

Investigation of Stability by Using Lyapunov Functions of Fixed Sign

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Abstract. We consider a nonautonomous system of ordinary differential equations admitting the trivial solution for which there exists a nonnegative Lyapunov function whose derivative is nonpositive due to the properties of the system. It is assumed that the trivial solution of the system is uniformly asymptotically stable in the set, where the Lyapunov function turns into zero. We prove theorems on the uniform stability and uniform asymptotic stability and present an example illustrating the proposed approach.

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1. Introduction

We consider a system of ordinary differential equations

$$\frac{dx}{dt} = X(t, x), \quad (1.1)$$

where $x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$, $t \in \mathbb{R}_+ = [0, \infty)$, $X = (X_1, X_2, \dots, X_n) : \mathbb{R}_+ \times B_H \rightarrow \mathbb{R}^n$, and $B_H = \{x \in \mathbb{R}^n : |x| \leq H\}$. The functions $X_i(t, x)$, $i = 1, \dots, n$, are assumed to be continuous and satisfying the Lipschitz condition with respect to x uniformly in t in the region $\mathbb{R}_+ \times B_H$, where $H > 0$ is a constant. Moreover, we set $X(t, 0) \equiv 0$. Under these assumptions, system (1.1) admits the trivial solution

$$x = 0. \quad (1.2)$$

In the analysis of the uniform asymptotic stability of solution (1.2) of system (1.1), it is customary to use Lyapunov's direct method. In

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this method, it is assumed that there exists a positive definite function $V(t, x)$ whose derivative $\frac{dV}{dt}$ is a negative definite function by virtue of system (1.1). At the same time, there are no standard methods for the construction of Lyapunov functions for systems of ordinary differential equations in general case. Therefore, it is of interest for applications to construct criteria of uniform asymptotic stability by using the functions with weaker properties as compared with the functions V and $\frac{dV}{dt}$. Thus, in [2, 5, 6, 10], criteria of this sort are obtained under the assumption that V is a positive definite function and $\frac{dV}{dt}$ is a nonpositive function. In [3, 4, 9], the conditions imposed on the function V are also weakened and, instead of positive definiteness, it is assumed that this function is nonnegative ($V \geq 0$). Moreover, in [3, 4], the right-hand sides of the system are assumed to be either autonomous or periodic in t . In [9], a special case of nonautonomous systems is used and the properties of the limiting equations are used to prove the theorems on stability and asymptotic stability. In the present paper, we continue the investigations originated in [3, 4, 9]. It is assumed that system (1.1) is nonautonomous and its trivial solution is uniformly asymptotically stable in the integral set $V(t, x) = 0$.

2. Basic Definitions

Consider system of ordinary differential equations (1.1) and denote $\Gamma = \mathbb{R}_+ \times B_H$, where H is a positive number. We now introduce several definitions used in [7, 8].

Definition 2.1. *A set M in the space (t, x) is called an integral set if, for any point $(t_0, x_0) \in M$, the inclusion $(t, x(t)) \in M$ holds for all $t \geq t_0$, where $x(t) = x(t, t_0, x_0)$ is the solution of system (1.1) with the initial data $x(t_0) = x_0$.*

Let $M \subset \mathbb{R}_+ \times \mathbb{R}^n$ and let M_s be the intersection of this set with the hyperplane $t = s$. By $\rho(x, M_s)$ we denote the distance from the point x to the set M_s .

By analogy with [4, 11], we introduce the following definitions:

Definition 2.2. *Solution (1.2) of system (1.1) is called stable with respect to the integral set M if, for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for $t \geq t_0$ whenever $(t_0, x_0) \in M$ and $\|x_0\| < \delta$.*

Definition 2.3. *If it is possible to choose the number δ in Definition 2.2 independent of t_0 (i.e., $\delta = \delta(\varepsilon)$), then solution (1.2) is called uniformly stable with respect to the set M .*

Definition 2.4. *The trivial solution of system (1.1) is called attractive with respect to the integral set M if, for any $t_0 \in \mathbb{R}_+$, one can find $\eta = \eta(t_0) > 0$ and, for any $\varepsilon > 0$ and $x_0 \in M_{t_0} \cap B_\eta$, one can find $\sigma = \sigma(\varepsilon, t_0, x_0) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geq t_0 + \sigma$.*

Definition 2.5. *The trivial solution of equations (1.1) is called uniformly attractive with respect to the set M if, for some $\eta > 0$ and any $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t_0 \in \mathbb{R}_+$, $x_0 \in M_{t_0} \cap B_\eta$, and $t \geq t_0 + \sigma$.*

Definition 2.6. *Solution (1.2) of system of differential equations (1.1) is called:*

- asymptotically stable with respect to the integral set M if it is stable and attractive with respect to M ;
- uniformly asymptotically stable with respect to the integral set M if it is uniformly stable and uniformly attractive with respect to M .

Definition 2.7. *We say that a function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a Hahn's function ($g \in \mathcal{K}$) if it is continuous, monotonically increasing, and such that $g(0) = 0$.*

Since, by definition, the right-hand sides of equations (1.1) satisfy the Lipschitz condition, there exists $L > 0$ such that

$$\|X(t, x_1) - X(t, x_2)\| \leq L\|x_1 - x_2\|, \quad x_1 \in B_H, \quad x_2 \in B_H.$$

Therefore, if $x(t, t_0, x_0) \in B_H$ and $y(t, t_0, y_0) \in B_H$, then, by using the estimate from [1, p. 14], we obtain

$$\|x(t, t_0, x_0) - y(t, t_0, y_0)\| \leq \delta e^{L(t-t_0)} \quad (2.1)$$

provided that $\|x_0 - y_0\| \leq \delta$.

3. Theorem on Uniform Stability

Theorem 3.1. *Assume that system (1.1) is such that there exists a continuous function $V(t, x) : \mathbb{R}_+ \times B_H \rightarrow \mathbb{R}$ with the following properties:*

- (a) V is periodic in t with period ω ;
- (b) $V(t, x) \geq 0$ and $V(t, 0) \equiv 0$;
- (c) the function V does not increase along the solutions of system (1.1);

(d) solution (1.2) of system (1.1) is uniformly asymptotically stable with respect to the integral set M , where

$$M = \{(t, x) : t \in \mathbb{R}_+, x \in B_H, V(t, x) = 0\}.$$

Then the trivial solution of system (1.1) is uniformly stable.

Proof. Denote

$$S(\alpha) = \{(t, x) \in \mathbb{R}_+ \times B_\alpha : V(t, x) = 0\},$$

$$S_t(\alpha) = \{x \in B_\alpha : V(t, x) = 0\}.$$

We prove the theorem by contradiction. To do this, we assume that solution (1.2) of system (1.1) is not uniformly stable. This means that there exists a positive number h and a sequence of initial data $\{(t_{0n}, x_{0n})\}_{n=1}^\infty$ such that $t_{0n} \geq 0$, $\lim_{n \rightarrow \infty} \|x_{0n}\| = 0$, and $\|x(t_{0n} + T_n^*, t_{0n}, x_{0n})\| \geq h$ for some $T_n^* > 0$. Let ε be an arbitrary positive number satisfying the condition $4\varepsilon < h$. By virtue of assumption (d) of the theorem, the trivial solution of system (1.1) is uniformly attractive with respect to M . Hence, for some $\eta > 0$, there exists $\sigma = \sigma(\varepsilon) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t_0 \in \mathbb{R}_+$, $x_0 \in S_{t_0}(\eta)$, and $t \geq t_0 + \sigma$. In what follows, we choose ε such that $4\varepsilon < \eta$. Assume that numbers $T_n > 0$ correspond to points of time such that

$$\|x(t_{0n} + T_n, t_{0n}, x_{0n})\| = 4\varepsilon \quad \text{and} \quad \|x(t_{0n} + t, t_{0n}, x_{0n})\| < 4\varepsilon \quad \text{for } t < T_n. \quad (3.1)$$

In view of inequality (2.1), we conclude that $\lim_{n \rightarrow \infty} T_n = +\infty$. We also consider a sequence $\{t_n\}_{n=1}^\infty$ satisfying the inequalities

$$0 < t_n < T_n, \quad \sigma \leq T_n - t_n \leq Q \quad (3.2)$$

for all $n \geq n_0 \in \mathbb{N}$, where \mathbb{N} is the set of natural numbers, Q is a constant satisfying the condition $Q > \sigma$ (thus, it is possible to set $Q = 2\sigma$), and $\{\tau_n\}_{n=1}^\infty$ is a sequence whose elements satisfy the conditions

$$\tau_n \in [0, \omega) \quad \text{and} \quad \tau_n = t_{0n} + t_n - p_n \omega, \quad (3.3)$$

where $p_n \in \mathbb{Z}_+$, and \mathbb{Z}_+ is the set of nonnegative integers.

Consider a sequence of points $\{y_n\}$ in the phase space, where

$$y_n = x(t_{0n} + t_n, t_{0n}, x_{0n}). \quad (3.4)$$

By assumptions (3.1) and (3.2), we conclude that $y_n \in B_{4\varepsilon}$ for any $n \geq n_0 \in \mathbb{N}$. The set $B_{4\varepsilon}$ is a bounded and closed subset of \mathbb{R}^n and,

hence, one can find a subsequence of the sequence $\{y_n\}$ convergent to an element $y_* \in B_{4\varepsilon}$. Similarly, the sequence $\{\tau_n\}$ contains a subsequence convergent to $\tau_* \in [0, \omega]$. Thus, without loss of generality, we can assume that the sequences themselves converge to the corresponding elements: $y_n \rightarrow y_*$ and $\tau_n \rightarrow \tau_*$ as $n \rightarrow \infty$.

In view of the properties of the function V and notations (3.3) and (3.4), we obtain

$$\begin{aligned} 0 \leq V(\tau_*, y_*) &= \lim_{n \rightarrow \infty} V(\tau_n, y_n) = \lim_{n \rightarrow \infty} V(t_{0n} + t_n, y_n) = \\ &= \lim_{n \rightarrow \infty} V(t_{0n} + t_n, x(t_{0n} + t_n, t_{0n}, x_{0n})) \leq \lim_{n \rightarrow \infty} V(t_{0n}, x_{0n}) = 0. \end{aligned}$$

Consequently, $V(\tau_*, y_*) = 0$ and $(\tau_*, y_*) \in S(4\varepsilon)$. By $k \in \mathbb{N}$ we denote a number such that the following conditions are satisfied:

$$k \geq n_0, \quad t_k \geq \sigma(\varepsilon), \quad |\delta_k| < \max\left\{1, \frac{\varepsilon}{Lh} e^{-LQ}\right\}, \quad \text{and} \quad \|y_k - y_*\| < \varepsilon e^{-QL}, \quad (3.5)$$

where $\delta_k = t_{0k} + t_k - p_k \omega - \tau_* = \tau_k - \tau_*$. It is clear that the indicated choice of k is possible because $\delta_n \rightarrow 0$, $\|y_n - y_*\| \rightarrow 0$, and $t_n \rightarrow +\infty$ as $n \rightarrow \infty$.

We have

$$\|x(t_{0k} + T_k, t_{0k}, x_{0k})\| \leq J_1 + J_2 + J_3,$$

where

$$J_1 = \|x(t_{0k} + T_k, t_{0k}, x_{0k}) - x(t_{0k} + T_k, t_{0k} + t_k, y_*)\|,$$

$$J_2 = \|x(t_{0k} + T_k, t_{0k} + t_k, y_*) - x(t_{0k} + T_k, t_{0k} + t_k - \delta_k, y_*)\|,$$

$$J_3 = \|x(t_{0k} + T_k, \tau_* + p_k \omega, y_*)\| = \|x(t_{0k} + T_k, t_{0k} + t_k - \delta_k, y_*)\|.$$

In view of the uniqueness of solutions, we have $x(t_{0k} + T_k, t_{0k}, x_{0k}) = x(t_{0k} + T_k, t_{0k} + t_k, x(t_{0k} + t_k, t_{0k}, x_{0k})) = x(t_{0k} + T_k, t_{0k} + t_k, y_k)$ and, consequently, by virtue of (2.1) and (3.5), we obtain

$$\begin{aligned} J_1 &= \|x(t_{0k} + T_k, t_{0k} + t_k, y_k) - x(t_{0k} + T_k, t_{0k} + t_k, y_*)\| \\ &\leq \|y_* - y_k\| e^{(T_k - t_k)L} \leq \|y_* - y_k\| e^{QL} < \varepsilon. \quad (3.6) \end{aligned}$$

We now estimate J_2 and J_3 by using inequalities (3.5):

$$\begin{aligned} J_2 &= \|x(t_{0k} + T_k, t_{0k} + t_k, y_*) \\ &\quad - x(t_{0k} + T_k, t_{0k} + t_k, x(t_{0k} + t_k, t_{0k} + t_k - \delta_k, y_*))\| \\ &\leq \|y_* - x(t_{0k} + t_k, t_{0k} + t_k - \delta_k, y_*)\| e^{LQ} \leq |\delta_k| L h e^{LQ} < \varepsilon; \quad (3.7) \end{aligned}$$

$J_3 < \varepsilon$ since $(\tau_* + p_k \omega, y_*) \in S(4\varepsilon)$ and $T_k - t_k \geq \sigma(\varepsilon)$.

Note that estimate (3.7) was obtained under the assumption that $\delta_k \geq 0$. Similarly, we can show that this estimate is also true for $\delta_k \leq 0$.

By using the deduced estimates, we find

$$\|x(t_{0k} + T_k, t_{0k}, x_{0k})\| < 3\varepsilon$$

in contradiction with assumption (3.1). This contradiction proves the uniform stability of the trivial solution of equations (1.1). \square

4. Theorem on Uniform Asymptotic Stability

Theorem 4.1. *Assume that system (1.1) admits the existence of a differentiable function $V(t, x) : \mathbb{R}_+ \times B_H \rightarrow \mathbb{R}$ such that:*

(a) V is periodic in t with period ω ;

(b) $V(t, x) \geq 0$ and $V(t, 0) \equiv 0$;

(c) $\frac{dV}{dt} \leq -c(V(t, x))$, $c \in \mathcal{K}$;

(d) the solution (1.2) of system (1.1) is uniformly asymptotically stable with respect to the integral set $S(H)$.

Then the trivial solution of system (1.1) is uniformly asymptotically stable.

Proof. Since the trivial solution of equations (1.1) is uniformly asymptotically stable with respect to $S(H)$, there exists $\eta > 0$ such that

$$\begin{aligned} (\forall \xi > 0) (\exists \sigma = \sigma(\xi) > 0) (\forall t_0 \in \mathbb{R}_+) (\forall x_0 \in S_{t_0}(\eta)) \\ (\forall t \geq t_0 + \sigma) \|x(t, t_0, x_0)\| < \xi. \end{aligned} \quad (4.1)$$

By virtue of Theorem 3.1, the solution (1.2) of system (1.1) is uniformly stable and, hence, one can find $\zeta > 0$ such that $\|x(t, t_0, x_0)\| < \eta$ for all $x_0 \in B_\zeta$, $t_0 \in \mathbb{R}_+$, and $t \geq t_0$. Let us show that the trivial solution of system (1.1) is uniformly attractive. We take arbitrary $\varepsilon > 0$. The fact that the trivial solution is uniformly stable implies the existence of $\delta = \delta(\varepsilon) > 0$ such that the inequality

$$\|x(t, t_0, x_0)\| < \varepsilon \quad (4.2)$$

is true for any $x_0 \in B_\delta$, $t_0 \in \mathbb{R}_+$, and $t \geq t_0$.

By using (4.1), we obtain

$$\begin{aligned} \left(\exists \sigma_1 = \sigma_1(\varepsilon) = \sigma\left(\frac{\delta}{3}\right) > 0 \right) (\forall t_0 \in \mathbb{R}_+) (\forall y_0 \in S_{t_0}(\eta)) \\ (\forall t \geq t_0 + \sigma_1) \|x(t, t_0, y_0)\| < \frac{\delta}{3}. \end{aligned} \quad (4.3)$$

It follows from inequality (2.1) that there exists

$$\gamma = \gamma(\varepsilon) = \frac{1}{3}\delta(\varepsilon)e^{-L\sigma_1(\varepsilon)} > 0$$

such that

$$\|x(t_0 + \sigma_1, t_0, x_0) - x(t_0 + \sigma_1, t_0, y_0)\| < \frac{\delta}{3}, \quad (4.4)$$

whenever $t_0 \in \mathbb{R}_+$ and $\|x_0 - y_0\| < \gamma$.

We now show that

$$\lim_{t \rightarrow \infty} \rho(x(t, t_0, x_0), S_t(\eta)) = 0 \quad (4.5)$$

for all solutions $x(t)$ originating in B_ζ ; moreover, limit relation (4.5) holds uniformly in $t_0 \in \mathbb{R}_+$ and $x_0 \in B_\zeta$. For this purpose, we first prove that there exists a Hahn's function a such that

$$V(t, x) \geq a(\rho(x, S_t(\eta))). \quad (4.6)$$

Denote

$$\psi(r) = \inf V(t, x) \quad \text{for } t \in [0, \omega], \quad \rho(x, S_t(\eta)) = r, \quad x \in B_\eta.$$

Thus, $\psi(r)$ is a continuous scalar function such that $\psi(0) = 0$, and $\psi(r) > 0$ for $r > 0$. By $a(r)$ we denote a continuous monotonically increasing function such that $a(0) = 0$ and $a(r) \leq \psi(r)$ for $0 < r < \eta$. It is clear that the function a constructed as indicated above satisfies inequality (4.6). Further, we show that

$$(\forall \alpha > 0) (\exists T = T(\alpha)) (\forall t_0 \in \mathbb{R}_+) (\forall x_0 \in B_\zeta) (\forall t \geq t_0 + T) \quad (4.7) \\ V(t, x(t, t_0, x_0)) < \alpha.$$

Denote $\sup_{\substack{t \in [0, \omega] \\ x \in B_H}} V(t, x) = P$. We now estimate the period of time for which the inequality $V(t, x(t, t_0, x_0)) \geq \alpha$ can hold. In this case,

$$V(t, x(t)) = V(t_0, x_0) + \int_{t_0}^t \dot{V} dt \leq P - c(\alpha)(t - t_0);$$

$$c(\alpha)(t - t_0) \leq P - V(t, x(t)) \leq P - \alpha; \quad t - t_0 \leq \frac{P - \alpha}{c(\alpha)}.$$

This means that inequality (4.7) is satisfied for $T = \frac{P - \alpha}{c(\alpha)}$.

By using properties (4.7) and (4.6), we can prove the validity of the limit relation (4.5):

$$(\forall \gamma > 0) (\exists T_1 = T_1(\gamma) > 0) (\forall t_0 \in \mathbb{R}_+) (\forall x_0 \in B_\zeta) (\forall t \geq t_0 + T_1) \\ \rho(x(t, t_0, x_0), S_t(\eta)) < \gamma.$$

This yields

$$(\forall t_0 \in \mathbb{R}_+) (\forall x_0 \in B_\zeta) (\exists y_0 \in S_{t_0+T_1}(\eta)) \|x(t_0 + T_1, t_0, x_0) - y_0\| < \gamma. \quad (4.8)$$

Inequalities (4.8) and (4.4) imply that

$$\|x(t_0 + T_1 + \sigma_1, t_0 + T_1, x(t_0 + T_1, t_0, x_0)) - x(t_0 + T_1 + \sigma_1, t_0 + T_1, y_0)\| < \frac{\delta}{3}.$$

This inequality can be rewritten in the form

$$\|x(t_0 + T_1 + \sigma_1, t_0, x_0) - x(t_0 + T_1 + \sigma_1, t_0 + T_1, y_0)\| < \frac{\delta}{3}. \quad (4.9)$$

By virtue of (4.3), we obtain

$$\|x(t_0 + T_1 + \sigma_1, t_0 + T_1, y_0)\| < \frac{\delta}{3}. \quad (4.10)$$

It follows from (4.9) and (4.10) that $\|x(t_0 + T_1 + \sigma_1, t_0, x_0)\| < \frac{2}{3}\delta < \delta$, whence, in view of the property of uniform stability (4.2), we conclude that the inequality

$$\|x(t, t_0 + T_1 + \sigma_1, x(t_0 + T_1 + \sigma_1, t_0, x_0))\| < \varepsilon$$

holds for $t > t_0 + T_1 + \sigma_1$. By virtue of the unique solvability of system (1.1), we can write

$$x(t, t_0 + T_1 + \sigma_1, x(t_0 + T_1 + \sigma_1, t_0, x_0)) = x(t, t_0, x_0).$$

Thus, it is proved that one can find $\zeta > 0$ such that, for any $\varepsilon > 0$, there exists $\sigma_* = \sigma_*(\varepsilon) = T_1(\varepsilon) + \sigma_1(\varepsilon)$ for which the inequality $\|x(t, t_0, x_0)\| < \varepsilon$ holds for all $t_0 \in \mathbb{R}_+$, $x_0 \in B_\zeta$, and $t \geq t_0 + \sigma_*$. This means that the trivial solution of system (1.1) is uniformly attractive. The theorem has been proved. \square

5. Example

We consider a system of ordinary differential equations

$$\begin{aligned}\frac{dx}{dt} &= \frac{x}{1+t^2} - xy^2 - x^3, \\ \frac{dy}{dt} &= x\left(\cos t + \frac{\sin t}{1+t^2}\right) + x^3(\sin^3 t - \sin t) - 3x^2y \sin^2 t + 2xy^2 \sin t - y^3\end{aligned}\quad (5.1)$$

and study the property of stability of its trivial solution

$$x = 0, \quad y = 0. \quad (5.2)$$

Note that system (5.1) admits an integral set M specified by the equality

$$y - x \sin t = 0.$$

We take the Lyapunov function in the form $V = (y - x \sin t)^2$. Its derivative is equal to

$$\frac{dV}{dt} = -2(y - x \sin t)^4 = -2V^2.$$

In the set M , the first equation of system (5.1) takes the form

$$\frac{dx}{dt} = \frac{x}{1+t^2} - x^3(1 + \sin^2 t).$$

The trivial solution of this equation is uniformly asymptotically stable, i.e., solution (5.2) of system (5.1) is uniformly asymptotically stable with respect to the integral set M . This means that all conditions of Theorem 4.1 are satisfied for system (5.1) and, thus, it is possible to conclude that the trivial solution of equations (5.1) is uniformly asymptotically stable.

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