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NEW DISSIPATED ENERGY FOR THE UNSTABLE THIN FILM EQUATION

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Abstract. The fluid thin film equation $h_t = -(h^n h_{xxx})_x - a_1 (h^m h_x)_x$ is known to conserve mass $\int h\,dx$, and in the case of $a_1 \le 0$, to dissipate entropy $\int h^{3/2-n}\,dx$ (see [8]) and the L^2-norm of the gradient $\int h_x^2\,dx$ (see [3]). For the special case of $a_1 = 0$ a new dissipated quantity $\int h^{\alpha} h_x^2\,dx$ was recently discovered for positive classical solutions by Laugesen (see [15]). We extend it in two ways. First, we prove that Laugesen's functional dissipates strong nonnegative generalized solutions. Second, we prove the full α-energy $\int \big(\frac{1}{2} h^{\alpha} h_x^2 - \frac{a_1\, h^{\alpha+m-n+2}}{(\alpha+m-n+1)(\alpha+m-n+2)}\big)\,dx$ dissipation for strong nonnegative generalized solutions in the case of the unstable porous media perturbation $a_1 > 0$ and the critical exponent $m = n + 2$.

1. Introduction.

It is well known that analysis of the existence, uniqueness and regularity of weak solutions for nonlinear evolution equations relies heavily on a priori estimates. Often, the physical energy or entropy which originate from the related model can provide non-increasing in time quantities. Unfortunately, it is far from obvious how to construct new non-increasing Lyapunov type functionals. A general algebraic approach to the construction of entropies in higher-order nonlinear PDEs can be found in [14] and can be applied to analyse thin film equations with stabilizing porus media type perturbations. In this paper, inspired by Laugesen's result [15] on dissipation, we prove that the energy functional introduced in [15] dissipates strong nonnegative generalized solutions. However, our method of the proof is only applicable to some subset of the Laugsen's dissipation region [15] (see the shaded area on Figure 1).

We study the longwave-unstable generalized thin film equation

$$h_t = -(h^n\, h_{xxx})_x - a_1\,(h^m\, h_x)_x, \tag{1.1}$$

where $h(x,t)$ gives the height of the evolving free-surface. The exponent n plays a stabilizing role due to fourth-order forward diffusion term and the exponent m plays a destabilizing role due to backward second-order diffusion term for the case

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when $a_1 > 0$. This class of equations originates from many physical/industrial applications involving air-fluid-solid interface. For example: the case $n = 1, \quad m = 1$ describes a thin jet in a Hele-Shaw cell [10], the case $n = 3, \quad m = -1$ describes Van der Waals driven rupture of thin films [19], the case $m = n = 3$ describes shape of fluid droplets hanging from a ceiling [11], and the case $n = 0, \quad m = 1$ describes solidification of a hyper-cooled melt (this is a modified Kuramoto-Sivashinsky equation) [4].

To prove that the nonnegativity property is preserved in nonlinear thin film equation $h_t = -(h^n h_{xxx})_x$ for $n \geq 1$ (case $a_1 = 0$) Bernis and Friedman [3] used set of dissipated and conserved quantities: mass conservation $\int h\,dx = M$, surface energy dissipation $\frac{d}{dt}\int h_x^2\,dx \leq 0$, and entropy dissipation $\frac{d}{dt}\int h^{2-n}\,dx \leq 0$. The new so-called β-entropy $\int h^{2-n+\beta}\,dx$ was introduced by Bertozzi and Pugh [5] and independently and simultaneously by Beretta, Bertsch, Dal Passo [1] to extend this result to $n > 0$. They also successfully used this new entropy to obtain exponential with respect to the L^∞-norm convergence toward the mean value steady state solution. To analyse this convergence rate in H^1-norm for the special case $n = 1,\ a_1 = 0$ Carlen and Ulusoy [9] used the dissipated energy $\int h^\alpha h_x^2\,dx$ constructed by Laugesen [15] for classical positive solutions. Exponential asymptotic convergence toward the mean value was also studied by Tudorascu in [18]. This list of connections between new properties of solutions in thin film PDEs proved by means of newly discovered dissipated quantities is far from complete.

In this paper we prove that there exists a subinterval I of $-1 < \alpha < 1$ (I depends on n only) and a nonnegative strong generalized solution such that for any $\alpha \in I$ the full α-energy

$$\mathcal{E}_0^{(\alpha)}(t) = \int_\Omega \left(\tfrac{1}{2}\,h^\alpha\,h_x^2 - \tfrac{a_1\,h^{\alpha+m-n+2}}{(\alpha+m-n+1)(\alpha+m-n+2)}\right)dx$$

dissipates. For the unstable porus media perturbation case $a_1 > 0$ this dissipation is proven under the assumptions that the total mass of the solution is less than or equal to the critical one, $m = n + 2$ and domain Ω is unbounded or h is compactly supported. For the stable case $a_1 \leq 0$ no such assumptions are needed.

We proceed as follows. First, we show the dissipation for the classical solutions of the regularized problem and then we take this dissipation to the limit. We prove dissipation of the full α-energy for positive classical solutions of the regularized problem for any value of the coefficient a_1 and without any additional assumptions about the total mass of the solution or its support. However our method of taking the dissipation to the limit due to the Bernis-Friedman method of regularization requires additional conditions for the case $a_1 > 0$.

2. **Auxiliary results to generalized weak solutions.**

We consider nonnegative weak solutions to the following initial–boundary problem:

$$(\mathrm{P})\begin{cases} h_t + (h^n h_{xxx} + a_1 h^m h_x)_x = 0 \text{ in } Q_T, & (2.1)\\ \dfrac{\partial^i h}{\partial x^i}(-a,t) = \dfrac{\partial^i h}{\partial x^i}(a,t) \text{ for } t > 0,\ i = \overline{0,3}, & (2.2)\\ h(0,x) = h_0(x) \geqslant 0, & (2.3)\end{cases}$$

where $h = h(t,x)$, $\Omega = (-a,a)$, $Q_T = (0,T)\times\Omega$, $n > 0$, $m > 0$, and $a_1 \in \mathbb{R}^1$. We define a generalized weak solution in the Bernis-Friedman sense (see, e.g. [1, 3]).

Definition 2.1 (generalized weak solution)**.** Let $n > 0$, $m > 0$, and $a_1 \in \mathbb{R}^1$. A generalized weak solution of problem (P) is a function h satisfying

$$h \in C^{1/2,1/8}_{x,t}(\overline{Q}_T) \cap L^\infty(0,T;H^1(\Omega)), \tag{2.4}$$

$$h_t \in L^2(0,T;(H^1(\Omega))'), \tag{2.5}$$

$$h \in C^{4,1}_{x,t}(\mathcal{P}), \ \ h^{\frac{n}{2}}(h_{xxx} + a_1 h^{m-n} h_x) \in L^2(\mathcal{P}), \tag{2.6}$$

where $\mathcal{P} = \overline{Q}_T \setminus (\{h = 0\} \cup \{t = 0\})$ and h satisfies (2.1) in the following sense:

$$\int_0^T \langle h_t(\cdot,t), \phi\rangle\, dt - \iint\limits_{\mathcal{P}} h^n(h_{xxx} + a_1 h^{m-n} h_x)\phi_x \, dxdt \ = 0 \tag{2.7}$$

for all $\phi \in C^1(Q_T)$ with $\phi(-a,\cdot) = \phi(a,\cdot)$;

$$h(\cdot,t) \to h(\cdot,0) = h_0 \text{ pointwise \& strongly in } L^2(\Omega) \text{ as } t \to 0, \tag{2.8}$$

$$h(-a,t) = h(a,t)\ \forall t \in [0,T] \text{ and } \frac{\partial^i h}{\partial x^i}(-a,t) = \frac{\partial^i h}{\partial x^i}(a,t) \tag{2.9}$$

for $i = \overline{1,3}$ at all points of the lateral boundary where $\{h \neq 0\}$.

Because the second term of (2.7) has an integral over $\mathcal{P}$ rather than over Q_T, the generalized weak solution is "weaker" than a standard weak solution. Also note that the first term of (2.7) uses $h_t \in L^2(0,T;(H^1(\Omega))')$; this is different from the definition of weak solution first introduced by Bernis and Friedman [3]; there, the first term was the integral of $h\phi_t$. The proof of the existence of generalized weak solutions follows the ideas of [1, 3, 5, 6, 7, 17].

Let

$$G_0^{(\beta)}(z) := \begin{cases} \dfrac{z^{\beta-n+2}}{(\beta-n+2)(\beta-n+1)} \text{ if } \beta - n \neq \{-1,-2\}, \\ z\ln z - z \text{ if } \beta - n = -1, \\ -\ln z \text{ if } \beta - n = -2, \end{cases} \tag{2.10}$$

$(G_0^{(\beta)}(z))'' = z^{\beta-n}$, and $G_0(z) := G_0^{(0)}(z)$.

Theorem 2.2. *Let $a_1 \in \mathbb{R}^1$, $n > 0$; $m \geqslant n/2$ for $a_1 > 0$, and $m > 0$ for $a_1 \leq 0$.*

(a) [Existence.] Let the nonnegative initial data $h_0 \in H^1(\Omega)$ satisfy

$$\int_\Omega G_0(h_0(x))\, dx < \infty, \tag{2.11}$$

and either 1) $h_0(-a) = h_0(a) = 0$ or 2) $h_0(-a) = h_0(a) \neq 0$ and $\frac{\partial^i h_0}{\partial x^i}(-a) = \frac{\partial^i h_0}{\partial x^i}(a)$ holds for $i = 1,2,3$. Then for some time $T_{loc} > 0$ there exists a nonnegative generalized weak solution, h, on $Q_{T_{loc}}$ in the sense of the definition 2.1. Furthermore,

$$h \in L^2(0,T_{loc};H^2(\Omega)). \tag{2.12}$$

Let

$$\mathcal{E}_0(T) := \int_\Omega \{\tfrac{1}{2}h_x^2(x,T) - a_1 D_0(h(x,T))\}\, dx, \tag{2.13}$$

where $D_0(z) := \frac{z^{m-n+2}}{(m-n+1)(m-n+2)}$. Then the weak solution satisfies

$$\mathcal{E}_0(T) + \iint\limits_{\{h>0\}} h^n(h_{xxx} + a_1 h^{m-n}h_x)^2\, dxdt \leqslant \mathcal{E}_0(0), \tag{2.14}$$

$$\iint\limits_{\{h>0\}} h^n h_{xxx}^2 \, dxdt \leqslant \ const < \infty. \tag{2.15}$$

for all $T \le T_{loc}$. The time of existence, T_{loc}, is determined by a_1, $|\Omega|$, $\int h_0$, $\|h_{0x}\|_2$, and $\int G_0(h_0)$. Moreover, $T_{loc} = +\infty$ for $a_1 \le 0$.

(b) [Regularity.] If the initial data from (a) also satisfies

$$\int_\Omega G_0^{(\beta)}(h_0(x))\, dx < \infty$$

for some $-1/2 < \beta < 1$, $\beta \neq 0$ then there exists $0 < T_{loc}^{(\beta)} \le T_{loc}$ such that the nonnegative generalized weak solution has the extra regularity

$$h^{\frac{\beta+2}{2}} \in L^2(0, T_{loc}^{(\beta)}; H^2(\Omega)) \text{ and } h^{\frac{\beta+2}{4}} \in L^2(0, T_{loc}^{(\beta)}; W_4^1(\Omega)). \tag{2.16}$$

The time of existence, $T_{loc}^{(\beta)}$, is determined by a_1, $|\Omega|$, $\int h_0$, $\|h_{0x}\|_2$, and $\int G_0^{(\beta)}(h_0)$. Moreover, $T_{loc}^{(\beta)} = +\infty$ for $a_1 \le 0$.

There is nothing special about the time T_{loc} in the Theorem 2.2. In the case $a_1 > 0$ and $n/2 \le m < n+2$ (or $m = n+2$ and $M \le M_c$), given a countable collection of times in $[0, T_{loc}]$, one can construct a weak solution for which these bounds will hold at those times. Also, we note that the analogue of Theorem 4.2 in [3] also holds: there exists a nonnegative weak solution with the integral representation

$$\int_0^T \langle h_t(\cdot,t), \phi\rangle \, dt + \iint\limits_{Q_T} (n h^{n-1} h_x h_{xx} \phi_x + h^n h_{xx}\phi_{xx})\, dxdt \tag{2.17}$$
$$- a_1 \iint\limits_{Q_T} h^m h_x \phi_x \, dxdt = 0.$$

3. **Dissipation of energy for nonnegative weak solutions.** The main result of the present paper is the following

Theorem 3.1. *Let $a_1 \in \mathbb{R}^1$, $1/2 < n < 3$; $m \geqslant n/2$ for $a_1 > 0$, and $m > 0$ for $a_1 \le 0$, and*

$$\mathcal{E}_0^{(\alpha)}(T) := \int_\Omega \{\tfrac{1}{2} h^\alpha h_x^2(x,T) - a_1 \tilde{D}_0(h(x,T))\}\, dx, \tag{3.1}$$

where $\tilde{D}_0(z) := \frac{z^{\alpha+m-n+2}}{(\alpha+m-n+1)(\alpha+m-n+2)}$, and $\mathcal{E}_0^{(0)}(T) = \mathcal{E}_0(T)$. Then there exists a non-empty subinterval I (see [15] for the explicit form of the I) of $0 \le \alpha < 1$ for $\frac{1}{2} < n < 3$, and of $\frac{3}{2} - n < \alpha < 0$ for $\frac{3}{2} < n < 3$ such that for any $\alpha \in I$ the nonnegative weak solution from Theorem 2.2 satisfies the following estimates:

(i) if $a_1 \leqslant 0$ then

$$\mathcal{E}_0^{(\alpha)}(T) \leqslant \mathcal{E}_0^{(\alpha)}(0); \tag{3.2}$$

(ii) if $a_1 > 0$ then

$$\mathcal{E}_0^{(\alpha)}(T) \leqslant \mathcal{E}_0^{(\alpha)}(0) + C_1 \iint\limits_{Q_T} h^{\alpha+3m-2n+2} dxdt \text{ for } m > n+2; \tag{3.3}$$

$$\mathcal{E}_0^{(\alpha)}(T) \leqslant \mathcal{E}_0^{(\alpha)}(0) + T(C_1 M^{\frac{2\alpha+5m-3n+4}{n+2-m}} + C_2 M^{\alpha+3m-2n+2}) \tag{3.4}$$

for $m < n+2$ and $\alpha > 2n - 3m - 1$;

$$\mathcal{E}_0^{(\alpha)}(T) \leqslant \mathcal{E}_0^{(\alpha)}(0) + C_3 T\, M^{\alpha+3m-2n+2} \tag{3.5}$$

for $m < n+2$ and $2n-3m-2 < \alpha \leqslant 2n-3m-1$;

$$\mathcal{E}_0^{(\alpha)}(T) \leqslant \mathcal{E}_0^{(\alpha)}(0) + C_2 T\, M^{\alpha+n+8} \text{ for } m = n+2 \text{ and } 0 < M \leqslant M_c. \tag{3.6}$$

Here $C_2 = 0$ if Ω is unbounded or h has compact support.

Remark 1 (extra regularity)**.** In particular, the extra regularity

$$h^{\frac{\alpha+2}{2}} \in L^\infty(0,T;H^1(\Omega))$$

follows directly from Theorem 3.1. Hence, $h^{\frac{\alpha+2}{2}}(\cdot,T) \in H^1(\Omega)$ for almost all $T \in [0,T_{loc}^{(\beta)}]$ and therefore $h^{\frac{\alpha+2}{2}}(\cdot,T) \in C^{1/2}(\overline{\Omega})$ for almost all $T \in [0,T_{loc}^{(\beta)}]$. Assume that T_0 is chosen such that $h^{\frac{\alpha+2}{2}}(\cdot,T_0) \in C^{1/2}(\overline{\Omega})$ and $h(x_0,T_0) = 0$ at some $x_0 \in \bar{\Omega}$. Then there exists a constant L such that

$$h^{\frac{\alpha+2}{2}}(x,T_0) = |h^{\frac{\alpha+2}{2}}(x,T_0) - h^{\frac{\alpha+2}{2}}(x_0,T_0)| \le L|x-x_0|^{1/2}.$$

Hence $h(x,T_0) \le L^{\frac{2}{\alpha+2}}|x-x_0|^{\frac{1}{\alpha+2}}$, i.e. $h(.,T) \in C^{\frac{1}{\alpha+2}}(\bar{\Omega})$ for almost every $T \in [0,T_{loc}^{(\beta)}]$.

Remark 2 (rate of decrease)**.** For $a_1 \le 0$ and $1/2 < n < 3$ we can generalize the results from [9, Theorem 1.1] in the following way:

$$\int_\Omega h^\alpha h_x^2(x,t)\,dx \le C(1+t)^{-\frac{1}{2}} \text{ for } \tfrac{n-4}{2} \le \alpha < 0,$$

whence $\|h-\bar{h}\|_\infty \le C(1+t)^{-\frac{1}{4}}$ for any nonnegative strong solution h. Here $C = C(a_1,\alpha,n,\bar{h},\mathcal{E}_0^{(\alpha)}(0))$, and $\bar{h} = \frac{1}{|\Omega|}\|h_0\|_1$. The proof is similar to [9].

3.1. **Regularized problem.** Given $\delta,\varepsilon > 0$, a regularized parabolic problem, similar to that of Bernis and Friedman [3], is considered:

$$(\mathrm{P}_{\delta,\varepsilon})\begin{cases} h_t + (f_{\delta\varepsilon}(h)(h_{xxx} + a_1 D_\varepsilon''(h)h_x))_x = 0, & (3.7)\\ \dfrac{\partial^i h}{\partial x^i}(-a,t) = \dfrac{\partial^i h}{\partial x^i}(a,t) \text{ for } t>0,\ i=\overline{0,3}, & (3.8)\\ h(x,0) = h_{0,\varepsilon}(x), & (3.9)\end{cases}$$

where

$$f_{\delta\varepsilon}(z) := f_\varepsilon(z) + \delta = \frac{|z|^{s+n}}{|z|^s + \varepsilon|z|^n} + \delta,\ D_\varepsilon''(z) := \frac{|z|^{m-n}}{1+\varepsilon|z|^{m-n}} \tag{3.10}$$

$\forall\, z \in \mathbb{R}^1$, $\varepsilon > 0$, $s \geqslant 4$. The $\delta > 0$ in (3.10) makes the problem (3.7) regular (i.e. uniformly parabolic). The parameter ε is an approximating parameter which has the effect of increasing the degeneracy from $f(h) \sim |h|^n$ to $f_\varepsilon(h) \sim h^s$. The nonnegative initial data, h_0, is approximated via

$$\begin{gathered} h_{0,\varepsilon} \in C^{4+\gamma}(\Omega),\ h_{0,\varepsilon} \geqslant h_0 + \varepsilon^\theta \text{ for some } 0 < \theta < \tfrac{2}{2s-3}, \\ \frac{\partial^i h_{0,\varepsilon}}{\partial x^i}(-a) = \frac{\partial^i h_{0,\varepsilon}}{\partial x^i}(a) \text{ for } i = \overline{0,3}, \\ h_{0,\varepsilon} \to h_0 \text{ strongly in } H^1(\Omega) \text{ as } \varepsilon \to 0. \end{gathered} \tag{3.11}$$

The ε term in (3.11) "lifts" the initial data so that it will be positive. The function $h_{0,\varepsilon}$ is smoothing the initial data from $H^1(\Omega)$ to $C^{4+\gamma}(\Omega)$.

Sketch of Proof: By Eĭdelman [12, Theorem 6.3, p.302], the regularized problem has the unique classical solution $h_{\delta\varepsilon} \in C_{x,t}^{4+\gamma,1+\gamma/4}(\Omega\times[0,\tau_{\delta\varepsilon}])$ for some time $\tau_{\delta\varepsilon} > 0$. For any fixed values of δ and ε, by Eĭdelman [12, Theorem 9.3, p.316] if one can

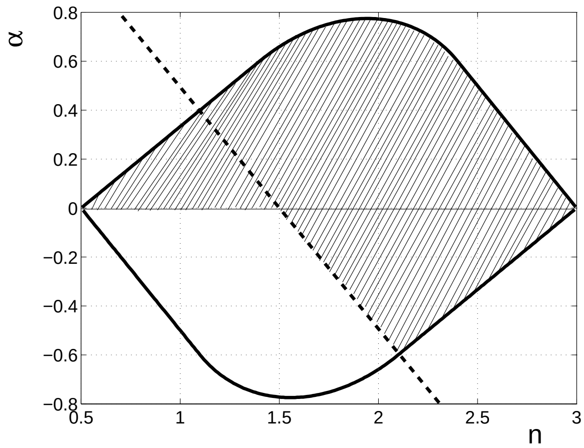

FIGURE 1. The dissipation region computed numerically by Matlab: α versus n. The dashed line corresponds to $\alpha = 3/2 - n$.

prove a uniform in time an a priori bound $|h_{\delta\varepsilon}(x,t)| \le A_\varepsilon < \infty$ for some longer time interval $[0, T_{loc,\varepsilon}]$ $(T_{loc,\varepsilon} > \tau_{\delta\varepsilon})$ and for all $x \in \Omega$ then Schauder-type interior estimates [12, Corollary 2, p.213] imply that the solution $h_{\delta\varepsilon}$ can be continued in time to be in $C^{4+\gamma,1+\gamma/4}_{x,t}(\Omega \times [0, T_{loc,\varepsilon}])$.

Although the solution $h_{\delta\varepsilon}$ is initially positive, there is no guarantee that it will remain nonnegative. The goal is to take $\delta \to 0$, $\varepsilon \to 0$ in such a way that 1) $T_{loc,\varepsilon} \to T_{loc} > 0$, 2) the solutions $h_{\delta\varepsilon}$ converge to a (nonnegative) limit, h, which is a generalized weak solution, and 3) h inherits certain a priori bounds. This is done by proving various a priori estimates for $h_{\delta\varepsilon}$ that are uniform in δ and ε and hold on a time interval $[0, T_{loc}]$ that is independent of δ and ε. As a result, $\{h_{\delta\varepsilon}\}$ will be a uniformly bounded and equicontinuous (in the $C^{1/2,1/8}_{x,t}$ norm) family of functions in $\bar{\Omega} \times [0, T_{loc}]$. Taking $\delta \to 0$ will result in a family of functions $\{h_\varepsilon\}$ that are classical, positive, unique solutions to the regularized problem with $\delta = 0$. Taking $\varepsilon \to 0$ will then result in the desired generalized weak solution h. This last step is where the possibility of non-unique weak solutions arise; see [1] for simple examples of how such constructions applied to $h_t = -(|h|^n h_{xxx})_x$ can result in two different solutions arising from the same initial data.

3.2. **Dissipation of energy for positive solutions.**

Lemma 3.2. *Let α belong to the full domain shown on Figure 1, and*

$$\mathcal{E}^{(\alpha)}_\varepsilon(T) := \int_\Omega \{\tfrac{1}{2} h^\alpha h_x^2(x,T) - a_1 \tilde{D}_\varepsilon(h(x,T))\}\, dx, \tag{3.12}$$

where $\tilde{D}''_\varepsilon(z) := z^\alpha D''_\varepsilon(z)$. Then the unique positive classical solution h_ε of the problem $(P_{0,\varepsilon})$ satisfies

$$\mathcal{E}^{(\alpha)}_\varepsilon(T) \leqslant \mathcal{E}^{(\alpha)}_\varepsilon(0) + \mu \iint\limits_{Q_T} h^{\alpha-2} f_\varepsilon(h) D''_\varepsilon(h) h_x^4 dxdt$$
$$+ \varepsilon\, k_1 \int_\Omega h^{\alpha-s-4} f^2_\varepsilon(h) h_x^6\, dx + \varepsilon^2 k_2 \int_\Omega h^{\alpha-2s-4} f^3_\varepsilon(h) h_x^6\, dx, \tag{3.13}$$

where $k_i = k_i(\alpha, n, s)$ are constants, and $\mu = \mu(\alpha, a_1)$ such that $\mu(0, a_1) = 0$ and $\mu(\alpha, a_1) \leqslant 0$ for $a_1 \leqslant 0$.

Note that, although we use the same convenient notations introduced in [15], the proof of Lemma 3.2 has essential differences from the proof of Theorem 1 of [15]. Indeed, we introduce new ideas in order to estimate the lower-order term in the equation (3.7). In particular, the new quantity N is introduced, the quantity R is modified, and so are the terms involving the regularization parameter ε in (3.19).

Proof of Lemma 3.2. To prove the bound (3.13), multiply (3.7) with $\delta = 0$ by $-\frac{\alpha}{2}h^{\alpha-1}h_x^2 - h^\alpha h_{xx} - a_1\tilde{D}_\varepsilon'(h)$, integrate over Ω, use integration by parts, apply the periodic boundary conditions (3.8), to find

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_\varepsilon^{(\alpha)}(t) = &-\int_\Omega h^\alpha f_\varepsilon(h)(h_{xxx} + a_1 D_\varepsilon''(h)h_x)^2\,dx\\ &-\frac{\alpha}{2}(\alpha-1)\int_\Omega h^{\alpha-2}h_x^3 f_\varepsilon(h)(h_{xxx} + a_1 D_\varepsilon''(h)h_x)\,dx\\ &-2\alpha\int_\Omega h^{\alpha-1}h_x h_{xx} f_\varepsilon(h)(h_{xxx} + a_1 D_\varepsilon''(h)h_x)\,dx. \qquad (3.14)\end{aligned}$$

The equality (3.14) can be rewritten as

$$\frac{d}{dt}\mathcal{E}_\varepsilon^{(\alpha)}(t) = -R^2 - 2\alpha\,RS - \frac{\alpha}{2}(\alpha-1)\,RL, \qquad (3.15)$$

where the quantities

$$\begin{aligned}R &:= \langle (h^\alpha f_\varepsilon(h))^{1/2}(h_{xxx} + a_1 D_\varepsilon''(h)h_x)| = |(h^\alpha f_\varepsilon(h))^{1/2}(h_{xxx} + a_1 D_\varepsilon''(h)h_x)\rangle,\\ S &:= \langle (h^{\alpha-2} f_\varepsilon(h))^{1/2}h_x h_{xx}| = |(h^{\alpha-2} f_\varepsilon(h))^{1/2}h_x h_{xx}\rangle,\\ L &:= \langle (h^{\alpha-4} f_\varepsilon(h))^{1/2}h_x^3| = |(h^{\alpha-4} f_\varepsilon(h))^{1/2}h_x^3\rangle,\\ N &:= \langle (h^{\alpha-2} f_\varepsilon(h)D_\varepsilon''(h))^{1/2}h_x^2| = |(h^{\alpha-2} f_\varepsilon(h)D_\varepsilon''(h))^{1/2}h_x^2\rangle,\end{aligned}$$

each represent half of an inner product in $L^2(\Omega)$. We will need the following integration by parts formulas

$$\begin{aligned}SL &= -\frac{1}{5}(\alpha-3)\int_\Omega h^{\alpha-4}f_\varepsilon(h)h_x^6\,dx - \frac{1}{5}\int_\Omega h^{\alpha-3}f_\varepsilon'(h)h_x^6\,dx\\ &= -\frac{1}{5}(\alpha+n-3)L^2 - \frac{1}{5}\varepsilon(s-n)\int_\Omega h^{\alpha-s-4}f_\varepsilon^2(h)h_x^6\,dx, \qquad (3.16)\end{aligned}$$

$$\begin{aligned}RL &= -(\alpha+n-2)SL - \varepsilon(s-n)\int_\Omega h^{\alpha-s-3}f_\varepsilon^2(h)h_x^4 h_{xx}\,dx - 3S^2 + a_1 N^2\\ &= \frac{1}{5}(\alpha+n-2)(\alpha+n-3)L^2 - 3S^2 + a_1 N^2 - \varepsilon(s-n)\int_\Omega h^{\alpha-s-3}f_\varepsilon^2(h)h_x^4 h_{xx}\,dx\\ &\quad + \frac{1}{5}\varepsilon(s-n)(\alpha+n-2)\int_\Omega h^{\alpha-s-4}f_\varepsilon^2(h)h_x^6\,dx\\ &= \frac{1}{5}(\alpha+n-2)(\alpha+n-3)L^2 - 3S^2 + a_1 N^2\\ &\quad + \frac{1}{5}\varepsilon(s-n)(2\alpha+3n-s-5)\int_\Omega h^{\alpha-s-4}f_\varepsilon^2(h)h_x^6\,dx\\ &\quad + \frac{2}{5}\varepsilon^2(s-n)^2\int_\Omega h^{\alpha-2s-4}f_\varepsilon^3(h)h_x^6\,dx. \qquad (3.17)\end{aligned}$$

Here we use the auxiliary equality $f'_\varepsilon(z) = nz^{-1}f_\varepsilon(z) + \varepsilon(s-n)z^{-(s+1)}f^2_\varepsilon(z)$. Thus, from (3.15) we have

$$\frac{d}{dt}\mathcal{E}^{(\alpha)}_\varepsilon(t) + \frac{\varepsilon^2}{5}\alpha(\alpha-1)(s-n)^2\int_\Omega h^{\alpha-2s-4}f^3_\varepsilon(h)h^6_x\,dx + \frac{a_1}{2}\alpha(\alpha-1)N^2$$
$$= -R^2 - 2\alpha\,RS - \frac{\alpha}{10}(\alpha-1)(\alpha+n-2)(\alpha+n-3)L^2 + \frac{3\alpha}{2}(\alpha-1)S^2$$
$$+\frac{\varepsilon}{10}\alpha(\alpha-1)(s-n)(s-2\alpha-3n+5)\int_\Omega h^{\alpha-s-4}f^2_\varepsilon(h)h^6_x\,dx. \quad (3.18)$$

Our next step is to express (3.18) as the negative of a sum of squares to obtain the energy dissipation. To achieve this, we use (3.16) and (3.17) to deduce that for all $\kappa\in\mathbb{R}^1$,

$$\frac{d}{dt}\mathcal{E}^{(\alpha)}_\varepsilon(t) = -(R+\alpha\,S+\kappa\,L)^2 + \beta(S+\frac{1}{5}(\alpha+n-3)\,L)^2 + \gamma\,L^2 + \mu\,N^2$$
$$+\varepsilon\,k_1\int_\Omega h^{\alpha-s-4}f^2_\varepsilon(h)h^6_x\,dx + \varepsilon^2 k_2\int_\Omega h^{\alpha-2s-4}f^3_\varepsilon(h)h^6_x\,dx, \quad (3.19)$$

where

$$k_1 = \frac{2}{25}(s-n)\Big(5\kappa(\alpha-s+3n-5) + \frac{5\alpha}{4}(\alpha-1)(s-2\alpha-3n+5)$$
$$+(\alpha+n-3)(\frac{\alpha}{2}(5\alpha-3)-6\kappa)\Big),\ k_2 = \frac{1}{5}(s-n)^2\Big(4\kappa-\alpha(\alpha-1)\Big),$$

$$\beta = \frac{\alpha}{2}(5\alpha-3)-6\kappa,\ \ \mu = a_1\Big(2\kappa-\frac{\alpha}{2}(\alpha-1)\Big),$$

$$\gamma = \kappa^2 - \frac{2}{25}\kappa(\alpha+n-3)\big(5(2-n)+3(\alpha+n-3)\big)$$
$$-\frac{\alpha}{50}(\alpha+n-3)\big(5(\alpha-1)(\alpha+n-2)-(5\alpha-3)(\alpha+n-3)\big)$$
$$= \kappa^2 - \frac{6}{25}\kappa(\alpha+n-3)\big(\alpha-\frac{2n-1}{3}\big) - \frac{3}{50}\alpha(\alpha+n-3)\big(\alpha-\frac{2n-1}{3}\big).$$

Now we have to choose the parameter κ in such a way that $\beta\leqslant 0$ and $\gamma\leqslant 0$. In this case, the parameter $\mu>0$ for $a_1>0$, and $\mu\leqslant 0$ for $a_1\leqslant 0$. According to [15], we can find κ such that $\beta\leqslant 0$, and $\gamma\leqslant 0$ when $1/2<n<3$, see also Figure 1 where this region was computed numerically by Matlab (see [15] for the explicit form of the domain).

3.3. **Limit process in (3.13).** Rewrite the integral $\iint\limits_{Q_T}\varepsilon h^{\alpha-s-4}_\varepsilon f^2_\varepsilon(h_\varepsilon)h^6_{\varepsilon,x}\,dxdt$ in the form

$$\iint\limits_{Q_T}\varepsilon h^{\alpha-s-4}_\varepsilon f^2_\varepsilon(h_\varepsilon)h^6_{\varepsilon,x}\,dxdt = \iint\limits_{Q_T}\frac{\varepsilon\,h^{\alpha+s-n}_\varepsilon}{(h^{s-n}_\varepsilon+\varepsilon)^2}h^{n-4}_\varepsilon h^6_{\varepsilon,x}\,dxdt.$$

Using the Young's inequality

$$ab\leqslant\frac{a^p}{p}+\frac{b^q}{q}\Leftrightarrow p\,ab\leqslant a^p+(p-1)b^q,\ \frac{1}{p}+\frac{1}{q}=1, \quad (3.20)$$

with $a = z^{\frac{s-n}{p}}$ and $b = \left(\frac{\varepsilon}{p-1}\right)^{\frac{1}{q}}$, we deduce

$$\varepsilon\frac{z^{\alpha+s-n}}{(z^{s-n}+\varepsilon)^2} \leqslant \varepsilon\frac{z^{\alpha}}{z^{s-n}+\varepsilon} \leqslant \frac{(p-1)^{\frac{1}{q}}}{p}\frac{\varepsilon\, z^{\alpha}}{z^{\frac{s-n}{p}}\varepsilon^{\frac{1}{q}}} = \frac{(p-1)^{\frac{1}{q}}}{p}\varepsilon^{\frac{q-1}{q}} z^{\frac{p\alpha-s+n}{p}},$$

choosing $p = \frac{s-n}{\alpha} > 1$ and $q = \frac{s-n}{s-n-\alpha} > 1$ ($\Rightarrow 0 < \alpha < s-n$), we find

$$\varepsilon\frac{z^{\alpha+s-n}}{(z^{s-n}+\varepsilon)^2} \leqslant \frac{\alpha}{s-n}\Big(\frac{s-n-\alpha}{\alpha}\Big)^{\frac{s-n-\alpha}{s-n}}\varepsilon^{\frac{\alpha}{s-n}}.$$

Similarly, we deal with the integral $\varepsilon^2\int_\Omega h_\varepsilon^{\alpha-2s-4} f_\varepsilon^3(h) h_{\varepsilon,x}^6\,dx$. Due to Lemma A.3 and (2.15), $\iint\limits_{Q_T} h_\varepsilon^{n-4} h_{\varepsilon,x}^6\,dxdt$ is uniformly bounded then

$$\Big|\iint\limits_{Q_T}(k_1\varepsilon h_\varepsilon^{\alpha-s-4} f_\varepsilon^2(h_\varepsilon) + k_2\varepsilon^2 h_\varepsilon^{\alpha-2s-4} f_\varepsilon^3(h_\varepsilon))h_{\varepsilon,x}^6\,dxdt\Big|$$
$$\leqslant C\,\varepsilon^{\frac{\alpha}{s-n}}\iint\limits_{Q_T} h_\varepsilon^{n-4} h_{\varepsilon,x}^6\,dxdt \leqslant C\,\varepsilon^{\frac{\alpha}{s-n}}, \quad (3.21)$$

where the positive constant C is independent of ε. Letting $\varepsilon \to 0$, from (3.21) we obtain

$$(k_1\varepsilon h_\varepsilon^{\alpha-s-4} f_\varepsilon^2(h_\varepsilon) + k_2\varepsilon^2 h_\varepsilon^{\alpha-2s-4} f_\varepsilon^3(h_\varepsilon))h_{\varepsilon,x}^6 \to 0 \text{ in } L^1(Q_T) \quad (3.22)$$

for $0 < \alpha < s-n$.

Now, we show (3.22) for the case of $\alpha < 0$. Rewrite the integral

$$\iint\limits_{Q_T}\varepsilon h_\varepsilon^{\alpha-s-4} f_\varepsilon^2(h_\varepsilon) h_{\varepsilon,x}^6\,dxdt$$

in the form

$$\iint\limits_{Q_T}\varepsilon h_\varepsilon^{\alpha-s-4} f_\varepsilon^2(h_\varepsilon) h_{\varepsilon,x}^6\,dxdt = \iint\limits_{Q_T}\frac{\varepsilon\, h_\varepsilon^{s-2}}{(h_\varepsilon^{s-n}+\varepsilon)^2} h_\varepsilon^{\alpha-2} h_{\varepsilon,x}^6\,dxdt.$$

Using the inequality (3.20) with $a = z^{\frac{s-n}{p}}$ and $b = \left(\frac{\varepsilon}{p-1}\right)^{\frac{1}{q}}$, we obtain

$$\varepsilon\frac{z^{s-2}}{(z^{s-n}+\varepsilon)^2} \leqslant \frac{(p-1)^{\frac{2}{q}}}{p^2}\frac{\varepsilon\, z^{s-2}}{z^{\frac{2(s-n)}{p}}\varepsilon^{\frac{2}{q}}} = \frac{(p-1)^{\frac{2}{q}}}{p^2}\varepsilon^{\frac{q-2}{q}} z^{\frac{p(s-2)-2(s-n)}{p}},$$

choosing $p = \frac{2(s-n)}{s-2+\alpha-\beta} < 2$ and $q = \frac{2(s-n)}{2(s-n)-s+2-\alpha+\beta} > 2$ ($\Rightarrow n > 2-\alpha+\beta$), we find

$$\varepsilon\frac{z^{s-2}}{(z^{s-n}+\varepsilon)^2} \leqslant \Big(\frac{s-2+\alpha-\beta}{2(s-n)}\Big)^2\Big(\frac{s-2(n-1)-\alpha+\beta}{s-2+\alpha-\beta}\Big)^{\frac{s-2(n-1)-\alpha+\beta}{s-n}}\varepsilon^{\frac{n-2+\alpha-\beta}{s-n}} z^{\beta-\alpha},$$

where $\beta \in (-1/2, 1)$ follows from (2.16). Similarly, we deal with the integral $\varepsilon^2\int_\Omega h_\varepsilon^{\alpha-2s-4} f_\varepsilon^3(h) h_{\varepsilon,x}^6\,dx$. Due to $h \in L^\infty(0,T;H^1(\Omega))$ and (2.16), $\iint\limits_{Q_T} h_\varepsilon^{\alpha-2} h_{\varepsilon,x}^6\,dxdt$

is uniformly bounded then

$$\Big|\iint\limits_{Q_T} (k_1\varepsilon h_\varepsilon^{\alpha-s-4}f_\varepsilon^2(h_\varepsilon) + k_2\varepsilon^2 h_\varepsilon^{\alpha-2s-4}f_\varepsilon^3(h_\varepsilon))h_{\varepsilon,x}^6\,dxdt\Big| \leqslant C\,\varepsilon^{\frac{n-2+\alpha-\beta}{s-n}} \iint\limits_{Q_T} h_\varepsilon^{\beta-2}h_{\varepsilon,x}^6\,dxdt \leqslant C\,\varepsilon^{\frac{n-2+\alpha-\beta}{s-n}} \iint\limits_{Q_T} h_\varepsilon^{\beta-2}h_{\varepsilon,x}^4\,dxdt \leqslant C\,\varepsilon^{\frac{n-2+\alpha-\beta}{s-n}}, \quad (3.23)$$

where the positive constant C is independent of ε. Letting $\varepsilon \to 0$, we obtain (3.22) for $\frac{3}{2} - n < \alpha < 0$ and $\frac{3}{2} < n < 3$. In view of the Lebesgue's theorem, we have

$$\iint\limits_{Q_T} h_\varepsilon^{\alpha-2}f_\varepsilon(h_\varepsilon)D_\varepsilon''(h_\varepsilon)h_{\varepsilon,x}^4\,dxdt \to \iint\limits_{Q_T} h^{\alpha+m-2}h_x^4\,dxdt \quad (3.24)$$

if $m > 0$ and $\alpha > -\frac{1}{2} - m$, due to $h \in L^\infty(0,T;H^1(\Omega))$ and (2.16).

Integrating (3.19) over the time interval, and letting $\varepsilon \to 0$, in view of (3.22) and (3.24), we obtain (3.13) for some subinterval I for $0 \le \alpha < 1$ and $\frac{1}{2} < n < 3$ or for $-1 < \alpha < 0$ and $\frac{3}{2} < n < 3$. Note that, the convergence on the left-hand side follows from Fatou's lemma and from the corresponding a priori estimate (see, for example, [3, 5, 7, 17]). □

3.4. **Proof of Theorem 3.1.** Taking the limit $\varepsilon \to 0$ we obtain

$$\mathcal{E}_0^{(\alpha)}(T) + \gamma \iint\limits_{Q_T} h^{\alpha+n-4}h_x^6 dxdt \leqslant \mathcal{E}_0^{(\alpha)}(0) + \mu \iint\limits_{Q_T} h^{\alpha+m-2}h_x^4 dxdt. \quad (3.25)$$

Now, we estimate $\iint\limits_{Q_T} h^{\alpha+m-2}h_x^4 dxdt$. Using the Hölder inequality, we obtain

$$\iint\limits_{Q_T} h^{\alpha+m-2}h_x^4 dxdt \leqslant \int_0^T \Big(\int_\Omega h^{\alpha+n-4}h_x^6 dx\Big)^{\frac{2}{3}} \Big(\int_\Omega h^{\alpha+3m-2n+2} dx\Big)^{\frac{1}{3}} dt. \quad (3.26)$$

Applying Lemma A.4 to $v = h^{\frac{\alpha+n+2}{6}}$ with $a = \frac{6(\alpha+3m-2n+2)}{\alpha+n+2}$, $d = 6$, $b = \frac{6}{\alpha+n+2} < a$ ($\Rightarrow \alpha > 2n - 3m - 1$), $i = 0$, and $j = 1$, we deduce

$$\int_\Omega h^{\alpha+3m-2n+2}dx \leqslant d_1\Big(\int_\Omega v_x^6 dx\Big)^{\frac{\alpha+3m-2n+1}{\alpha+n+7}} \Big(\int_\Omega h\,dx\Big)^{\frac{3(2\alpha+5m-3n+4)}{\alpha+n+7}} + d_2\Big(\int_\Omega h\,dx\Big)^{\alpha+3m-2n+2} \leqslant c_1 M^{\frac{3(2\alpha+5m-3n+4)}{\alpha+n+7}} \Big(\int_\Omega h^{\alpha+n-4}h_x^6 dx\Big)^{\frac{\alpha+3m-2n+1}{\alpha+n+7}} + c_2 M^{\alpha+3m-2n+2}. \quad (3.27)$$

Substituting (3.27) in (3.26), we find

$$\iint\limits_{Q_T} h^{\alpha+m-2}h_x^4 dxdt \leqslant c_1 M^{\frac{2\alpha+5m-3n+4}{\alpha+n+7}} \int_0^T \Big(\int_\Omega h^{\alpha+n-4}h_x^6 dx\Big)^{\frac{\alpha+m+5}{\alpha+n+7}} dt + c_2 M^{\frac{\alpha+3m-2n+2}{3}} \int_0^T \Big(\int_\Omega h^{\alpha+n-4}h_x^6 dx\Big)^{\frac{2}{3}} dt. \quad (3.28)$$

If $m < n+2$ then, using Young's inequality, from (3.28) we arrive at

$$\iint\limits_{Q_T} h^{\alpha+m-2} h_x^4 dxdt \leqslant \epsilon \iint\limits_{Q_T} h^{\alpha+n-4} h_x^6 dxdt + C(\epsilon) T (c_1 M^{\frac{2\alpha+5m-3n+4}{n+2-m}} + c_2 M^{\alpha+3m-2n+2}). \quad (3.29)$$

Substituting (3.29) in (3.25), and choosing ϵ small enough, we obtain

$$\mathcal{E}_0^{(\alpha)}(T) \leqslant \mathcal{E}_0^{(\alpha)}(0) + T(C_1 M^{\frac{2\alpha+5m-3n+4}{n+2-m}} + C_2 M^{\alpha+3m-2n+2}) \quad (3.30)$$

for $\alpha > 2n-3m-1$ and $m < n+2$. Here $C_2 = 0$ if Ω is unbounded or h is compactly supported. In particular, if $0 < \alpha + 3m - 2n + 2 \leqslant 1$, i. e. $2n-3m-2 < \alpha \leqslant 2n-3m-1$ then, using the Hölder inequality and applying Young's inequality, from (3.26) we obtain

$$\iint\limits_{Q_T} h^{\alpha+m-2} h_x^4 dxdt \leqslant \epsilon \iint\limits_{Q_T} h^{\alpha+n-4} h_x^6 dxdt + C(\epsilon) |\Omega|^{2n-3m-1-\alpha} T\, M^{\alpha+3m-2n+2} \quad (3.31)$$

for $2n-3m-2 < \alpha \leqslant 2n-3m-1$. Substituting (3.31) in (3.25), and choosing ϵ small enough, we obtain

$$\mathcal{E}_0^{(\alpha)}(T) \leqslant \mathcal{E}_0^{(\alpha)}(0) + C_3 T\, M^{\alpha+3m-2n+2}. \quad (3.32)$$

If $m = n+2$ then, using Young's inequality, from (3.28) we deduce

$$\iint\limits_{Q_T} h^{\alpha+m-2} h_x^4 dxdt \leqslant c_1 M^2 \iint\limits_{Q_T} h^{\alpha+n-4} h_x^6 dxdt + \epsilon \iint\limits_{Q_T} h^{\alpha+n-4} h_x^6 dxdt + C(\epsilon) T\, M^{\alpha+n+8}. \quad (3.33)$$

Substituting (3.33) in (3.25), and choosing ϵ enough small, we obtain

$$\mathcal{E}_0^{(\alpha)}(T) \leqslant \mathcal{E}_0^{(\alpha)}(0) + C_2 T\, M^{\alpha+n+8} \quad (3.34)$$

for $\alpha > -n-7$, $m = n+2$ and $M \leqslant M_c$. Here $C_2 = 0$ if Ω is unbounded or h is compactly supported.

Appendix A.

Lemma A.3. *([13, 2]) Let $\Omega \subset \mathbb{R}^N$, $N < 6$, be a bounded convex domain with smooth boundary, and let $n \in \big(2 - \sqrt{1 - \frac{N}{N+8}}, 3\big)$ for $N > 1$, and $\frac{1}{2} < n < 3$ for $N = 1$. Then the following estimate holds for any positive functions $v \in H^2(\Omega)$ such that $\nabla v \cdot \vec{n} = 0$ on $\partial\Omega$ and $\int_\Omega v^n |\nabla \Delta v|^2 < \infty$:*

$$\int_\Omega \varphi^6 \{ v^{n-4} |\nabla v|^6 + v^{n-2} |D^2 v|^2 |\nabla v|^2 \} \leqslant c \Big\{ \int_\Omega \varphi^6 v^n |\nabla \Delta v|^2 + \int_{\{\varphi > 0\}} v^{n+2} |\nabla \varphi|^6 \Big\},$$

where $\varphi \in C^2(\Omega)$ is an arbitrary nonnegative function such that the tangential component of $\nabla \varphi$ is equal to zero on $\partial\Omega$, and the constant $c > 0$ is independent of v.

Lemma A.4. *([16]) If $\Omega \subset \mathbb{R}^N$ is a bounded domain with piecewise-smooth boundary, $a > 1$, $b \in (0, a)$, $d > 1$, and $0 \leqslant i < j$, $i, j \in \mathbb{N}$, then there exist positive constants d_1 and d_2 ($d_2 = 0$ if Ω is unbounded) depending only on Ω, d, j, b, and N such that the following inequality is valid for every $v(x) \in W^{j,d}(\Omega) \cap L^b(\Omega)$:*

$$\left\| D^i v \right\|_{L^a(\Omega)} \leqslant d_1 \left\| D^j v \right\|^{\theta}_{L^d(\Omega)} \|v\|^{1-\theta}_{L^b(\Omega)} + d_2 \|v\|_{L^b(\Omega)}, \quad \theta = \frac{\frac{1}{b} + \frac{i}{N} - \frac{1}{a}}{\frac{1}{b} + \frac{j}{N} - \frac{1}{d}} \in \left[\frac{i}{j}, 1\right).$$

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