal{E}}_\Omega(u_0) + c_1(u_0)T_1^{\frac{1}{4}}$, $g(t) := \big(\int\limits_\Omega |\nabla U^{\frac{\alpha+n+1}{4}}|^4\big)^{\frac{1}{4}} \times$ $\times \big(\int\limits_\Omega |D^2 U^{\frac{\alpha+n+1}{2}}|^2\big)^{\frac{1}{2}} + \big(\int\limits_\Omega |\nabla U^{\frac{\alpha+n+1}{4}}|^4\big)^{\frac{3}{4}}$, $c_i(u_0)$ are independent of $\|\nabla u_0\|_{L^2(\Omega)}$, and ξ is defined in (5.2). By virtue of Lemma B.9 and estimate (5.1), inequality (5.5) yields

$$\begin{aligned}\tilde{\mathcal{E}}_\Omega(U(t)) &\leqslant A_1(\tilde{\mathcal{E}}_\Omega(u_0),T_1)\exp\Big\{c_2(u_0)\int\limits_0^t g(\tau)\,d\tau\Big\}\\ &\leqslant A_1(\tilde{\mathcal{E}}_\Omega(u_0),T_1)\exp\big\{B_1(T_1)\big\} \quad \forall\, t \leqslant T_1,\ B_1(T_1) := c_3(u_0)T_1^{\frac{1}{4}} \qquad (5.6)\end{aligned}$$

(here, $A_1(.,.)$ is defined by (5.5)), provided that $\frac{N(\xi-1)}{4(N+2)} \leqslant 1 \Rightarrow \lambda \leqslant \frac{N+2}{N} + \frac{3(\alpha+n)}{4}$; $c_3(u_0)$ is independent of $\|\nabla u_0\|_{L^2(\Omega)}$.

For $N=3$ and $\lambda > \frac{3(\alpha+n)+5}{4}$, using inequalities (5.2) and (5.4), we get

$$\begin{aligned}\tilde{\mathcal{E}}_\Omega(U(t)) &\leqslant \tilde{\mathcal{E}}_\Omega(u_0(x)) + c\int\limits_0^t g(\tau)\,\Big(\tilde{\mathcal{E}}_\Omega(U(\tau))\Big)^{\frac{\xi}{8}}\,d\tau\\ &\quad + c\int\limits_0^t g(\tau)\bigg(\int\limits_\Omega |\nabla U^{\frac{\alpha+n+1}{4}}|^4\bigg)^{\frac{\nu}{4}}\big(\tilde{\mathcal{E}}_\Omega(U(\tau))\big)^{\frac{\lambda+1}{\alpha+n+3}}\,d\tau \quad \forall\, t \leqslant T_1.\end{aligned}$$

Since $\frac{\xi}{8} > \frac{\lambda+1}{\alpha+n+3}$, the last inequality yields

$$\tilde{\mathcal{E}}_\Omega(U(t)) \leqslant \tilde{\mathcal{E}}_\Omega(u_0(x)) + c\int\limits_0^t \tilde{g}(\tau)\big\{1 + \big(\tilde{\mathcal{E}}_\Omega(U(\tau))\big)^{\frac{\xi}{8}}\big\}\,d\tau \quad \forall\, t \leqslant T_1, \ (5.7)$$

where $\tilde{g}(t) := g(t)\big\{1 + \big(\int\limits_\Omega |\nabla U^{\frac{\alpha+n+1}{4}}|^4\big)^{\frac{\nu}{4}}\big\}$, ξ is defined in (5.2), ν is defined in (5.4), and $g(t)$ is defined by (5.5). By virtue of relation (5.1) and the Hölder inequality, we have

$$\int_0^t \tilde{g}(\tau)d\tau \leqslant B_2(T_1) \quad \forall\, t \leqslant T_1,\ B_2(T_1) := c_4(u_0)T_1^{\omega} + c_5(u_0)T_1^{\frac{1}{4}}, \quad (5.8)$$

where $\omega = \frac{\alpha+n-\lambda+2}{\alpha+n+3}$, $\lambda < \alpha+n+2$, and $c_i(u_0)$ are independent of $\|\nabla u_0\|_{L^2(\Omega)}$. Using inequality (5.7), we get

$$\tilde{\mathcal{E}}_\Omega(U(t)) \leqslant A_2(\tilde{\mathcal{E}}_\Omega(u_0), T_1) + c\int_0^t \tilde{g}(\tau)\left(\tilde{\mathcal{E}}_\Omega(U(\tau))\right)^{\frac{\xi}{8}} d\tau$$

where $A_2(\tilde{\mathcal{E}}_\Omega(u_0), T_1) := \tilde{\mathcal{E}}_\Omega(u_0) + B_2(T_1)$ and ω is defined in (5.8). By virtue of relation (5.8) and Lemma B.9, this yields

$$\begin{aligned}\tilde{\mathcal{E}}_\Omega(U(t)) &\leqslant A_2(\tilde{\mathcal{E}}_\Omega(u_0), T_1)\exp\Big\{c\int_0^t \tilde{g}(\tau)d\tau\Big\}\\ &\leqslant A_2(\tilde{\mathcal{E}}_\Omega(u_0), T_1)\exp\{B_2(T_1)\}, \quad (5.9)\end{aligned}$$

provided that $\frac{\xi}{8} \leqslant 1 \Rightarrow \lambda \leqslant \frac{3(\alpha+n)+7}{4}$. Thus, the a priori estimates (5.6) and (5.9) are uniform in $R_0 > 0$ ($\Omega = B(0, 4R_0)$) if the following conditions are satisfied:

$$\begin{aligned} &m > 0,\ \tfrac{1}{8} < n < 2,\\ &\max\left\{\tfrac{3(\alpha+n)-1}{4}, 1\right\} < \lambda \leqslant \tfrac{N+2}{N} + \tfrac{3(\alpha+n)}{4} \text{ for } N < 3,\\ &\max\left\{\tfrac{3(\alpha+n)-1}{4}, 1\right\} < \lambda \leqslant \tfrac{3(\alpha+n)+7}{4} \text{ for } N = 3.\end{aligned} \quad (5.10)$$

It is easy to verify that, for any n, m, and λ from (1.23), there exists a positive $\alpha \in \tilde{\Delta}_{n,\lambda}$ for which relations (5.10) are true.

Using the refined entropy inequality (5.1), we establish an estimate independent of $R_0 > 0$ for the initial energy $G_T(\frac{3}{2}R_0)$ defined by (4.7), thus refining an estimate for the motion of the boundary of the support. Using Lemma B.5 for the estimation of $G_T^{(1)}(\frac{3}{2}R_0)$ ($G_T^{(1)}(s)$ from (4.2)), by analogy with (4.3) we get

$$\begin{aligned}G_T^{(1)}(\tfrac{3}{2}R_0) &\leqslant c\Bigg(\iint_{Q_T} |\nabla U^{\frac{\alpha+m+1}{2}}|^2\Bigg)^{\theta_1}\Bigg(\int_0^T\Big(\int_\Omega U^{\alpha+1}\Big)^{1+\frac{m}{\alpha+1}}\Bigg)^{1-\theta_1}\\ &+ cT\Bigg(\sup_{t\in(0,T)}\int_\Omega U^{\alpha+1}\Bigg)^{\frac{\alpha+m+1}{\alpha+1}} \overset{(5.1)}{\leqslant} c(T + T^{1-\theta_1})\|u_0\|^h_{L^{\alpha+1}(\Omega)}\end{aligned}$$

$$+ c(T + T^{1-\theta_1})\|u_0\|^{\alpha+m+1}_{L^{\alpha+1}(\Omega)} \leqslant c(u_0)(T + T^{1-\theta_1}) \quad \forall T \leqslant T_1,$$

where $h = \frac{(mN+2(m+\alpha+1))(\alpha+1)}{mN+2(\alpha+1)}$ and $c(u_0)$ depends only on $\|u_0\|_{L^{\alpha+1}(\Omega)}$. By analogy, we estimate the energy functions $G^{(i)}_T(\frac{3}{2}R_0)$, $i = 2, 3$ ($G^{(i)}_T(s)$ from (4.2)). As a result, we obtain

$$G_T(\tfrac{3}{2}R_0) \leqslant c_6(u_0)f_3(T), \;\; f_3(T) \text{ is defined by (4.7)}$$

and, hence, relation (4.9) holds for $\tilde{T} = \Gamma^{-1}\big(\frac{R_0}{c_7(u_0)}\big)$. This implies that $\operatorname{supp} U(t,.) \subset B(0, R_0{+}c_7(u_0)\Gamma(t)) \; \forall\, 0{\leqslant}t{\leqslant}T_1{=}\min\big\{T^{(1)}_{loc}, \Gamma^{-1}\big(\frac{R_0}{c_7(u_0)}\big)\big\}$, where $\Gamma(t)$ is defined in Theorem 1.2 and $c_7(u_0)$ depends only on $\|u_0\|_{L^{\alpha+1}(\Omega)}$ and is independent of $R_0 > 0$. Also note that the constructed solution $U_1(t, x) := U(t, x)$, $(t, x) \in (0, T_1) \times \mathbb{R}^N$, satisfies estimates (5.1), (5.6), and (5.9).

We now construct an extension of the solution $U(t, x)$ for $t \geqslant T_1$. Denote $R_1 := R_0 + c_7(u_0)\Gamma(T_1)$ (it is obvious that $R_1 \leqslant 2R_0$) and consider problem (N) in the domain $\Omega_2 := B(0, 4R_1)$. Following the proofs of Theorems 1.1 and 1.2, we construct a solution $U_2(t, x)$ of problem (N) in the domain $(T_1, T_1 + T_2) \times \Omega_2$ with the initial condition $U_2(T_1, x) = U_1(T_1, x)$. Here $T_2 := \min\big\{T^{(2)}_{loc},\, \Gamma^{-1}\big(\frac{R_1}{c_7(u_0)}\big) + T_1\big\}$, $T^{(2)}_{loc} = T^{(2)}_{loc}(\mathcal{E}_\Omega(U_1(T_1)), \|U_1(T_1)\|_{L^{\alpha+1}(\Omega_2)}, \Omega_2)$, $\mathcal{E}_\Omega(.)$ is defined in assertion (h) of Theorem A.1. Furthermore, $\operatorname{supp} U_2(t,.) \subset B(0, R_1 + c_7(U_1(T_1))\Gamma(t - T_1)) \; \forall\, t \in [T_1, T_1{+}T_2]$, and the following estimates analogous to estimates (5.1), (5.6), and (5.9) are true:

$$\|U_2(t,.)\|_{L^{\alpha+1}(\Omega_2)} \leqslant \|U_1(T_1,.)\|_{L^{\alpha+1}(\Omega_2)} \; \forall\, t \in [T_1, T_1 + T_2], \tag{5.11}$$

$$\tilde{\mathcal{E}}_\Omega(U_2(t,.)) \leqslant A_i(\tilde{\mathcal{E}}_\Omega(U_1(T_1)), T_1)\exp\{B_i(T_1)\} \; \forall\, t \in [T_1, T_1{+}T_2], \tag{5.12}$$

where $A_i(.,.), B_i(.)$ $(i = 1, 2)$ are defined by relations (5.5)–(5.9). By virtue of estimates (5.1), (5.6), (5.9), (5.11), and (5.12), it is easy to verify that the function

$$U(t, x) := \begin{cases} U_1(t, x), \; (t, x) \in [0, T_1] \times \mathbb{R}^N, \\ U_2(t, x), \; (t, x) \in [T_1, T_1 + T_2] \times \Omega_2, \\ 0, \; (t, x) \in [T_1, T_1 + T_2] \times \{\mathbb{R}^N \backslash \Omega_2\}, \end{cases}$$

is a solution of the Cauchy problem in the domain $(0, T_1 + T_2) \times \mathbb{R}^N$ with the initial condition $U(0, x) = u_0(x)$. Therefore, by virtue of Theorem 1.2, we can refine the estimate for the support of this solution:

$\operatorname{supp} U(t,.) \subset B(0, R_0 + c_7(u_0)\Gamma(t)) \ \forall\, 0 \leqslant t \leqslant T_1 + T_2$; furthermore, the following estimates analogous to estimates (5.1), (5.6), and (5.9) are true:

$$\|U(t,.)\|_{L^{\alpha+1}(\Omega_2)} \leqslant \|u_0\|_{L^{\alpha+1}(\Omega_2)} \ \forall\, t \leqslant T_1 + T_2, \tag{5.13}$$

$$\tilde{\mathcal{E}}_\Omega(U(t,.)) \leqslant A_i(\tilde{\mathcal{E}}_\Omega(u_0), T_1+T_2) \exp\{B_i(T_1+T_2)\} \ \forall\, t \leqslant T_1+T_2, \tag{5.14}$$

where $A_i(.,.), B_i(.)$ $(i = 1, 2)$ are defined by relations (5.5)–(5.9). We now use the value $U(T_1+T_2, x)$ as an initial function in the Neumann problem in the domain $(T_1+T_2, T_1+T_2+T_3) \times \Omega_3$. Here $T_3 := \min\bigl\{T^{(3)}_{loc}(\mathcal{E}_\Omega(U(T_1+T_2)), \|U(T_1+T_2)\|_{L^{\alpha+1}(\Omega_3)}, \Omega_3), \Gamma^{-1}\bigl(\frac{R_2}{c_7(u_0)}\bigr) + T_1 + T_2\bigr\}$, $\Omega_3 := B(0, 4R_2)$, and $R_2 := R_0 + c_7(u_0)\Gamma(T_1 + T_2)$. By virtue of Theorems 1.1, 1.2, we obtain a solution $U_3(t, x)$ such that $\operatorname{supp} U_3(t,.) \Subset \Omega_3 \ \forall\, t \in [T_1+T_2, T_1+T_2+T_3]$. Therefore, extending this solution by zero outside Ω_3, we obtain the corresponding solution of the Cauchy problem in $(0, T_1+T_2+T_3) \times \mathbb{R}^N$ with a unique estimate of the support:

$$\operatorname{supp} U(t,.) \subset B(0, R_0 + c_7(u_0)\Gamma(t)) \ \forall\, 0 \leqslant t \leqslant T_1 + T_2 + T_3.$$

We repeat this procedure k times and pass to the limit as $k \to \infty$. As a result, we obtain a solution $U(t, x)$ for all $t \leqslant T^* = \sum\limits_{i=1}^{\infty} T_i$. Let us show that $T^* = \infty$. Assume that this is not true. Then, for an arbitrarily small $\varepsilon > 0$, we can choose a number $k = k(\varepsilon)$ such that $\sum\limits_{i=1}^{k} T_i = T', T^* - T' < \varepsilon$, and the solution $U(t, x)$ is constructed on the interval $(0, T')$. Then, using $U(T', x)$ as an initial function and following Theorems 1.1 and 1.2, we construct a solution $U_{k+1}(t, x)$ of problem (N) in the domain $(T', T' + T_{k+1}) \times \Omega_{k+1}$. Here $\Omega_{k+1} := B(0, 4R_{k+1}), R_{k+1} := R_0 + c_7(u_0)\Gamma(T')$, and $T_{k+1} := \min\bigl\{T^{(k+1)}_{loc}(\mathcal{E}_\Omega(U(T')), \|U(T')\|_{L^{\alpha+1}(\Omega_{k+1})}, \Omega_{k+1}), \Gamma^{-1}\bigl(\frac{R_{k+1}}{c_7(u_0)}\bigr) + T'\bigr\}$. Since the support of this solution is finite in Ω_{k+1} for all $t \in [T', T' + T_{k+1}]$, we conclude that $U_{k+1}(t, x)$ is a solution of the corresponding Cauchy problem that defines an extension of the solution $U(t, x)$ to the interval $(0, T' + T_{k+1})$. By virtue of the definition of T_{k+1}, it is easy to see that T_{k+1} does not depend on k and ε and depends only on T^*; furthermore, $T_{k+1} = T_{k+1}(T^*) \to 0$ only if $T^* \to \infty$. Therefore, we can extend our solution $U(t, x)$ to the interval $(0, T^* - \varepsilon + T_{k+1}(T^*))$, where $T^* - \varepsilon + T_{k+1}(T^*) > T^*$, which contradicts the assumption. This means that $T^* = \infty$. Theorem 1.3 is proved.

Appendix A

A.1. Construction of the Solutions $u_{\delta\sigma}$

Theorem A.1. *Suppose that $N \in \{1,2,3\}$, $m > -1$, $n > 0$, and, furthermore, $n < m$ if $N = 3$ and $m \geqslant 5$. Then, for arbitrary $\delta > 0$ and $\sigma > 0$ and finite $T > 0$, there exists a pair of functions $(u_{\delta\sigma}, \overrightarrow{J}_{\delta\sigma})$ that is a solution of problem $(N_{\delta\sigma})$ in the following sense:*

(a) $(u_{\delta\sigma}, \overrightarrow{J}_{\delta\sigma}) \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2_*(\Omega)) \times L^2(Q_T)$*;*

(b) the pair $(u_{\delta\sigma}, \overrightarrow{J}_{\delta\sigma})$ satisfies the following integral identity:

$$\int\limits_0^T \langle \partial_t u_{\delta\sigma}(t), \zeta(t)\rangle_{(H^1)^*,H^1}\, dt = \iint\limits_{Q_T} \overrightarrow{J}_{\delta\sigma}\nabla\zeta + \iint\limits_{Q_T} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma})\nabla u_{\delta\sigma}\zeta$$

with an arbitrary function $\zeta \in L^2(0,T;H^1(\Omega))$;

(c) the vector-function $\overrightarrow{J}_{\delta\sigma}$ satisfies the equality

$$\overrightarrow{J}_{\delta\sigma} = m_{\delta\sigma}(u_{\delta\sigma})[\nabla\Delta u_{\delta\sigma} - u_{\delta\sigma}^{m-n}\nabla u_{\delta\sigma}]$$

in the following weak sense:

$$\iint\limits_{Q_T} \overrightarrow{J}_{\delta\sigma}\cdot\vec{\eta} = \tfrac{1}{2}\iint\limits_{Q_T} m''_{\delta\sigma}(u_{\delta\sigma})|\nabla u_{\delta\sigma}|^2\nabla u_{\delta\sigma}\vec{\eta}$$
$$+ \tfrac{1}{2}\iint\limits_{Q_T} m'_{\delta\sigma}(u_{\delta\sigma})|\nabla u_{\delta\sigma}|^2 \operatorname{div}\vec{\eta} + \iint\limits_{Q_T} m'_{\delta\sigma}(u_{\delta\sigma})\,\langle\nabla u_{\delta\sigma}, D\vec{\eta}, \nabla u_{\delta\sigma}\rangle$$
$$+ \iint\limits_{Q_T} m_{\delta\sigma}(u_{\delta\sigma})\nabla u_{\delta\sigma}\nabla\operatorname{div}\vec{\eta} + \iint\limits_{Q_T} H_{\delta\sigma}(u_{\delta\sigma})\operatorname{div}\vec{\eta}$$

for all $\vec{\eta} \in L^\infty(0,T;W^2_\infty(\Omega))$: $\vec{\eta}\cdot\vec{n} = 0$ on $(0,T)\times\partial\Omega$; here, $H_{\delta\sigma}(z) := \int\limits_0^z \tau^{m-n} m_{\delta\sigma}(\tau)\,d\tau$;

(d) $u_{\delta\sigma}(t,.) \underset{t\to 0}{\Rightarrow} u_{0\delta\sigma}(.)$ *in* $L^2(\Omega)$*;*

(e) if $N = 1$ and $s \geqslant 4$, then $u_{\delta\sigma}(t,.) > 0$ in $(0,T)\times\overline{\Omega}$; if $N = 2$ and $s > 4$ or $N = 3$ and $s > 8$, then $u_{\delta\sigma}(t,.) > 0$ in $\overline{\Omega}$ for almost all $t > 0$;

(f) for almost all t and s from (e), the following inequality is true:

$$\int\limits_\Omega G_0(u_{\delta\sigma}(t))dx + \iint\limits_{Q_t}\left[|\Delta u_{\delta\sigma}|^2 + u_{\delta\sigma}^{m-n}|\nabla u_{\delta\sigma}|^2\right]$$
$$\leqslant \int\limits_\Omega G_0(u_{0\delta\sigma}(x))dx + \iint\limits_{Q_t} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma})g_0(u_{\delta\sigma})\nabla u_{\delta\sigma}$$

where $G_0(z) := \int\limits_1^z g_0(\tau)d\tau = \int\limits_1^z \int\limits_1^v \frac{d\tau dv}{m_{\delta\sigma}(\tau)}$;

(*g*) $u_{\delta\sigma}(t,.) \in C^{\beta}(\overline{\Omega})$ *for almost all* $t > 0$ *and any* $\beta \in \left[0, \frac{4-N}{2}\right)$, $N < 4$;

(*h*) *for almost all* $0 \leqslant t_1 < t_2$ ($t_1 = 0$ *if* $m - n + 2 \leqslant 0$) *and any* $q' \in (1,2)$ ($q' = 2$ *if* $N = 1$), *the following estimate is true for the flow* $\overrightarrow{J}_{\delta\sigma}$:

$$\int\limits_{t_1}^{t_2} \|\overrightarrow{J}_{\delta\sigma}(t)\|^2_{L^{q'}(\Omega)} dt \leqslant 2 \sup_{t\in(t_1,t_2)} \|m_{\delta\sigma}(u_{\delta\sigma}(t))\|_{L^{\frac{q'}{2-q'}}(\Omega)}$$
$$\times\left[\mathcal{E}_\Omega(u_{\delta\sigma}(t_1)) - \mathcal{E}_\Omega(u_{\delta\sigma}(t_2)) + \mathcal{I}_{\delta\sigma}(u_{\delta\sigma})\right] \quad (A.1)$$

where

$$\mathcal{I}_{\delta\sigma}(u_{\delta\sigma}) := \tilde{\mathcal{I}}_{\delta\sigma}(u_{\delta\sigma}) + \iint\limits_{Q_{t_1}^{t_2}} \{\tfrac{1}{2} m_{\delta\sigma}(u_{\delta\sigma})|\nabla u_{\delta\sigma}|^2 + \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) u_{\delta\sigma} \nabla u_{\delta\sigma}\},$$

$$\tilde{\mathcal{I}}_{\delta\sigma}(u_{\delta\sigma}) := \iint\limits_{Q_{t_1}^{t_2}} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma})\{-\Delta u_{\delta\sigma} + \Psi'_0(u_{\delta\sigma})\}\nabla u_{\delta\sigma},$$

$$\mathcal{E}_\Omega(u_{\delta\sigma}(t)) := \tilde{\mathcal{E}}_\Omega(u_{\delta\sigma}(t)) + \tfrac{1}{2}\|u_{\delta\sigma}(t)\|^2_{L^2(\Omega)},$$

$\Psi_0(z)$ *is defined by* (1.14), *and* $\tilde{\mathcal{E}}_\Omega(u_{\delta\sigma}(t))$ *is defined by* (1.20);

(*i*) *for almost all* $0 \leqslant t_1 < t_2$ ($t_1 = 0$ *if* $m - n + 2 \leqslant 0$), *the following inequalities are true:*

$$\mathcal{E}_\Omega(u_{\delta\sigma}(t_2)) \leqslant \mathcal{E}_\Omega(u_{\delta\sigma}(t_1)) + \mathcal{I}_{\delta\sigma}(u_{\delta\sigma}), \quad \mathcal{I}_{\delta\sigma}(u_{\delta\sigma}) \textit{ from } (A.1), \quad (A.2)$$

$$\tilde{\mathcal{E}}_\Omega(u_{\delta\sigma}(t_2)) \leqslant \tilde{\mathcal{E}}_\Omega(u_{\delta\sigma}(t_1)) + \tilde{\mathcal{I}}_{\delta\sigma}(u_{\delta\sigma}), \quad \tilde{\mathcal{I}}_{\delta\sigma}(u_{\delta\sigma}) \textit{ from } (h); \quad (A.3)$$

(*j*) *under conditions* (*e*), *for any* $\alpha \in \left(\frac{1}{2} - n, 2 - n\right) \setminus \{0, -1\}$ *and* $T > 0$, *there exist positive constants* c_1 *and* $C_2(\Omega)$ *such that*

$$\sup_{t\in[0,T]} \int\limits_\Omega G^{(\alpha)}_{\delta\sigma}(u_{\delta\sigma}(t))\, dx + c_1^{-1} \iint\limits_{Q_T} \{|D^2 u_{\delta\sigma}^{\frac{\alpha+n+1}{2}}|^2 + |\nabla u_{\delta\sigma}^{\frac{\alpha+n+1}{4}}|^4\}$$
$$+ c_1^{-1} \iint\limits_{Q_T} u_{\delta\sigma}^{\alpha+m-1} |\nabla u_{\delta\sigma}|^2 \leqslant \int\limits_\Omega G^{(\alpha)}_{\delta\sigma}(u_{0\delta\sigma})\, dx + C_2(\Omega) \iint\limits_{Q_T} u_{\delta\sigma}^{\alpha+n+1}$$
$$+ \iint\limits_{Q_T} \overrightarrow{\chi}\, b'_{\delta\sigma}(u_{\delta\sigma}) g^{(\alpha)}_{\delta\sigma}(u_{\delta\sigma}) \nabla u_{\delta\sigma} \quad (A.4)$$

where $G^{(\alpha)}_{\delta\sigma}(z) := \frac{1}{\alpha(\alpha+1)}z^{\alpha+1} + \frac{\delta}{(s-\alpha-n)(s-\alpha-n-1)}z^{\alpha+n+1-s} + \frac{\sigma}{(\alpha+n)(\alpha+n+1)}$ $\times z^{\alpha+n+1}$, $g^{(\alpha)}_{\delta\sigma}(z) := (G^{(\alpha)}_{\delta\sigma}(z))'$, $(g^{(\alpha)}_{\delta\sigma}(z))' = \frac{z^{\alpha+n-1}}{m_{\delta\sigma}(z)}$, *and* $C_2(\Omega) = 0$ *if the domain* Ω *is convex.*

A.2. Proof of Corollary 1.1

We write the following two simple corollaries of the interpolation Lemmas B.4 and B.5:

$$\int\limits_{\Omega} v^a \leqslant c\Big(\int\limits_{\Omega} |\nabla v|^2\Big)^{\frac{\theta_1 a}{2}} \Big(\int\limits_{\Omega} v^b\Big)^{\frac{(1-\theta_1)a}{b}} + c\Big(\int\limits_{\Omega} v^b\Big)^{\frac{a}{b}} \tag{A.5}$$

where $\theta_1 = \frac{2N(a-b)}{a(2N+b(2-N))}$ and $\frac{\theta_1 a}{2} + \frac{(1-\theta_1)a}{b} = \frac{2a+N(2-b)}{2N+b(2-N)} > 1$, $b < a$ (< 6 for $N = 3$), and

$$\int\limits_{\partial\Omega} v^a \leqslant c\Big(\int\limits_{\Omega} |\nabla v|^2\Big)^{\frac{\theta_2 a}{2}} \Big(\int\limits_{\Omega} v^b\Big)^{\frac{(1-\theta_2)a}{b}} + c\Big(\int\limits_{\Omega} v^b\Big)^{\frac{a}{b}} \tag{A.6}$$

where $\theta_2 = \frac{2(aN-b(N-1))}{a(2N+b(2-N))}$, $a > \frac{b(N-1)}{N}$, and $\frac{\theta_2 a}{2} + \frac{(1-\theta_2)a}{b} = \frac{2a+(N-1)(2-b)}{2N+b(2-N)} > 1$, $\frac{b+2}{2} < a$ (< 4 for $N = 3$). Denote

$$E_b(v(t,.)) = \int\limits_{\Omega} \{|\nabla v(t,.)|^2 + v^b(t,.)\}dx. \tag{A.7}$$

We successively estimate the right-hand sides of inequalities (1.18) and (1.20). Since $n < 2$, the interval $\tilde{\Delta}_{n,\lambda} \cap (0, +\infty)$ ($\tilde{\Delta}_{n,\lambda}$ is defined in (1.12)) is nonempty, and we can choose a strictly positive parameter α. Applying inequality (A.6) to the function $v = u$ for $a = \lambda + \alpha$ and $b = \alpha + 1$ and integrating with respect to t, we get

$$\Big|\int\limits_0^T \int\limits_{\partial\Omega} \overrightarrow{\chi} \cdot \vec{n}\, \mathcal{B}^{(\alpha)}(u)\Big| \overset{(1.6)}{\leqslant} c \int\limits_0^T \int\limits_{\partial\Omega} u^{\lambda+\alpha} \leqslant c \int\limits_0^T \{E^{s_1}_{\alpha+1}(u(t)) + E^{s_2}_{\alpha+1}(u(t))\}\, dt$$

for $\max\{1, \frac{3-\alpha}{2}\} < \lambda$ ($< 4 - \alpha, N = 3$). Here, $s_1 = \frac{2(\lambda+\alpha)+(1-\alpha)(N-1)}{2N+(\alpha+1)(2-N)}$, $s_2 = \frac{\lambda+\alpha}{\alpha+1}$, $s_i > 1$, and $\mathcal{B}^{(\alpha)}(z)$ is defined in (1.18). Applying inequality (A.6) to $v = u$ for $a = 4\lambda - 3(\alpha + n) + 1$ and $b = \alpha + 1$ and integrating with respect to t, we obtain (see (2.19))

$$\Big| \iint\limits_{Q_T} \overrightarrow{\chi} b'(u)\Delta u \nabla u \Big| \overset{(1.6)}{\leqslant} \gamma \iint\limits_{Q_T} \big| D^2 u^{\frac{\alpha+n+1}{2}} \big|^2 + \gamma\, B_0$$
$$+ c \iint\limits_{Q_T} u^{4\lambda-3(\alpha+n)+1} \leqslant \gamma \iint\limits_{Q_T} \big| D^2 u^{\frac{\alpha+n+1}{2}} \big|^2 + \gamma\, B_0$$
$$+ c \int\limits_0^T \{E^{s_3}_{\alpha+1}(u(t)) + E^{s_4}_{\alpha+1}(u(t))\} dt \quad \forall \gamma > 0$$

for $\alpha + \frac{3n}{4} < \lambda$ $(< \frac{3(\alpha+n)+5}{4}$, $N = 3)$. Here, B_0 is defined in (2.3), $s_3 = \frac{4\lambda-3(\alpha+n)+1}{\alpha+1}$, $s_4 = \frac{2(4\lambda-3(\alpha+n)+1)+N(1-\alpha)}{2N+(\alpha+1)(2-N)}$, and $s_i > 1$. Integrating inequality (A.6) with $v = u$, $a = \lambda+m-n+1$, $b = \alpha+1$, and $m-n+1 > 0$ with respect to t, we get

$$\Big| \int\limits_0^T \int\limits_{\partial\Omega} \overrightarrow{\chi} \cdot \vec{n}\, \tilde{\Psi}_0(u) \Big| \overset{(1.14)}{\underset{(1.6)}{\leqslant}} c \int\limits_0^T \int\limits_{\partial\Omega} u^{\lambda+m-n+1}$$
$$\leqslant c \int\limits_0^T \{E^{s_5}_{\alpha+1}(u(t)) + E^{s_6}_{\alpha+1}(u(t))\}\, dt, \quad E_b \text{ is defined in (A.7)},$$

for $\max\{n - m + \frac{\alpha+1}{2}, n - m + \alpha\} < \lambda$ $(< n - m + 3$, $N = 3)$. Here, $s_5 = \frac{2(\lambda+m-n+1)+(1-\alpha)(N-1)}{2N+(\alpha+1)(2-N)}$, $s_6 = \frac{\lambda+m-n+1}{\alpha+1}$, $s_i > 1$, and $\tilde{\Psi}_0(z)$ is defined in (1.20).

Adding inequalities (1.18) and (1.20) together and taking into account the estimates obtained above and the fact that $u_0(x) \equiv 0$, we obtain

$$E_{\alpha+1}(u(T,x)) \leqslant c \int\limits_0^T \{E^{s_{\min}}_{\alpha+1}(u(t)) + E^{s_{\max}}_{\alpha+1}(u(t))\}\, dt \;\forall\, T \leqslant T_{loc}. \quad \text{(A.8)}$$

Here, $1 < s_{\min} := \min\{s_i\} \leqslant s_{\max} := \max\{s_i\}$ $(i = \overline{1,6})$ if $\max\{1, \frac{3-\alpha}{2}, \alpha+\frac{3n}{4}, n-m+\frac{\alpha+1}{2}, n-m+\alpha\} < \lambda$ $(< \min\{4-\alpha, \frac{3(\alpha+n)+5}{4}, n-m+3\}$ for $N = 3)$, $\alpha \in \tilde{\Delta}_{n,\lambda} \cap (0, +\infty)$, and $m-n+1 > 0$. In the case where $N = 3$ and $\min\{4-\alpha, \frac{3(\alpha+n)+5}{4}, n-m+3\} \leqslant \lambda < \alpha+n+2$ $(\alpha \in \tilde{\Delta}_{n,\lambda} \cap (0, +\infty))$, an estimate of the form (A.8) is obtained if one estimates the right-hand sides of (1.18) and (1.20) using inequalities (2.3) and (2.4) (see Sec. 2.2).

Combining all restrictions imposed on the parameters α and $\lambda > 0$, we establish that inequality (A.8) is true if

$$\max\{1, \tfrac{3-\alpha}{2}, \alpha + \tfrac{3n}{4}, n - m + \tfrac{\alpha+1}{2}, n - m + \alpha\} < \lambda\,(< \alpha + n + 2,\ N = 3). \tag{A.9}$$

It is easy to verify that, for all m, n, and λ, by virtue of (1.21) there exists α from the interval $\tilde{\Delta}_{n,\lambda} \cap (0, +\infty)$ such that relation (A.9) is true. Then, by virtue of Lemma B.9, relation (A.8) implies that $u(t, x) \equiv 0$ $\forall\, 0 \leqslant t \leqslant T_{loc}$ (T_{loc} is defined in Theorem 1.1) if inequality (1.21) is true.

Appendix B

Generalized Lebesgue Lemma. Suppose that a set $E \subset \mathbb{R}^N$ is measurable, $g_n \underset{n\to\infty}{\Rightarrow} g$ in $L^q(E)$ with $1 \leqslant q < \infty$, and $f_n, f : E \to \mathbb{R}^N$ are measurable functions such that $f_n \underset{n\to\infty}{\to} f$ a.e. in E and $|f_n|^p \leqslant |g_n|^q$ a.e. in E with $1 \leqslant p < \infty$. Then $f_n \underset{n\to\infty}{\Rightarrow} f$ in $L^p(E)$.

Lemma B.1. ([21]) *Suppose that X, Y, and Z are Banach spaces, $X \Subset Y \subset Z$, and X and Z are reflexive. Then the imbedding $\{u \in L^{p_0}(0, T; X) : \partial_t u \in L^{p_1}(0, T; Z), 1 < p_i < \infty, i = 0, 1\} \Subset L^{p_0}(0, T; Y)$ is compact.*

Lemma B.2. ([24]). *Suppose that X, Y, and Z are Banach spaces and $X \Subset Y \subset Z$. Then the imbedding $\{u \in L^\infty(0, T; X) : \partial_t u \in L^p(0, T; Z),$ $p > 1\} \Subset C(0, T; Y)$ is compact.*

Lemma B.3. ([15]). *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with piecewise-smooth boundary $\partial\Omega$ and $\tilde{m}$, $p \geqslant 1$. Then there exists a constant $d > 0$ that depends only on $\tilde{m}$, p, N and the structure of $\partial\Omega$ and is such that, for all $v \in L^\infty(0, T; L^{\tilde{m}}(\Omega)) \cap L^p(0, T; W^1_p(\Omega))$, the following inequality is true:*

$$\|v\|_{L^q(Q_T)} \leqslant d\Big(1 + \frac{T}{|\Omega|^{\frac{N(p-\tilde{m})+\tilde{m}p}{N\tilde{m}}}}\Big)^{\frac{1}{q}}\Big(\operatorname*{ess\,sup}_{0<t<T} \|v(t)\|_{L^{\tilde{m}}(\Omega)} + \|\nabla v\|_{L^p(Q_T)}\Big)$$

where $q = \frac{p(N+\tilde{m})}{N}$.

Lemma B.4. ([22]). *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary C^1, $a > 1$, $b > 0$, $a > \frac{N-1}{N} b > 0$, and $d > 1$, then there exist positive constants d_1 and d_2 ($d_2 = 0$ if the domain Ω is unbounded) that depend only on Ω, d, b, and N and are such that, for any function $v(x) \in W^1_d(\Omega) \cap L^b(\Omega)$, the following inequality is true:*

$$\|v\|_{L^a(\partial\Omega)} \leqslant d_1 \|\nabla v\|^{\theta}_{L^d(\Omega)} \|v\|^{1-\theta}_{L^b(\Omega)} + d_2 \|v\|_{L^b(\Omega)}\,,\ \theta = \frac{\frac{1}{b} - \frac{N-1}{aN}}{\frac{1}{b} + \frac{1}{N} - \frac{1}{d}} \in (0, 1).$$

Lemma B.5. ([22]). *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with piecewise-smooth boundary, $a > 1$, $b \in (0, a)$, $d > 1$, and $0 \leqslant i < j$, $i, j \in \mathbb{N}$, then there exist positive constants d_1 and d_2 ($d_2 = 0$ if the domain Ω is unbounded) that depend only on Ω, d, j, b, and N and are such that, for any function $v(x) \in W_d^j(\Omega) \cap L^b(\Omega)$, the following inequality is true:*

$$\left\| D^i v \right\|_{L^a(\Omega)} \leqslant d_1 \left\| D^j v \right\|_{L^d(\Omega)}^{\theta} \|v\|_{L^b(\Omega)}^{1-\theta} + d_2 \|v\|_{L^b(\Omega)}$$

where $\theta = \frac{\frac{1}{b} + \frac{i}{N} - \frac{1}{a}}{\frac{1}{b} + \frac{j}{N} - \frac{1}{d}} \in \left[\frac{i}{j}, 1\right)$.

Lemma B.6. ([13]). *Suppose that $N \in \{1, 2, 3\}$, $0 < n$ (< 4 if $N = 3$), $\beta \in \left(\frac{3}{4}, \frac{3}{2}\right)$, and $\mathcal{F}$ is a bounded subset of $L^\infty(0, T; H^1(\Omega))$ such that the following conditions are satisfied:*

(i) for any $f \in \mathcal{F}$, there exists a vector-function $\overrightarrow{J}(f)$ such that

$$f_t = -\operatorname{div} \overrightarrow{J}(f) \text{ in } L^2(0, T; (W_q^1(\Omega))^*) \quad \forall q > \tfrac{4N}{2N + n(2-N)},$$

$$\left\{ \overrightarrow{J}(f) \right\}_{f \in \mathcal{F}} \text{ is bounded in } L^2(0, T; L^{q'}(\Omega)) \; \forall q' < \tfrac{4N}{2N + n(N-2)};$$

(ii) $\{f^\beta\}_{f \in \mathcal{F}}$ is bounded in $L^2(0, T; H^2(\Omega))$.

Then $\{f^\beta\}_{f \in \mathcal{F}}$ is relatively compact in $L^2(0, T; H^1(\Omega))$.

Lemma B.7. ([21]). *Suppose that Q is a bounded domain in $\mathbb{R}_t^+ \times \mathbb{R}_x^N$ and f_n and f_0 are functions from $L^q(Q)$, $1 < q < \infty$, such that $\|f_n\|_{L^q(Q)} \leqslant c$ and $f_n \underset{n \to \infty}{\to} f_0$ a.e. in Q. Then $f_n \underset{n \to \infty}{\rightharpoonup} f_0$ in $L^q(Q)$.*

Lemma B.8. ([17]). *Let $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m, m \geqslant 1$, $\beta = \prod\limits_{j=1}^{m} \beta_j$, and $\overline{\beta_i} = \frac{\beta}{\beta_i} = \prod\limits_{j=1, j \neq i}^{m} \beta_j$. Suppose that nonnegative nondecreasing functions $G_i(s)$ satisfy the conditions*

$$G_i(s - \delta) \leqslant d_i \left(\sum_{i=1}^{m} \frac{G_i(s)}{\delta^{\alpha_i}} \right)^{\beta_i} \; \forall \, s < s_1, \; \delta > 0, \; i = \overline{1, m}$$

with real numbers $d_i > 0$, $\beta_i > 1$, and $\alpha_i > 0$ for $i = \overline{1, m}$. Let $G(s) := \sum\limits_{i=1}^{m} \overline{(d_i^{\overline{\beta_i}})} \, (G_i(s))^{\overline{\beta_i}}$. Then there exists a positive constant $d > 1$ depending on m, α_i, and β_i and such that $G_i(s_0) \equiv 0$ for all $i = \overline{1, m}$ where $s_0 \leqslant s_1 - d \sum\limits_{i=1}^{m} \left(d_i^{\overline{\beta_i}} \overline{(d_i^{\overline{\beta_i}})}^{\,1-\beta_i} (G(s_1))^{\beta_i - 1} \right)^{\frac{1}{\alpha_i \beta}}$.

Lemma B.9. ([12]). *Suppose that $v(t)$ is a nonnegative summable function on $[0,T]$ that, for almost all $t \in [0,T]$, satisfies the integral inequality*

$$v(t) \leqslant k + m \int_0^t h(\tau) g(v(\tau))\, d\tau$$

where $k \geqslant 0, m \geqslant 0$, $h(\tau)$ is summable on $[0,T]$, and $g(\tau)$ is a positive function for $\tau > 0$. Then

$$v(t) \leqslant G^{-1}\left(G(k) + m \int_0^t h(\tau)\, d\tau \right)$$

for almost all $t \in [0,T]$. Here $G(v) = \int\limits_{v_0}^{v} \frac{d\tau}{g(\tau)}$, $v > v_0 > 0$.

References

[1] E. Beretta, M. Bertsch, R. Dal Passo, *Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation* // Arch. Rat. Mech. Anal. **129** (1995), No. 2, 175–200.

[2] F. Bernis, *Viscous flows, fourth order nonlinear degenerate parabolic equations and singular elliptic problems* // In J. I. Diaz, M. A. Herrero, A. Linan, and J. L. Vazquez, eds, Free boundary problems: theory and applications, Pitman Research Notes in Mathematics. Longman, Harlow, **323** (1995), 40–56.

[3] F. Bernis, *Finite speed of propagation and continuity of the interface for thin viscous flows* //Adv. Diff. Equ. **1** (1996), No. 3, 337–368.

[4] F. Bernis, *Finite speed of propagation for thin viscous flows when* $2 \leqslant n < 3$ // C. R. Acad. Sci. Paris; Ser. I Math **322** (1996), 1169–1174.

[5] F. Bernis, A. Friedman, *Higher order nonlinear degenerate parabolic equations* // J. Diff. Equ. **83** (1990), 179–206.

[6] A. L. Bertozzi, A. Münch, M. Shearer, *Undercompressive shocks in thin film flows* // Physica D. **134** (1999), 431–464.

[7] A. L. Bertozzi, A. Münch, M. Shearer, K. Zumbrun, *Stability of compressive and undercompressive thin film traveling waves. The dynamics of thin fluid films* // Euro. J. Appl. Math. **12** (2001), No. 3, 253–291.

[8] A. L. Bertozzi, M. Pugh, *The lubrication approximation for thin viscous films: the moving contact line with a porous media cutoff of the Van der Waals interactions* // Nonlinearity. **7** (1994), 1535–1564.

[9] A. L. Bertozzi, M. Pugh, *Long-wave instabilities and saturation in thin film equations* // Commun. Pure Appl. Math. **51** (1998), No. 6, 625–661.

[10] A. L. Bertozzi, M. Shearer, *Existence of undercompressive traveling waves in thin film equations* // SIAM J. Math. Anal. **32** (2000), 194–213.

[11] M. Bertsch, R. Dal Passo, H. Garcke, G. Grün, *The thin viscous flow equation in higher space dimension* // Adv. Diff. Equ. **3** (1998), 417–440.

[12] I. Bihari, *A generalization of a lemma of Bellman and its applications to uniqueness problems of differential equations* // Acta Math. Hung. **7** (1956), 81–94.

[13] R. Dal Passo, H. Garcke, G. Grün, *On a fourth-order degenerate parabolic equation: Global entropy estimates, existence and qualitative behavior of solutions* // SIAM J. Math. Anal. **29** (1998), No. 2, 321–342.

[14] R. Dal Passo, L. Giacomelli, A. Shishkov, *The thin film equation with nonlinear diffusion* // Commun. in Partial Diff. Equ. **26 (9&10)** (2001), 1509–1557.

[15] E. DiBenedetto *Degenerate parabolic equations*, Springer-Verlag, Berlin, 1993.

[16] C. M. Elliot, H. Garcke, *On the Cahn-Hilliard equation with degenerate mobility* // SIAM J. Math. Anal. **27** (1996), No. 2, 404–423.

[17] L. Giacomelli, A. Shishkov, *Propagation of support in one-dimensional convected thin-film flow* // Rome, Preprint IAC "Mauro Picone"; Sept. 2003, 22 pp. (to appear in Ind. Univ. Math. J.).

[18] G. Grün, *Degenerate parabolic differential equations of fourth order and plasticity model with non-local hardening* // Z. Anal. Anwendungen. **14** (1995), 541–574.

[19] J. Hulshof, A. Shishkov, *The thin film equation with $2 \leqslant n < 3$: Finite speed of propagation in terms of the L^1–norm* // Adv. Diff. Equ. **3** (1998), 625–642.

[20] R. Kersner, A. Shishkov, *Existence of free-boundaries in thin-film theory* // Donetsk, Preprint IAMM NASU; No. **6** (1996), 15 pp.

[21] J.-L. Lions, *Quelques methodes de resolution des problemes aux limites non lineaires*, Dunod Gauthier–Villars, 1969.

[22] L. Nirenberg, *An extended interpolation inequality* // Ann. Scuola Norm. Sup. Pisa **20** (1966), 733–737.

[23] A. Oron, S. H. Davis, G. Bankoff, *Long-scale evolution of thin liquid films* // Rev. Mod. Phys. **69** (1997), No. 3, 931–980.

[24] J. Simon, *Compact sets in the space $L^p(0,T;B)$* // Annali di Mathematica Pure ed Applicata. **146** (1988), 65–96.

[25] R. M. Taranets, *Propagation of perturbations in equations of thin capillary films with nonlinear absorption* // Tr. Inst. Prikl. Mat. Mekh. **8** (2003), 181–194.

[26] R. M. Taranets, A. E. Shishkov, *Waiting time phenomenon in thin-film equations* // Ukr. Mat. Zh. **55** (2003), No. 7, 935–952.