Siberian Mathematical Journal, Vol. 47, *No.* 4, *pp.* , 2006
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PROPAGATION OF PERTURBATIONS IN THIN CAPILLARY FILM EQUATIONS WITH NONLINEAR DIFFUSION AND CONVECTION

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UDC 517.95

Abstract: We study the evolution of the support of an arbitrary strong generalized solution to the Cauchy problem for the thin film equation with nonlinear diffusion and convection. We find an upper bound exact (in a sense) for the propagation speed of the support of this solution.

Keywords: thin film equation, convection, Cauchy problem, support propagation

Introduction. In this article we study the behavior of the support of a strong generalized solution to the following Cauchy problem:

$$u_t + \operatorname{div}(u^n \nabla \Delta u - u^m \nabla u) = \vec{\chi} \cdot \nabla b(u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \tag{1}$$

$$u(0,x) = u_0(x) \in H^1(\mathbb{R}^N), \quad u_0 \geq 0, \quad \operatorname{supp} u_0 \text{ is a compact set}, \tag{2}$$

$$|b'(z)| \leq c|z|^{\lambda-1} \ \forall z \in \mathbb{R}^1, \quad b(0) = 0, \quad \lambda > 0, \ c > 0, \tag{3}$$

where $u = u(t,x)$, $N \leq 3$, $n > 0$, $m \in \mathbb{R}^1$, and $\vec{\chi} \in \mathbb{R}^N$. The fourth-order degenerate parabolic equation (1) is a typical representative of the broad class of nonlinear equations of the form:

$$u_t + \operatorname{div}(f(u)\nabla\Delta u + \nabla A(u)) = g(t,x,u,\nabla u). \tag{4}$$

These equations appear in modeling of various processes of elasticity theory, fluid dynamics, and binary alloy dynamics [1–6]. For example, equation (1) with $b(u) = 0$ describes the evolution of a liquid film propagating over a rigid surface under the action of the surface tension forces, viscosity, and gravity (for $m = n$). Observe that for $m < 0$ the second-order term in (1) corresponds to the Van der Waals intermolecular forces and for $m > 0$ ($m \neq n$), to the internal diffusion forces. The fourth-order term in (1) describes the influence of the surface tension capillary forces. The exponent $n > 0$ characterizes the behavior of the interface with the rigid surface. For example, the case $n = 3$ corresponds to contact without slip, while $n \in (0,3)$ corresponds to motion of a fluid with partial slip. The case $n = m = 1$ describes the change of the size of the domain occupied by the fluid in a Hele–Shaw half-space cell in the viscous regime [7]. The case $n = 1$ and $m = 0$ corresponds to the principal term as $u \to 0$ in the Cahn–Hilliard binary alloy equation with the logarithmic free energy. The convection term in (1) describes the action of different potentials: for example, the gravity on an inclined plane ($\vec{\chi} b(u) = (u^n, 0, \dots, 0)$), the surface tension caused by changes of the temperature regimes ($\vec{\chi} b(u) = (-u^{n-1}, 0, \dots, 0)$) [2, 8, 9], etc.

The mathematical study of fourth-order degenerate thin film type equations originated with [10] where, in particular, a nonnegative weak generalized solution was constructed for the Neumann problem with an arbitrary nonnegative initial function for the one-dimensional equation:

$$u_t + \operatorname{div}(|u|^n \nabla \Delta u) = 0. \tag{5}$$

The solution by the authors is “weaker” than the usual generalized solution in the sense that the domain of existence of the corresponding integrals in the integral identity is smaller than the whole domain in which the problem is studied. Observe that the nonnegativity property of the solution is enjoyed by the thin film type equations and is not valid for general high-order equations. Later, the nonnegative generalized solutions were constructed for the boundary value problems for multidimensional equation (5),

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Donetsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 47, No. 4, pp. 914–931, July–August, 2006.
Original article submitted April 12, 2005.

and a series of the qualitative properties depending on the parameters n and N of the so-constructed solutions were described for instance:

the finite propagation speed property for $N = 1$ and $0 < n < 2$ [11,12], $2 \leq n < 3$ [13,14], $n \geq 4$ [15] and for $N \in \{2, 3\}$ and $1/8 < n < 2$ [16], $2 \leq n < 3$ [16,17];

the finite inverse speed property for $N = 1$ and $1/2 < n < 3/2$ [11];

the waiting time phenomenon for $N = 1$ and $0 < n < 3$ and for $N \in \{2, 3\}$ and $1/8 < n < 2$ [18].

In [15] it was demonstrated that weak solutions do not possess the uniqueness property. To specify the uniqueness class, the strong generalized solution were introduced to be the weak solutions satisfying some special integral estimate which is referred to as the entropy estimate in the literature. The conjecture is still open whether this very class of strong solutions is the uniqueness class. One of the important properties of an arbitrary strong solution is the finite propagation speed property.

In [6] nonnegative weak solutions were constructed for the Neumann problem for (4) with $g = g(t, x, u)$ having at most linear growth in u, $|A'(u)| \leq d_0 f(u)$, and $f(u)$ having power growth in a neighborhood of $u = 0$. In the case of $g = g(t, x, u) \sim |u|^{\lambda-1}u$ $(\lambda > 0)$, $A'(u) = -|u|^m$, and $f(u) = |u|^n$ equation (4) was studied in [19,20]. In [21], for the Cauchy problem for the multidimensional equation (1) with $b(u) = 0$, the nonnegative generalized solutions were constructed with a weak initial trace ($m > 0$, $n \in (1/8, 2)$), the conditions on the parameters n and m were found which guarantee the finite ($m > 0$ and $n \in (1/8, 2)$) and infinite ($m < 0$ and $0 < m - n + 2 < 1/2$) speed of propagation, and also the exact asymptotic expansion for the motion of the support of a solution was obtained.

In [2,8,9] stability of solutions of the traveling wave type was studied for the one-dimensional equation (1) for $n = m = 3$, $\chi = 1$, and $b(u) = u^3 - u^2$. In [22] a nonnegative local generalized solution was constructed for the one-dimensional equation (1) without the diffusion term for $n \in (0, 3)$, $\lambda > \max\{3n/4 - 1, 1/8\}$, and $b(u)$ in (3); moreover, under the additional condition $\lambda < 9/2$, a nonnegative global generalized solution to (1)–(3) was constructed. In the same article the finite propagation speed property was proven and an upper bound for the speed was found at large and small times. In [23], for the multidimensional equation (1) with $m > 0$, $n \in (1/8, 2)$, and $\max\{1, (3n - 1)/4\} < \lambda < (5N + 8)/(4N) + \min\{n, 5/4\}$ if $N < 3$ and $\max\{1, (3n - 1)/4\} < \lambda < 2 + \min\{n, 5/4\}$ if $N = 3$, a nonnegative generalized solution to (1)–(3) was constructed which possesses the finite propagation speed property.

In this article we find an upper bound for the speed of the support of a solution to (1)–(3). The method for proving consists in studying the behavior of solutions to some special functional inequalities and is a generalization of the method of a small parameter of [24]. In turn, this method bases on application of the Saint-Venant principle. Detailed information on the Saint-Venant principle and its application to studying partial differential equations can be found in the survey [25].

Notations. Given an $(N \times N)$-matrix A and vectors $a, b \in \mathbb{R}^N$, define $\langle a, A, b\rangle := \sum_{i,j=1}^{N} a_i A_{ij} b_j$; χ_A is the characteristic function of a set A; for an arbitrary measurable function $v(t, x)$ we define the positivity set $\mathrm{P} := \mathrm{P}(v) = \{v > 0\} = \{(t, x) \in \mathrm{Dom}(v) : v(t, x) > 0\}$; $C_c^k(\Omega) := \{v \in C^k(\Omega) : \operatorname{supp} v \subset \Omega\}$; $H^k(\Omega) := W_2^k(\Omega)$; $c(u_0)$ and $c_i(u_0)$ are positive constants depending on the given parameters of (1)–(3) and the initial function $u_0(x)$; and d and d_i are positive constants independent of the parameters of (1)–(3). We will drop the differential signs whenever the domain of integration is clear from the context.

Throughout the article we refer to Lemmas A.1–A.4 whose statements are given in the appendix.

Definition of a strong solution. A solution to (1)–(3) is understood in the following sense:

Definition 1. Let $N \leq 3$, $m > -1$, $n > 0$, and $\lambda > 0$. A nonnegative function $u(t, x) \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}^+; H^1_{\mathrm{loc}}(\mathbb{R}^N)\big)$ is called a *solution* to (1)–(3) if

(i) $\chi_{\mathrm{P}} u^{n-2}|\nabla u|^3$, $\chi_{\mathrm{P}} u^{n-1}|\nabla u|^2$, $u^n|\nabla u|$, u^{m+1}, and $b(u)$ belong to $L^1_{\mathrm{loc}}\big([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}^N)\big)$, where $\mathrm{P} = \mathrm{P}(u)$;

(ii) for all $\zeta \in C_c^3([0,\infty) \times \mathbb{R}^N)$ the following holds:

$$-\int\limits_0^\infty \int\limits_{\mathbb{R}^N} u\zeta_t - \int\limits_{\mathbb{R}^N} u_0\zeta(0,x) + \int\limits_0^\infty \int\limits_{\mathbb{R}^N} \vec{\chi}\nabla\zeta b(u) = \frac{n(n-1)}{2} \iint\limits_{\mathrm{P}} u^{n-2}|\nabla u|^2 \nabla u \nabla\zeta$$
$$+\frac{n}{2} \iint\limits_{\mathrm{P}} u^{n-1}|\nabla u|^2 \Delta\zeta + n \iint\limits_{\mathrm{P}} u^{n-1}\langle \nabla u, D^2\zeta, \nabla u\rangle$$
$$+\int\limits_0^\infty \int\limits_{\mathbb{R}^N} u^n \nabla u \nabla \Delta\zeta + \frac{1}{m+1} \int\limits_0^\infty \int\limits_{\mathbb{R}^N} u^{m+1}\Delta\zeta.$$

Remark 1. A solution satisfying Definition 1 is called a *weak generalized solution* to (1)–(3). The concept of weak solution to multidimensional thin film equations was proposed in [5, 6, 26, 21].

We state the existence theorem of a strong solution to (1)–(3) which was proved in [23]:

Theorem 1. *Suppose that* $N \le 3$, $m > 0$, $1/8 < n < 2$,

$$\max\{1, (3n-1)/4\} < \lambda < (5N+8)/4N + \min\{n, 5/4\}, \text{ if } N < 3;$$
$$\max\{1, (3n-1)/4\} < \lambda < 2 + \min\{n, 5/4\}, \text{ if } N = 3,$$

and $u_0(x) \in H^1(\mathbb{R}^N) \cap L^{m-n+2}(\mathbb{R}^N)$ *is a nonnegative function with* $\operatorname{supp} u_0 \subset B(0, R_0)$, $R_0 < +\infty$. *Then* (1)–(3) *has a solution* $u(t,x)$ *in the sense of Definition* 1 *such that*

(i) $\operatorname{supp} u(t,.)$ *is compact for almost all* $t > 0$ *and there is an increasing function* $\Gamma(t) \in C[0,+\infty)$, $\Gamma(0) = 0$, *such that* $\operatorname{supp} u(t,.) \subset B(0, R_0 + c(u_0)\Gamma(t))$ $\forall t > 0$;

(ii) *for* $\alpha \in \Delta_{n,\lambda} := ((1/2-n)_+, \min\{(n+1)/3, 2-n\})$ *satisfying the conditions*

$$\max\left\{\frac{3(\alpha+n)-1}{4}, 1\right\} < \lambda \le \frac{N+2}{N} + \frac{3(\alpha+n)}{4} \quad \text{for } N < 3;$$
$$\max\left\{\frac{3(\alpha+n)-1}{4}, 1\right\} < \lambda \le \frac{3(\alpha+n)+7}{4} \quad \text{for } N = 3$$

(*it is easy to verify that, under the conditions of the theorem, the set of such* α*'s is nonempty*), *we have*

$$u^{m-n+2} \in L^\infty_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^N)), \quad u^{\frac{\alpha+n+1}{4}} \in L^4_{\mathrm{loc}}\big([0,\infty); W^1_4(\mathbb{R}^N)\big),$$
$$u^{\frac{\alpha+n+1}{2}} \in L^2_{\mathrm{loc}}([0,\infty); H^2(\mathbb{R}^N)), \quad u^{\frac{\alpha+m+1}{2}} \in L^2_{\mathrm{loc}}([0,\infty); H^1(\mathbb{R}^N));$$

(iii) *the following is valid for almost all* $0 \le t_1 < t_2$ *and an arbitrary nonnegative function* $\zeta \in C^2([t_1,t_2] \times \mathbb{R}^N)$:

$$\frac{1}{\alpha(\alpha+1)} \int\limits_{\mathbb{R}^N} \zeta^4 u^{\alpha+1}(t_2,x)\,dx - \frac{1}{\alpha(\alpha+1)} \int\limits_{t_1}^{t_2} \int\limits_{\mathbb{R}^N} (\zeta^4)_t u^{\alpha+1}$$
$$+c_3^{-1} \int\limits_{t_1}^{t_2} \int\limits_{\mathbb{R}^N} \zeta^4 \{|\nabla u^{\frac{\alpha+m+1}{2}}|^2 + |\nabla u^{\frac{\alpha+n+1}{4}}|^4 + |D^2 u^{\frac{\alpha+n+1}{2}}|^2\}$$
$$\le \frac{1}{\alpha(\alpha+1)} \int\limits_{\mathbb{R}^N} \zeta^4 u^{\alpha+1}(t_1,x)\,dx + c_3 \int\limits_{t_1}^{t_2} \int\limits_{\{\zeta(t)>0\}} u^{\alpha+m+1}(\zeta^2|\nabla\zeta|^2 + \zeta^3|\Delta\zeta|)$$

$$+c_3\int\limits_{t_1}^{t_2}\int\limits_{\{\zeta(t)>0\}} u^{\alpha+n+1}(|\nabla\zeta|^4+\zeta^2|\Delta\zeta|^2)-\int\limits_{t_1}^{t_2}\int\limits_{\{\zeta(t)>0\}} \vec{\chi}\mathscr{B}^{(\alpha)}(u)\nabla\zeta^4, \qquad (6)$$

where $\mathscr{B}^{(\alpha)}(z) := \alpha^{-1}\int_0^z b'(\tau)\tau^\alpha d\tau$ *and* α *is taken from* (ii);

(iv) $u(t,.) \underset{t\to 0}{\to} u_0(.)$ *strongly in* $L^2(\mathbb{R}^N)$.

Remark 2. In [23] existence of a time-local strong solution to (1)–(3) was proved under the conditions $N \le 3$, $m > 0$, $n \in (1/8, 2)$ and

$$1 < \lambda < \kappa + 1 + \max\{n+\kappa, m\} \quad \text{for } N < 3,$$
$$1 < \lambda < \min\{(4n+7)/3, 4\} \text{ for } N = 3; \quad \kappa := \frac{2}{N}\min\{(n+4)/3, 3-n\}.$$

The perturbation propagation speed in the Cauchy problem. Introduce the following notations:

$$\Omega(s) = \{x = (x_1, x') : x_1 > d_1(s + |x'|),\ 0 < d_1 < \infty\}, \quad Q_T(s) = \Omega(s)\times(0,T),$$
$$K_T(s,\delta) = Q_T(s)\backslash Q_T(s+\delta), \quad K(s,\delta) = \Omega(s)\backslash\Omega(s+\delta), \quad K := \int\limits_{\mathbb{R}^N} u_0(x)\,dx,$$
$$h_0(s) := \int\limits_{\Omega(s)} u_0^{\alpha+1}(x)\,dx \quad \forall s \in \mathbb{R}^1,\ \delta > 0.$$

Without loss of generality we may assume that

$$\operatorname{supp} u_0 \cap \{x_1 \ge d_2|x'|\} = \varnothing, \quad \vec{\chi} = \overrightarrow{(\chi_1, 0, 0)},$$

where $d_2 \in [0, d_1)$ is some fixed number. Hence,

$$h_0(s) \equiv 0 \quad \forall s \ge 0.$$

The following theorem contains some upper bound for the speed of the support of a solution to (1)–(3):

Theorem 2. *Let* $u(t,x)$ *be an arbitrary strong solution to* (1)–(3) *of Theorem* 1 *with a compactly-supported initial function* $u_0(x)$ *such that* $0 \le u_0(x) \in H^1(\mathbb{R}^N)\cap L^{m-n+2}(\mathbb{R}^N)$. *Then the following upper bounds hold for the front* $\Gamma(t) = \sup\{|x| : x \in \operatorname{supp} u(t,.)\}$ *of the support:*

(i) *if* $\chi_1 \in \mathbb{R}^1\backslash\{0\}$ *and* $b(u)$ *satisfies* (3) *then*

$$\Gamma(t) \le \begin{cases} c_1(u_0)\max\{\Gamma_0(t), t^{1-\frac{N(\lambda-1)}{nN+4}}\}\ \forall t > 0 & \text{for } 1 < \lambda \le n+1+4/N, \\ c_1(u_0)\max\{\Gamma_0(t), t^{1-\frac{N(\lambda-1)}{mN+2}}\}\ \forall t > 0 & \text{for } 1 < \lambda \le m+1+2/N, \end{cases}$$

where $\Gamma_0(t) = \max\{t^{1/(nN+4)},\ t^{1/(mN+2)}\}$;

(ii) *if* $\chi_1 > 0$, $b(u) \ge du^\lambda$, *and* $\lambda > \max\{n+1, m+1\}$ *then*

$$\Gamma(t) \le c_2(u_0)\min\{\Gamma_0(t), \max\{t^{\frac{\lambda-n-1}{4(\lambda-1)-n}}, t^{\frac{\lambda-m-1}{2(\lambda-1)-m}}\}\} \quad \forall t > 0,$$

where $\Gamma_0(t)$ *is defined in* (i);

(iii) *if* $\chi_1 > 0$, $b(u) \ge du^\lambda$, *and* $\lambda < \min\{n+1, m+1\}$ *then*

$$\Gamma(t) \le c_3(u_0)\min\{\Gamma_0(t), 1\} \quad \forall t > 0;$$

(iv) *if* $\chi_1 > 0$, $b(u) \ge du^{\lambda}$, *and* $\lambda = n+1 = m+1$ *then*

$$\Gamma(t) \le c_4(u_0)\min\{\Gamma_0(t), 1+\log(1+t)\} \quad \forall t > 0,$$

where the positive constants $c_i(u_0)$ *depend only on* m, n, λ, N, $\|u_0\|_{L^1(\mathbb{R}^N)}$ *(moreover,* $c_3(u_0)$ *and* $c_4(u_0)$ *depend on* $\|u_0\|_{L^{\alpha+1}(\mathbb{R}^N)}$*) and are independent of the norm of* $u_0(x)$ *in* $H^1(\mathbb{R}^N) \cap L^{m-n+2}(\mathbb{R}^N)$*, although the initial function belongs to this space.*

Remark 3. In the absence of the convection ($\chi_1 = 0$) and in the situation when $\chi_1 > 0$, $b(u) \ge du^{\lambda}$, and $n+1 < \lambda < m+1$ (or $m+1 < \lambda < n+1$), the bound for the speed of motion of support coincides with that in [21] and has the following form: $\Gamma(t) \le c(u_0)\Gamma_0(t)$ $\forall t > 0$, where the constant $c(u_0)$ depends only on m, n, N, and $\|u_0\|_{L^1(\mathbb{R}^N)}$ and $\Gamma_0(t)$ is defined in (i) of Theorem 2.

Auxiliary lemmas.

Lemma 1. *Let* $u(t,x)$ *be a strong solution to* (1)–(3)*. Then the following are valid for arbitrary* $\alpha \in \Delta_{n,\lambda}$*:*

$$\begin{aligned} L_T^{(1)}(s+\delta) :=& \sup_{t\in(0,T)} \int\limits_{\Omega(s+\delta)} u^{\alpha+1}(t) + \frac{1}{T}\iint\limits_{Q_T(s+\delta)} u^{\alpha+1} \\ &+c_3^{-1}\iint\limits_{Q_T(s+\delta)} [|D^2 u^{\frac{\alpha+n+1}{2}}|^2 + |\nabla u^{\frac{\alpha+n+1}{4}}|^4 + |\nabla u^{\frac{\alpha+m+1}{2}}|^2] \\ \le& c\Bigg(\delta^{-2}\iint\limits_{K_T(s,\delta)} u^{\alpha+m+1} + \delta^{-4}\iint\limits_{K_T(s,\delta)} u^{\alpha+n+1} + |\chi_1|\delta^{-1}\iint\limits_{K_T(s,\delta)} u^{\alpha+\lambda} + h_0(s)\Bigg) \\ :=& R_T^{(1)}(s,\delta) \quad \forall s \in \mathbb{R}^1,\ \delta > 0,\ T > 0,\ \chi_1 \in \mathbb{R}^1; \end{aligned} \tag{7}$$

$$\begin{aligned} &\sup_{t\in(0,T)} \int\limits_{\Omega(s+\delta)} u^{\alpha+1}(t) + \frac{1}{T}\iint\limits_{Q_T(s+\delta)} u^{\alpha+1} + \frac{cd_3\chi_1}{\delta}\iint\limits_{K_T(s+\delta/4,\delta/4)} u^{\alpha+\lambda} \\ &+c_3^{-1}\iint\limits_{Q_T(s+\delta)} [|D^2 u^{\frac{\alpha+n+1}{2}}|^2 + |\nabla u^{\frac{\alpha+n+1}{4}}|^4 + |\nabla u^{\frac{\alpha+m+1}{2}}|^2] \\ &\le c\Bigg(\delta^{-2}\iint\limits_{K_T(s,\delta)} u^{\alpha+m+1} + \delta^{-4}\iint\limits_{K_T(s,\delta)} u^{\alpha+n+1} + h_0(s)\Bigg) \\ &:= R_T^{(2)}(s,\delta) \quad \forall s \in \mathbb{R}^1,\ \delta > 0,\ T > 0,\ \chi_1 \ge 0,\ d_3 > 0. \end{aligned} \tag{8}$$

Proof. Introduce the nonnegative cut-off functions $\zeta(\tau)$ and $\varphi(\tau)$ in the space $C^2(\mathbb{R}^1)$ with the following properties:

$$\varphi(\tau) = \begin{cases} 0, & \text{if } \tau \le 0, \\ 1 & \text{if } \tau \ge \frac{15d_1}{16}, \end{cases} \qquad \zeta(\tau) = \begin{cases} \tau & \text{if } \tau \ge \frac{1}{8}, \\ \frac{1}{16} & \text{if } \tau < \frac{1}{32}, \end{cases}$$

$$0 \le \varphi(\tau) \le 1, \quad \frac{d^2\zeta}{d\tau^2} \ge 0 \quad \forall \tau \in \mathbb{R}^1;$$

moreover,

$$\frac{d}{d\tau}\varphi(\tau) \ge 0 \quad \forall \tau \in \mathbb{R}^1, \quad \frac{d}{d\tau}\varphi(\tau) \ge d_0 > 0 \quad \forall \tau \in \left(\frac{3d_1}{16}, \frac{3d_1}{4}\right).$$

Define the family of basic cut-off functions:

$$\varphi_{s,\delta}(x) = \varphi\left(-d_1\zeta\left(\frac{|x'|}{\delta}\right) + \frac{x_1 - d_1 s}{\delta}\right) \quad \forall s \in \mathbb{R}^1,\ \delta > 0.$$

By the above properties of $\varphi(\tau)$ and $\zeta(\tau)$, the functions of the above family satisfy the relations

$$\varphi_{s,\delta}(x) = \begin{cases} 0 & \forall x : x_1 \le d_1\big(s + \delta\zeta(\frac{|x'|}{\delta})\big), \\ 1 & \forall x : x_1 \ge d_1\big(s + \delta\big(\zeta(\frac{|x'|}{\delta}) + \frac{15}{16}\big)\big), \end{cases} \qquad 0 \le \varphi_{s,\delta}(x) \le 1\ \forall x \in \mathbb{R}^N.$$

The inequalities

$$|\nabla\varphi_{s,\delta}| \le \frac{d}{\delta}, \quad |\Delta\varphi_{s,\delta}| \le \frac{d}{\delta^2}$$

hold for all x such that $d_1(s + \delta\zeta(|x'|/\delta)) \le x_1 \le d_1(s + \delta(\zeta(|x'|/\delta) + 15/16))$; and for all x such that $d_1(s + \delta(\zeta(|x'|/\delta) + 3/16)) \le x_1 \le d_1(s + \delta(\zeta(|x'|/\delta) + 3/4))$ we have

$$(\varphi_{s,\delta}(x))_{x_1} \ge \frac{d_3}{\delta} > 0.$$

Moreover, the following inclusions are obvious:

$$\begin{gathered} \Omega(s+\delta) \subset \{x = (x_1, x') : x_1 \ge d_1(s + \delta(\zeta(|x'|/\delta) + 15/16))\} \subset \Omega(s + 15/16\delta) \\ \subset \{x : x_1 \ge d_1(s + \delta\zeta(|x'|/\delta))\} \subset \Omega(s), \\ K(s,\delta) \supset \{x : d_1(s + \delta\zeta(|x'|/\delta)) \le x_1 \le d_1(s + \delta(\zeta(|x'|/\delta) + 15/16))\}, \\ K(s + \delta/4, \delta/4) \subset \{x : d_1(s + \delta(\zeta(|x'|/\delta) + 3/16)) \le x_1 \le d_1(s + \delta(\zeta(|x'|/\delta) + 3/4))\}. \end{gathered}$$

In particular, these inclusions imply that $\operatorname{supp}\varphi_{s,\delta}(x) \Subset \Omega(s)$.

Putting

$$\zeta^4(x,t) = \varphi^4_{s,\delta}(x)\exp(-tT^{-1}) \quad \forall T > 0$$

in the local entropy estimate (6) and making simple transformations, we come to (7) and (8). $\square$

Lemma 2. *Let $u(x,t)$ be an arbitrary strong solution to (1)–(3). Introduce the energy functions which are connected with this solution:*

$$E_T(s) := \iint\limits_{Q_T(s)} u^{\alpha+m+1}, \quad I_T(s) := \iint\limits_{Q_T(s)} u^{\alpha+n+1}, \quad F_T(s) := \iint\limits_{Q_T(s)} u^{\alpha+\lambda}. \tag{9}$$

Then the following estimates for the decay rate are valid:

$$\begin{gathered} E_T(s) \le c_1(u_0)Ts^{-N(\alpha+m)}, \quad I_T(s) \le c_2(u_0)Ts^{-N(\alpha+n)}, \\ F_T(s) \le \begin{cases} c_3(u_0)Ts^{-\frac{N(\lambda+\alpha-1)}{N(n-\lambda+1)+4}}, & \text{if } 1 < \lambda \le n + 1 + \frac{4}{N}, \\ c_3(u_0)Ts^{-\frac{N(\lambda+\alpha-1)}{N(m-\lambda+1)+2}}, & \text{if } 1 < \lambda \le m + 1 + \frac{2}{N}, \end{cases} \end{gathered} \tag{10}$$

for arbitrary $s > 0$ and $T > 0$, where $c_i(u_0) = c_i(m, n, \lambda, N, K)$ and $K = \|u_0\|_{L^1(\mathbb{R}^N)}$.

Proof. Applying the interpolation inequality of Lemma A.1 in the domain $K(s,\delta)$ to the function $v_1 := u^{(\alpha+m+1)/2}$ for $a = d = 2$, $b = 2/(\alpha + m + 1)$, $i = 0$, and $j = 1$ and integrating the result with respect to time from 0 to T, we obtain

$$\iint\limits_{K_T(s,\delta)} u^{\alpha+m+1} \le c_4\delta^{-N(\alpha+m)}TK^{\alpha+m+1} + c_5T^{1-\mu_1}K^{(1-\mu_1)(\alpha+m+1)}\Bigg(\iint\limits_{K_T(s,\delta)} |\nabla u^{\frac{\alpha+m+1}{2}}|^2\Bigg)^{\mu_1}, \tag{11}$$

where $\mu_1 = \frac{N(\alpha+m)}{N(\alpha+m)+2}$. Similarly, applying the inequality of Lemma A.1 in $K(s,\delta)$ to the function $v_2 := u^{(\alpha+n+1)/2}$ for $a = d = 2$, $b = 2/(\alpha+n+1)$, $i = 0$, and $j = 2$ and integrating the result with respect to time, we find that

$$\iint\limits_{K_T(s,\delta)} u^{\alpha+n+1} \le c_6 \delta^{-N(\alpha+n)} T K^{\alpha+n+1} + c_7 T^{1-\mu_2} K^{(1-\mu_2)(\alpha+n+1)} \Bigg(\iint\limits_{K_T(s,\delta)} \big|D^2 u^{\frac{\alpha+n+1}{2}}\big|^2 \Bigg)^{\mu_2}, \qquad (12)$$

where $\mu_2 = \frac{N(\alpha+n)}{N(\alpha+n)+4}$. Applying the inequality of Lemma A.1 in the domain $K(s,\delta)$ to the function v_1 for $a = 2(\alpha+\lambda)/(\alpha+m+1)$, $d = 2$, $b = 2/(\alpha+m+1)$, $i = 0$, and $j = 1$ and the function v_2 for $a = 4(\alpha+\lambda)/(\alpha+n+1)$, $d = 4$, $b = 4/(\alpha+n+1)$, $i = 0$, and $j = 1$ and then integrating he result with respect to time, we obtain

$$\iint\limits_{K_T(s,\delta)} u^{\alpha+\lambda} \le c_8 \delta^{-N(\alpha+\lambda-1)} T K^{\alpha+\lambda} + c_9 T^{1-\mu_{i+2}} c(K) \Bigg(\iint\limits_{K_T(s,\delta)} |\nabla v_i|^{2i} \Bigg)^{\mu_{i+2}}, \qquad (13)$$

$i = 1, 2$, where $\mu_3 = \frac{N(\alpha+\lambda-1)}{N(\alpha+m)+2} \le 1 \Rightarrow 1 < \lambda \le m+1+\frac{2}{N}$ and $\mu_4 = \frac{N(\alpha+\lambda-1)}{N(\alpha+n)+4} \le 1 \Rightarrow 1 < \lambda \le n+1+\frac{4}{N}$. Inserting (11)–(13) in (7) and applying Young's "ε"-inequality, we find that

$$L_T^{(1)}(s+\delta) \le \varepsilon \iint\limits_{K_T(s,\delta)} \big[|D^2 u^{\frac{\alpha+n+1}{2}}|^2 + |\nabla u^{\frac{\alpha+n+1}{4}}|^4 + |\nabla u^{\frac{\alpha+m+1}{2}}|^2\big]$$

$$+ c(\varepsilon, K) T (\delta^{-(N(\alpha+m)+2)} + \delta^{-(N(\alpha+n)+4)} + \delta^{-\frac{N(\alpha+n)+4}{N(n-\lambda+1)+4}}) \quad \forall \varepsilon > 0,\ s \ge 0,\ \delta > 0,$$

or

$$L_T^{(1)}(s+\delta) \le \varepsilon \iint\limits_{K_T(s,\delta)} \big[|D^2 u^{\frac{\alpha+n+1}{2}}|^2 + |\nabla u^{\frac{\alpha+n+1}{4}}|^4 + |\nabla u^{\frac{\alpha+m+1}{2}}|^2\big]$$

$$+ c(\varepsilon, K) T (\delta^{-(N(\alpha+m)+2)} + \delta^{-(N(\alpha+n)+4)} + \delta^{-\frac{N(\alpha+m)+2}{N(m-\lambda+1)+2}}) \quad \forall \varepsilon > 0,\ s \ge 0,\ \delta > 0,$$

where $L_T^{(1)}(s+\delta)$ is defined in (7). Choosing

$$\varepsilon \in (0, \min\{2^{-(N(\alpha+n)+4)}, 2^{-(N(\alpha+n)+4)}, 2^{-\frac{N(\alpha+n)+4}{N(n-\lambda+1)+4}}, 2^{-\frac{N(\alpha+m)+2}{N(m-\lambda+1)+2}}\})$$

sufficiently small, by the standard iteration procedure we establish that

$$L_T^{(1)}(s_0+\delta_0) \le c(K) T U_i(\delta_0) \quad \forall s_0 \ge 0,\ \delta_0 > 0,\ i = 1, 2, \qquad (14)$$

where

$$U_1(\delta_0) := \delta_0^{-(N(\alpha+m)+2)} + \delta_0^{-(N(\alpha+n)+4)} + \delta_0^{-\frac{N(\alpha+m)+2}{N(m-\lambda+1)+2}},$$

$$U_2(\delta_0) := \delta_0^{-(N(\alpha+m)+2)} + \delta_0^{-(N(\alpha+n)+4)} + \delta_0^{-\frac{N(\alpha+n)+4}{N(n-\lambda+1)+4}}.$$

Let $\delta \to +\infty$ in (11)–(13) and use (14) for $s_0 = 0$ and $\delta_0 = s > 0$. We eventually obtain

$$I_T(s) \le c(K) T U^{\mu_1}(s),\ E_T(s) \le c(K) T U^{\mu_2}(s),\ F_T(s) \le c(K) T U^{\mu_{i+2}}(s), \quad i = 1, 2,$$

for every $s > 0$. Hence, (10) follows. $\square$

Lemma 3. *Suppose that* $\chi_1 \geq 0$, $b(u) \geq du^{\lambda}$, *and* $u(t,x)$ *is an arbitrary strong solution to* (1)–(3). *Then the following holds for arbitrary* $s \geq 0$, $\delta > 0$, *and* $T > 0$*:*

$$\sup_{t\in(0,T)} \int\limits_{\Omega(s+\delta)} u^{\alpha+1}(t) + \frac{1}{T} \iint\limits_{Q_T(s+\delta)} u^{\alpha+1} + \frac{cd_3\chi_1}{\delta} \iint\limits_{Q_T(s+\delta)} u^{\alpha+\lambda}$$
$$+c_3^{-1} \iint\limits_{Q_T(s+\delta)} [|D^2 u^{\frac{\alpha+n+1}{2}}|^2 + |\nabla u^{\frac{\alpha+n+1}{4}}|^4 + |\nabla u^{\frac{\alpha+m+1}{2}}|^2]$$
$$\leq c\Big(\delta^{-2} \iint\limits_{Q_T(s)} u^{\alpha+m+1} + \delta^{-4} \iint\limits_{Q_T(s)} u^{\alpha+n+1}\Big) := \widetilde{R}_T^{(2)}(s,\delta). \qquad (15)$$

Proof. Given $s_0 \geq 0$ and $\delta > 0$, we put $s_i = s_{i-1} + \delta/4$, $i \in \mathbb{N}$. Putting $s = s_i$ in (8) and summing the resulting inequalities, we arrive at (15). $\square$

Lemma 4. *Suppose that* $\chi_1 \geq 0$, $b(u) \geq du^{\lambda}$, *and* $E_T(s)$ *and* $I_T(s)$ *are the function from* (9) *connected with an arbitrary strong solution* $u(t,x)$ *to* (1)–(3). *Then the following estimates for the decay rate hold for arbitrary* $s > 0$ *and* $T > 0$*:*

$$I_T(s) \leq c_{10} T \Gamma^N(T) s^{-\frac{3(\alpha+n+1)}{\lambda-n-1}}, \quad E_T(s) \leq c_{11} T \Gamma^N(T) s^{-\frac{\alpha+m+1}{\lambda-m-1}} \qquad (16)$$

if $\lambda > \max\{n+1, m+1\}$, *where* $c_i = c_i(m,n,\lambda,N)$ *and* $\Gamma(T)$ *is from Theorem 2;*

$$I_T(s) \leq c_{12}(u_0) s^{-\xi_1}, \ E_T(s) \leq c_{13}(u_0) s^{-\xi_2} \quad \text{if } \lambda < \min\{n+1, m+1\},$$
$$I_T(s) = E_T(s) \leq c_{14}(u_0) T^{1-\theta_i} d_4^{-s} \quad \text{if } \lambda = n+1 = m+1 \qquad (17)$$

for every $s > 0$, *where*

$$\theta_1 = mN/(mN + 2(\alpha+1)), \quad \theta_2 = nN/(nN + 4(\alpha+1)),$$
$$c_i(u_0) = c_i(m,n,\lambda,N,\|u_0\|_{L^{\alpha+1}(\mathbb{R}^N)}), \quad d_4 > 1,$$

and

$$s^{-\xi_1} = \max\{s^{-4\frac{(\alpha+1)(N(m-\lambda+1)+2(\alpha+1))}{N(m-\lambda+1)(N(n-\lambda+1)+4(\alpha+1))}}, s^{-12\frac{\alpha+1}{N(n-\lambda+1)}}\},$$
$$s^{-\xi_2} = \max\{s^{-\frac{6(\alpha+1)(N(n-\lambda+1)+4(\alpha+1))}{N(n-\lambda+1)(N(m-\lambda+1)+2(\alpha+1))}}, s^{-2\frac{\alpha+1}{N(m-\lambda+1)}}\}.$$

Proof. Applying Young's "ε"-inequality with the exponents $(\alpha+\lambda)/(\alpha+m+1) > 1$ ($\Rightarrow \lambda > m+1$), $(\alpha+\lambda)/(\lambda-m-1)$ and $(\alpha+\lambda)/(\alpha+n+1) > 1$ ($\Rightarrow \lambda > n+1$), $(\alpha+\lambda)/(\lambda-n-1)$, we estimate the integrals on the right-hand side of (15):

$$c\delta^{-2} \iint\limits_{Q_T(s)} u^{\alpha+m+1} \leq \varepsilon \frac{c\chi_1}{\delta} \iint\limits_{Q_T(s)} u^{\alpha+\lambda}$$
$$+c(\varepsilon) T \chi_1^{-\frac{\alpha+m+1}{\lambda-m-1}} \delta^{\frac{\alpha+\lambda}{\lambda-m-1}(\frac{\alpha+m+1}{\alpha+\lambda}-2)} \operatorname{mes}\{\Omega(s) \cap \operatorname{supp} u(t,.)\} \quad \forall \varepsilon \in (0,1);$$

$$c\delta^{-4} \iint\limits_{Q_T(s)} u^{\alpha+n+1} \leq \varepsilon \frac{c\chi_1}{\delta} \iint\limits_{Q_T(s)} u^{\alpha+\lambda}$$
$$+c(\varepsilon) T \chi_1^{-\frac{\alpha+n+1}{\lambda-n-1}} \delta^{\frac{\alpha+\lambda}{\lambda-n-1}(\frac{\alpha+n+1}{\alpha+\lambda}-4)} \operatorname{mes}\{\Omega(s) \cap \operatorname{supp} u(t,.)\} \quad \forall \varepsilon \in (0,1).$$

Inserting the above estimates in (15), choosing

$$\varepsilon \in (0, \min\{2^{\frac{\alpha+\lambda}{\lambda-m-1}(\frac{\alpha+m+1}{\alpha+\lambda}-2)}, 2^{\frac{\alpha+\lambda}{\lambda-n-1}(\frac{\alpha+n+1}{\alpha+\lambda}-4)}\})$$

and iterating the so-obtained inequality for $s = s_i$, $\delta = \delta_i$, $s_i = s_{i-1} + \delta_{i-1} - \delta_i$, and $\delta_i = 2^{-i}\delta_0$, $i = \overline{1, \infty}$, $s_0 \geq 0$, $\delta_0 > 0$, we establish that

$$\begin{aligned} &\sup_{t\in(0,T)} \int\limits_{\Omega(s_0+\delta_0)} u^{\alpha+1}(t) + \frac{1}{T} \iint\limits_{Q_T(s_0+\delta_0)} u^{\alpha+1} + \frac{cd_3\chi_1}{\delta} \iint\limits_{Q_T(s_0+\delta_0)} u^{\alpha+\lambda} \\ &+ c_3^{-1} \iint\limits_{Q_T(s_0+\delta_0)} [|D^2 u^{\frac{\alpha+n+1}{2}}|^2 + |\nabla u^{\frac{\alpha+n+1}{4}}|^4 + |\nabla u^{\frac{\alpha+m+1}{2}}|^2] \\ &\leq cT\big(\chi_1^{-\frac{\alpha+m+1}{\lambda-m-1}} \delta_0^{\frac{\alpha+\lambda}{\lambda-m-1}(\frac{\alpha+m+1}{\alpha+\lambda}-2)} + \chi_1^{-\frac{\alpha+n+1}{\lambda-n-1}} \delta_0^{\frac{\alpha+\lambda}{\lambda-n-1}(\frac{\alpha+n+1}{\alpha+\lambda}-4)}\big) \\ &\times(\Gamma(T) - s_0)_+^N \quad \forall \delta_0 > 0,\ s_0 \geq 0, \end{aligned} \tag{18}$$

where $\lambda > \max\{n+1, m+1\}$. Apply Hölder's inequality with the exponents $(\alpha+\lambda)/(\alpha+m+1) > 1$, $(\alpha+\lambda)/(\lambda-m-1)$ and $(\alpha+\lambda)/(\alpha+n+1) > 1$, $(\alpha+\lambda)/(\lambda-n-1)$ to the functions $E_T(s)$ and $I_T(s)$. Using (18) with $s_0 = 0$ and $\delta_0 = s > 0$, we deduce

$$\begin{aligned} E_T(s) &\leq \Big(\iint\limits_{Q_T(s)} u^{\alpha+\lambda}\Big)^{\frac{\alpha+m+1}{\alpha+\lambda}} \Big(\iint\limits_{Q_T(s)} 1\Big)^{\frac{\lambda-m-1}{\alpha+\lambda}} \leq c[T\Gamma^N(T)]^{\frac{\lambda-m-1}{\alpha+\lambda}} \\ &\times [Ts^{1+\frac{\alpha+\lambda}{\lambda-m-1}(\frac{\alpha+m+1}{\alpha+\lambda}-2)}\Gamma^N(T)]^{\frac{\alpha+m+1}{\alpha+\lambda}} = cT\Gamma^N(T)s^{-\frac{\alpha+m+1}{\lambda-m-1}} \quad \text{if } \lambda > m+1, \end{aligned}$$

and

$$\begin{aligned} I_T(s) &\leq \Big(\iint\limits_{Q_T(s)} u^{\alpha+\lambda}\Big)^{\frac{\alpha+n+1}{\alpha+\lambda}} \Big(\iint\limits_{Q_T(s)} 1\Big)^{\frac{\lambda-n-1}{\alpha+\lambda}} \leq c[T\Gamma^N(T)]^{\frac{\lambda-n-1}{\alpha+\lambda}} \\ &\times [Ts^{1+\frac{\alpha+\lambda}{\lambda-n-1}(\frac{\alpha+n+1}{\alpha+\lambda}-4)}\Gamma^N(T)]^{\frac{\alpha+n+1}{\alpha+\lambda}} = cT\Gamma^N(T)s^{-\frac{3(\alpha+n+1)}{\lambda-n-1}} \quad \text{if } \lambda > n+1. \end{aligned}$$

Thereby, (16) are established.

Now, validate (17). Applying Lemma A.2 in the domain $Q_T(s+\delta)$ to the function $v_1 := u^{(\alpha+m+1)/2}$ for $p = 2$, $r = 2(\alpha+\lambda)/(\alpha+m+1)$, $q = 2(\alpha+1)/(\alpha+m+1)$, and $1-b_1 = 2(\alpha+1)/(N(m-\lambda+1)+2(\alpha+1))$, we obtain

$$\begin{aligned} E_T(s+\delta) &:= \iint\limits_{Q_T(s+\delta)} u^{\alpha+m+1} \leq c\Big(\iint\limits_{Q_T(s+\delta)} |\nabla u^{\frac{\alpha+m+1}{2}}|^2\Big)^{b_1} \Big(\iint\limits_{Q_T(s+\delta)} u^{\alpha+\lambda}\Big)^{1-b_1} \\ &\times \sup_{t\in(0,T)} \Big(\int\limits_{\Omega(s+\delta)} u^{\alpha+1}(t)\Big)^{\frac{m-\lambda+1}{\alpha+1}(1-b_1)} \overset{(15)}{\leq} c\delta^{1-b_1}(\widetilde{R}_T^{(2)}(s,\delta))^{1+\ell_1}, \end{aligned} \tag{19}$$

where $\ell_1 = 2(m-\lambda+1)/(N(m-\lambda+1)+2(\alpha+1))$ and $\lambda < m+1$. Similarly, applying Lemma A.2 in the domain $Q_T(s+\delta)$ to the function $v_2 := u^{(\alpha+n+1)/4}$ for $p = 4$, $r = 4(\alpha+\lambda)/(\alpha+n+1)$, $q = 4(\alpha+1)/(\alpha+n+1)$, and $1-b_2 = 4(\alpha+1)/(N(n-\lambda+1)+4(\alpha+1))$, we infer

$$\begin{aligned} I_T(s+\delta) &:= \iint\limits_{Q_T(s+\delta)} u^{\alpha+n+1} \leq c\Big(\iint\limits_{Q_T(s+\delta)} |\nabla u^{\frac{\alpha+n+1}{4}}|^2\Big)^{b_2} \Big(\iint\limits_{Q_T(s+\delta)} u^{\alpha+\lambda}\Big)^{1-b_2} \\ &\times \sup_{t\in(0,T)} \Big(\int\limits_{\Omega(s+\delta)} u^{\alpha+1}(t)\Big)^{\frac{n-\lambda+1}{\alpha+1}(1-b_2)} \overset{(15)}{\leq} c\delta^{1-b_2}\big(\widetilde{R}_T^{(2)}(s,\delta)\big)^{1+\ell_2}, \end{aligned} \tag{20}$$

where $\ell_2 = 4(n-\lambda+1)/(N(n-\lambda+1)+4(\alpha+1))$ and $\lambda < n+1$. Using the global entropy estimate ((6) with $\zeta \equiv 1$), from (19) and (20) we derive

$$E_T(s+\delta) \le c(u_0)\Big(\iint\limits_{Q_T(s+\delta)} u^{\alpha+\lambda}\Big)^{1-b_1}, \ b_1 = \frac{N(m-\lambda+1)}{N(m-\lambda+1)+2(\alpha+1)}, \ \lambda < m+1;$$

$$I_T(s+\delta) \le c(u_0)\Big(\iint\limits_{Q_T(s+\delta)} u^{\alpha+\lambda}\Big)^{1-b_2}, \ b_2 = \frac{N(n-\lambda+1)}{N(n-\lambda+1)+4(\alpha+1)}, \ \lambda < n+1,$$

where $c(u_0) = c(n,m,\lambda,N,\|u_0\|_{L^{\alpha+1}(\mathbb{R}^N)})$. Let $Z_T(s) := E_T^{1-b_2}(s) + I_T^{1-b_1}(s)$. Then the above inequalities and estimate (15) imply that

$$Z_T(s+\delta) \le c(u_0)\big(\delta^{-\beta} Z_T^{1-b_1}(s) + \delta^{-3\beta} Z_T^{1-b_2}(s)\big), \quad \beta = (1-b_1)(1-b_2).$$

Hence, applying Young's "ε"-inequality, we find that

$$Z_T(s+\delta) \le \varepsilon Z_T(s) + c(u_0)(\varepsilon^{-\frac{1-b_1}{b_1}}\delta^{-\frac{\beta}{b_1}} + \varepsilon^{-\frac{1-b_2}{b_2}}\delta^{-\frac{3\beta}{b_2}}) \quad \forall \varepsilon \in (0,1).$$

Since the function $Z_T(s)$ decreases in s, we iterate this inequality for $s = s_i$ and $\delta = \delta_i$, with $s_0 \ge 0$, $\delta_0 > 0$, $s_i = s_{i-1}+\delta_{i-1}-\delta_i$, and $\delta_i = 2^{-i}\delta_0$, $i = \overline{1,\infty}$. Choosing ε in the interval $(0, \min\{2^{-\beta/b_1}, 2^{-3\beta/b_2}\})$, we obtain

$$Z_T(s_0+\delta_0) \le c(u_0)\big(\delta_0^{-\frac{\beta}{b_1}} + \delta_0^{-\frac{3\beta}{b_2}}\big),$$

whence, putting $s_0 = 0$ and $\delta_0 = s > 0$, we arrive at the estimate

$$Z_T(s) \le c(u_0)(s^{-\frac{\beta}{b_1}} + s^{-\frac{3\beta}{b_2}}) \quad \forall s > 0.$$

Recalling the definition of $Z_T(s)$, we obtain

$$E_T(s) \le c(u_0)\max\{s^{-\frac{1-b_1}{b_1}}, s^{-\frac{3(1-b_1)}{b_2}}\} \quad \forall s > 0,$$

$$I_T(s) \le c(u_0)\max\{s^{-\frac{1-b_2}{b_1}}, s^{-\frac{3(1-b_2)}{b_2}}\} \quad \forall s > 0,$$

provided that $\lambda < \min\{n+1, m+1\}$, whence the sought estimates for the decay rate result.

In the case of $\lambda = n+1 = m+1$ it follows from (15) that

$$\frac{cd_3\chi_1\delta^3}{1+\delta^2} \iint\limits_{Q_T(s+\delta)} u^{\alpha+n+1} \le \iint\limits_{Q_T(s)} u^{\alpha+n+1} \quad \forall s \ge 0, \ \delta > 0,$$

thereby for every $i \in \mathbb{N}$

$$\Big(\frac{cd_3\chi_1\delta^3}{1+\delta^2}\Big)^i \iint\limits_{Q_T(s+i\delta)} u^{\alpha+n+1} \le \iint\limits_{Q_T(s)} u^{\alpha+n+1}.$$

Choosing δ_0 such that $d = \big(cd_3\chi_1\delta_0^3/\big(1+\delta_0^2\big)\big)^{1/\delta_0} > 1$, we obtain

$$d^{i\delta_0} I_T(s+i\delta_0) \le I_T(s) \quad \forall s \ge 0, \ \delta_0 > 0;$$

consequently,

$$I_T(s+\sigma) \le d^{\delta_0-\sigma} I_T(s) \quad \forall \sigma > 0, \ s \ge 0, \ d > 1.$$

Putting $s = 0$ and $\sigma = s > 0$ in the above estimate, we have

$$I_T(s) = E_T(s) \le d^{\delta_0 - s} I_T(0) \quad \forall s > 0.$$

Applying the interpolation inequality from Lemma A.1 to the function v_1 for $a = d = 2$, $b = 2(\alpha + 1)/(\alpha + m + 1)$, $i = 0$, $j = 1$, and $\theta_1 = mN/(mN + 2(\alpha + 1))$ (v_2 for $a = d = 4$, $b = 4(\alpha + 1)/(\alpha + n + 1)$, $i = 0$, $j = 1$, and $\theta_2 = nN/(nN + 4(\alpha + 1))$), integrating the result with respect to t and using (6) with $\zeta = 1$, we find that

$$I_T(0) = E_T(0) \le c(u_0) T^{1-\theta_i}.$$

Thus, the proof of Lemma 4 is complete. $\square$

We have carried out all auxiliary arguments and are ready now to prove Theorem 2.

Proof of Theorem 2. (i) Let $I_T(s)$, $E_T(s)$, and $F_T(s)$ be defined in (9). Apply the interpolation inequality of Lemma A.1 in the domain $\Omega(s + \delta)$ to the functions $v_1 := u^{(\alpha+m+1)/2}$ for $a = d = 2$, $b = 2(\alpha + 1)/(\alpha + m + 1)$, $i = 0$, $j = 1$, and $\theta_1 = mN/(mN + 2(\alpha + 1))$; $v_2 := u^{(\alpha+n+1)/4}$ for $a = d = 4$, $b = 4(\alpha + 1)/(\alpha + n + 1)$, $i = 0$, $j = 1$, and $\theta_2 = nN/(nN + 4(\alpha + 1))$; v_1 for $a = 2(\alpha + \lambda)/(\alpha + m + 1)$, $d = 2$, $b = 2(\alpha + 1)/(\alpha + m + 1)$, $i = 0$, $j = 1$, $\tilde{\theta}_3 = N(\lambda - 1)(\alpha + m + 1)[(\alpha + \lambda)(mN + 2(\alpha + 1))]^{-1}$, and $\lambda > 1$; and v_2 for $a = 4(\alpha + \lambda)/(\alpha + n + 1)$, $d = 4$, $b = 4(\alpha + 1)/(\alpha + n + 1)$, $i = 0$, $j = 1$, $\tilde{\theta}_4 = N(\lambda - 1)(\alpha + n + 1)[(\alpha + \lambda)(nN + 4(\alpha + 1))]^{-1}$, and $\lambda > 1$. Integrate the resulting inequality with respect to t. Using (7), we arrive at the relations

$$E_T(s + \delta) \le cT^{1-\theta_1} \big(R_T^{(1)}(s, \delta)\big)^{1+k_1}, \quad \theta_1 = \frac{mN}{mN + 2(\alpha + 1)}, \ k_1 = \frac{2m}{mN + 2(\alpha + 1)}, \tag{21}$$

$$I_T(s + \delta) \le cT^{1-\theta_2} \big(R_T^{(1)}(s, \delta)\big)^{1+k_2}, \quad \theta_2 = \frac{nN}{nN + 4(\alpha + 1)}, \ k_2 = \frac{4n}{nN + 4(\alpha + 1)}, \tag{22}$$

$$F_T(s + \delta) \le cT^{1-\theta_{i+2}} \big(R_T^{(1)}(s, \delta)\big)^{1+k_{i+2}}, \quad i = 1, 2,$$

where $T > 0$, $R_T^{(1)}(s, \delta)$ is defined in (7), and

$$\theta_3 = \frac{N(\lambda - 1)}{mN + 2(\alpha + 1)}, \ k_3 = \frac{2(\lambda - 1)}{mN + 2(\alpha + 1)} \quad \text{if } 1 < \lambda \le m + 1 + \frac{2(\alpha + 1)}{N},$$

$$\theta_4 = \frac{N(\lambda - 1)}{nN + 4(\alpha + 1)}, \ k_4 = \frac{4(\lambda - 1)}{nN + 4(\alpha + 1)} \quad \text{if } 1 < \lambda \le n + 1 + \frac{4(\alpha + 1)}{N}.$$

Introduce the auxiliary functions connected with the energy functions (9):

$$A_T(s + \delta) := E_T^{(1+k_2)(1+k_{i+2})}(s + \delta) \le cT^{(1-\theta_1)(1+k_2)(1+k_{i+2})} \big(R_T^{(1)}(s, \delta)\big)^{\beta_1},$$

$$B_T(s + \delta) := I_T^{(1+k_1)(1+k_{i+2})}(s + \delta) \le cT^{(1-\theta_2)(1+k_1)(1+k_{i+2})} \big(R_T^{(1)}(s, \delta)\big)^{\beta_1},$$

$$C_T(s + \delta) := F_T^{(1+k_1)(1+k_2)}(s + \delta) \le cT^{(1-\theta_{i+2})(1+k_1)(1+k_2)} \big(R_T^{(1)}(s, \delta)\big)^{\beta_1},$$

$$D_T(s + \delta) := T^h A_T(s + \delta) + B_T(s + \delta) + T^\mu C_T(s + \delta), \quad i = 1, 2,$$

where

$$\beta_1 := (1 + k_1)(1 + k_2)(1 + k_{i+2}),$$
$$h := [(1 - \theta_2)(1 + k_1) - (1 - \theta_1)(1 + k_2)](1 + k_{i+2}),$$
$$\mu := [(1 - \theta_2)(1 + k_{i+2}) - (1 - \theta_{i+2})(1 + k_2)](1 + k_1), \quad i = 1, 2.$$

Using the above inequalities and the inequality $(a-b)^{\alpha+1} \leq a^{\alpha+1} - b^{\alpha+1}\ \forall\alpha > 0,\ a > b > 0$, for the function $D_T(s)$ we obtain

$$\begin{aligned} D_T(s+\delta) \leq cT^{(1-\theta_2)(1+k_1)}(R_T^{(1)}(s,\delta))^{\beta_1} \\ \leq c[\delta^{-2\beta_1}T^{(1-\theta_2)(1+k_1)(1+k_{i+2})-h(1+k_1)}(\Delta D_T(s))^{1+k_1} \\ +\delta^{-4\beta_1}T^{(1-\theta_2)(1+k_1)(1+k_{i+2})}(\Delta D_T(s))^{1+k_2} \\ +\delta^{-\beta_1}T^{(1-\theta_2)(1+k_1)(1+k_{i+2})-\mu(1+k_1)}(\Delta D_T(s))^{1+k_{i+2}}] \\ \leq c\big[\delta^{-2\beta_1}T^{(1-\theta_2)(1+k_1)(1+k_{i+2})-h(1+k_1)}D_T^{k_1}(s) + \delta^{-4\beta_1}T^{(1-\theta_2)(1+k_1)(1+k_{i+2})}D_T^{k_2}(s) \\ +\delta^{-\beta_1}T^{(1-\theta_2)(1+k_1)(1+k_{i+2})-\mu(1+k_1)}D_T^{k_{i+2}}(s)\big]\Delta D_T(s), \end{aligned} \tag{23}$$

where $\Delta D_T(s) = D_T(s) - D_T(s+\delta)$. Now, we choose the parameter $\delta > 0$. Let

$$\delta_T^{(1)}(s) := \big[T^{(1-\theta_2)(1+k_1)(1+k_{i+2})-h(1+k_1)}D_T^{k_1}(s)\big]^{\frac{1}{2\beta_1}} = T^{\frac{(1-\theta_2)(1+k_{i+2})-h}{2(1+k_2)(1+k_{i+2})}}D_T^{\frac{k_1}{2\beta_1}}(s),$$

$$\delta_T^{(2)}(s) := \big[T^{(1-\theta_2)(1+k_1)(1+k_{i+2})}D_T^{k_2}(s)\big]^{\frac{1}{4\beta_1}} = T^{\frac{1-\theta_2}{4(1+k_2)}}D_T^{\frac{k_2}{4\beta_1}}(s),$$

$$\delta_T^{(i+2)}(s) := \big[T^{(1-\theta_2)(1+k_1)(1+k_{i+2})-\mu(1+k_1)}D_T^{k_{i+2}}(s)\big]^{\frac{1}{\beta_1}} = T^{\frac{(1-\theta_2)(1+k_1)-\mu}{(1+k_1)(1+k_2)}}D_T^{\frac{k_{i+2}}{\beta_1}}(s),$$

$$S_T^{(2)} := \big\{s \in \mathbb{R}^+ : \delta_T^{(2)}(s) \geq \max\big\{\delta_T^{(1)}(s), \delta_T^{(i+2)}(s)\big\}\big\},$$

$$S_T^{(i+2)} := \mathbb{R}^+ \backslash \big\{S_T^{(1)} \cup S_T^{(2)}\big\},$$

$$J_T(s) := \max\big\{\delta_T^{(1)}(s), \delta_T^{(2)}(s), \delta_T^{(i+2)}(s)\big\}, \quad i = 1,2.$$

Putting $\delta = \delta_T^{(1)}(s)$ in (23), we obtain

$$D_T\big(s+\delta_T^{(1)}(s)\big) \leq \frac{3c}{1+3c}D_T(s) \quad \forall s \in S_T^{(1)}. \tag{24}$$

Indeed, if $\delta_T^{(1)}(s) \geq \delta_T^{(k)}(s) > 0$ then

$$\big(\delta_T^{(1)}(s)\big)^{-4\beta}T^{(1-\theta_2)(1+k_1)(1+k_{i+2})}D_T^{k_2}(s) \leq \big(\delta_T^{(2)}(s)\big)^{-4\beta}T^{(1-\theta_2)(1+k_1)(1+k_{i+2})}D_T^{k_2}(s) = 1,$$

$$\begin{aligned} \big(\delta_T^{(1)}(s)\big)^{-\beta}T^{(1-\theta_2)(1+k_1)(1+k_{i+2})-\mu(1+k_1)}D_T^{k_{i+2}}(s) \\ \leq \big(\delta_T^{(i+2)}(s)\big)^{-\beta}T^{(1-\theta_2)(1+k_1)(1+k_{i+2})-\mu(1+k_1)}D_T^{k_{i+2}}(s) = 1. \end{aligned}$$

If $\delta_T^{(1)}(s) \geq \delta_T^{(k)}(s) = 0$ then $D_T(s) = 0$, $\delta_T^{(1)}(s) = 0$, and the assertion of the theorem is obvious. Similarly, we consider the cases $\delta = \delta_T^{(k)}(s)$:

$$D_T\big(s+\delta_T^{(k)}(s)\big) \leq \frac{3c}{1+3c}D_T(s) \quad \forall s \in S_T^{(k)},\ k = \overline{2,4}. \tag{25}$$

Combining (24) and (25), we obtain

$$D_T(s+J_T(s)) = \big\{D_T\big(s+\delta_T^{(k)}(s)\big), s \in S_T^{(k)}, k = \overline{1,4}\big\} \leq \frac{3c}{1+3c}D_T(s) \quad \forall s \in \mathbb{R}^+.$$

Taking the power $k_1/(2\beta_1)$ of both sides of the above inequality and multiplying the result by

$$T^{[(1-\theta_2)(1+k_1)(1+k_{i+2})-h(1+k_1)]/(2\beta_1)},$$

we obtain the inequality for $\delta_T^{(1)}(s)$, while taking the power $k_2/(4\beta_1)$ (k_{i+2}/β_1) and multiplying the result by $T^{(1-\theta_2)(1+k_1)(1+k_{i+2})/(4\beta_1)}$ $(T^{[(1-\theta_2)(1+k_1)(1+k_{i+2})-\mu(1+k_1)]/\beta_1})$, we find the inequality for $\delta_T^{(2)}(s)$ $(\delta_T^{(i+2)}(s))$. Combining the estimates obtained for $\delta_T^{(k)}(s)$, $k=\overline{1,4}$, we arrive at the basic functional relation for the function $J_T(s)$:

$$J_T(s+J_T(s)) \le \gamma J_T(s) \quad \forall s \in \mathbb{R}^+,\ 0<\gamma=\text{const}<1. \tag{26}$$

By Lemma A.3, it follows from (26) that

$$J_T(s) \equiv 0 \quad \forall s \ge s_0 + \frac{1}{1-\gamma} J_T(s_0). \tag{27}$$

Hence, we obtain an upper bound for the motion of the boundary of the support:

$$\Gamma(T) \le s_0 + \frac{1}{1-\gamma} J_T(s_0) = c \min_{s_0\in\mathbb{R}^+} \{\max_{j=\overline{1,9}} \{F_j(s_0)\}\} \quad \forall s_0 \ge 0. \tag{28}$$

Let $\widetilde{F}_i(s_0)$ be the right-hand sides of the estimates for $F_i(s_0)$ $(i=\overline{1,9})$ obtained by means of the estimates (10) for the decay rate. Using Lemma A.4, we minimize the functions $\widetilde{F}_i(s_0)$ over s_0. Eventually, we have

$$\min \widetilde{F}_1(s_0) = \widetilde{F}_1\big(s_{\text{opt}}^{(1)}\big) = c(u_0) s_{\text{opt}}^{(1)}, \quad \min \widetilde{F}_5(s_0) = \widetilde{F}_5\big(s_{\text{opt}}^{(5)}\big) = c(u_0) s_{\text{opt}}^{(5)},$$
$$\min \widetilde{F}_9(s_0) = \widetilde{F}_9\big(s_{\text{opt}}^{(9)}\big) = c(u_0) s_{\text{opt}}^{(9)},$$

where

$$s_{\text{opt}}^{(1)} = c(u_0) T^{1/(mN+2)}, \quad s_{\text{opt}}^{(5)} = c(u_0) T^{1/(nN+4)},$$
$$s_{\text{opt}}^{(9)} = \begin{cases} c(u_0) T^{[N(m-\lambda+1)+2]/(mN+2)} & \text{if } 1<\lambda\le m+1+2/N, \\ c(u_0) T^{[N(n-\lambda+1)+4]/(nN+4)} & \text{if } 1<\lambda\le n+1+4/N. \end{cases}$$

Carrying out simple but bulky computations, we find easily that

$$s_{\text{opt}}^{(k)} \le c(u_0) \max\big\{s_{\text{opt}}^{(1)}, s_{\text{opt}}^{(5)}, s_{\text{opt}}^{(9)}\big\}, \quad k\in\{2,3,4,6,7,8\}.$$

Thereby, assertion (i) is proven.

(ii) Using (8) and the scheme of (i), for the energy functions $I_T(s)$ and $E_T(s)$ in (9) we obtain the basic functional inequality of the form (26) with

$$J_T(s) := \max\big\{T^{\frac{1-\theta_2-\tilde{h}}{2(1+k_2)}} \big(T^{\tilde{h}} E_T^{1+k_2}(s) + I_T^{1+k_1}(s)\big)^{\frac{k_1}{2\beta_2}},$$
$$T^{\frac{1-\theta_2}{4(1+k_2)}} \big(T^{\tilde{h}} E_T^{1+k_2}(s) + I_T^{1+k_1}(s)\big)^{\frac{k_2}{4\beta_2}}\big\},$$

where $\beta_2 := (1+k_1)(1+k_2)$ and $\tilde{h} := (1-\theta_2)(1+k_1) - (1-\theta_1)(1+k_2)$. By Lemma A.3, we arrive at (27). This yields an upper bound for the motion of the boundary of the support:

$$\Gamma(T) \le c \min_{s_0\in\mathbb{R}^+} \{\max_{j=\overline{1,4}} \{F_j(s_0)\}\} \quad \forall s_0 \ge 0. \tag{29}$$

Using (16) and Lemma A.4, for $\Gamma(T)$ we infer

$$\Gamma(T) \le c s_{\text{opt}}^{(1)} = c T^{\frac{\lambda-m-1}{2(\lambda-1)-m}}, \quad \Gamma(T) \le c s_{\text{opt}}^{(4)} = c T^{\frac{\lambda-n-1}{4(\lambda-1)-n}},$$

$$\Gamma(T) \le cs_{\text{opt}}^{(2)}{=}cT^{\mu_1}, \;\; \mu_1{=}\frac{(\lambda-n-1)[(1-\theta_1)(1+k_2)+k_1\theta_2]}{3k_1(n+\alpha+1)+2(1+k_2)(\lambda-n-1)-Nk_1(\lambda-n-1)},$$

$$\Gamma(T) \le cs_{\text{opt}}^{(3)}{=}cT^{\mu_2}, \;\; \mu_2{=}\frac{(\lambda-m-1)[(1-\theta_2)(1+k_1)+k_2\theta_1]}{k_2(m+\alpha+1)+4(1+k_1)(\lambda-m-1)-Nk_2(\lambda-m-1)}.$$

It is easy to show that

$$s_{\text{opt}}^{(k)} \le \max\bigl\{s_{\text{opt}}^{(1)}, s_{\text{opt}}^{(4)}\bigr\}, \quad k=2,3.$$

This proves (ii).

(iii) Now, consider the case of $\chi_1 > 0$ and $\lambda < \min\{n+1, m+1\}$. Introduce the following auxiliary function for the energy functions $I_T(s)$ and $E_T(s)$ from (9):

$$D_T(s+\delta) := E_T^{1+\ell_2}(s+\delta) + I_T^{1+\ell_1}(s+\delta), \quad \text{with } \ell_i \text{ taken from (19) and (20).}$$

From (19) and (20) we derive

$$\begin{aligned} D_T(s+\delta) &\le c(\delta^{(1-b_1)(1+\ell_2)} + \delta^{(1-b_2)(1+\ell_1)})\bigl(\widetilde{R}_T^{(2)}(s,\delta)\bigr)^{\beta_3} \\ &\le c\bigl(\delta^{\kappa_1-2\beta_3} D_T^{\ell_1}(s) + \delta^{\kappa_1-4\beta_3} D_T^{\ell_2}(s)\bigr)\Delta D_T(s), \quad \text{with } b_i \text{ taken from (19) and (20),} \end{aligned}$$

where $\beta_3 := (1+\ell_1)(1+\ell_2)$, $\delta^{\kappa_1} = \max\{\delta^{(1-b_1)(1+\ell_2)}, \delta^{(1-b_2)(1+\ell_1)}\}$, and $\widetilde{R}_T^{(2)}(s,\delta)$ is taken from (15). Arguing as in (i), establish an inequality like (26) for

$$J_T(s) := c\max\Bigl\{\bigl(E_T^{1+\ell_2}(s) + I_T^{1+\ell_1}(s)\bigr)^{\frac{\ell_1}{2\beta_3-\kappa_1}}, \bigl(E_T^{1+\ell_2}(s) + I_T^{1+\ell_1}(s)\bigr)^{\frac{\ell_2}{4\beta_3-\kappa_1}}\Bigr\}.$$

By Lemma A.3, (26) implies (27) and an analog of (29). Estimate the right-hand side of (29), using (17), and then minimize the resulting estimates by Lemma A.4. Since the constants $c_{12}(u_0)$ and $c_{13}(u_0)$ in (17) are independent of T, after simple transformations we arrive at the estimate

$$\Gamma(T) \le c_{16}(u_0) \quad \forall T > 0,$$

where $0 < c_{16}(u_0) = c_{16}(m, n, \lambda, N, \|u_0\|_{L^{\alpha+1}(\mathbb{R}^N)})$, which yields (iii).

(iv) In the case of $\chi_1 > 0$ and $\lambda = n+1 = m+1$ the energy functions $I_T(s)$ and $E_T(s)$ from (9) coincide; therefore, applying the interpolation inequality of Lemma A.1 in the domain $\Omega(s+\delta)$ to the function v_1 for $a=d=2$, $b = 2(\alpha+1)/(\alpha+m+1)$, $i=0$, $j=1$, and $\theta_1 = mN/(mN+2(\alpha+1))$ (v_2 for $a=d=4$, $b=4(\alpha+1)/(\alpha+n+1)$, $i=0$, $j=1$, and $\theta_2 = nN/(nN+4(\alpha+1))$), integrating the result with respect to t, and using (15), we obtain

$$\begin{aligned} I_T(s+\delta) = E_T(s+\delta) &\le cT^{1-\theta_i}\bigl(\widetilde{R}_T^{(2)}(s,\delta)\bigr)^{1+k_i} \\ &\le cT^{1-\theta_i}\delta^{-\kappa_2(1+k_i)} I_T^{1+k_i}(s), \quad \delta^{-\kappa_2} = \max\{\delta^{-2}, \delta^{-4}\}, \; i=1,2, \end{aligned}$$

where θ_i and k_i are taken from (21) and (22) and $\widetilde{R}_T^{(2)}(s,\delta)$, from (15). Putting

$$\delta = \delta_T(s) = \bigl[(1+c)^{-1}T^{1-\theta_i} I_T^{k_i}(s)\bigr]^{\frac{1}{\kappa_2(1+k_i)}}$$

in this inequality, we find that

$$I_T(s+\delta_T(s)) \le \frac{c}{1+c} I_T(s) \quad \forall s > 0.$$

Hence, after simple transformations it follows that

$$\delta_T(s+\delta_T(s)) \le \gamma\delta_T(s) \quad \forall s \in \mathbb{R}^+, \; 0 < \gamma < 1.$$

By Lemma A.3, the above inequality implies that

$$\delta_T(s) \equiv 0 \quad \forall s \geq s_0 + \frac{1}{1-\gamma}\delta_T(s_0).$$

Hence, from (17), we see

$$\Gamma(T) \leq s_0 + c(u_0)T^{1-\theta_i} d_4^{-\eta_0 s_0}, \quad \eta_0 = \frac{k_i}{\kappa_2(1+k_i)}, \quad d_4 > 1.$$

Minimizing the right-hand side of this inequality over s_0 by means of Lemma A.4, we obtain

$$\Gamma(T) \leq \left\{ \begin{array}{rl} c(u_0)T^{1-\theta_i} & \text{if } c(u_0)T^{1-\theta_i} < \eta_0^{-1} \\ c(1+\log(c(u_0)\eta_0 T^{1-\theta_i})) & \text{if } c(u_0)T^{1-\theta_i} > \eta_0^{-1} \end{array} \right\} \leq c(u_0)(1+\log(1+T)) \quad \forall T > 0,$$

where $c(u_0)$ depends on $\|u_0\|_{L^{\alpha+1}(\mathbb{R}^N)}$. Thereby, Theorem 2 is proven completely.

A. Appendix.

Lemma A.1 [27]. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with piecewise smooth boundary, $a > 1$, $b \in (0,a)$, $d > 1$, and $0 \leq i < j$, $i,j \in \mathbb{N}$, then there exist positive constants d_1 and d_2 ($d_2 = 0$ if Ω is unbounded) depending only on Ω, d, j, b, and N such that the following inequality is valid for every $v(x) \in W_d^j(\Omega) \cap L^b(\Omega)$:*

$$\|D^i v\|_{L^a(\Omega)} \leq d_1 \|D^j v\|^{\theta}_{L^d(\Omega)} \|v\|^{1-\theta}_{L^b(\Omega)} + d_2 \|v\|_{L^b(\Omega)}, \quad \theta = \frac{\frac{1}{b} + \frac{i}{N} - \frac{1}{a}}{\frac{1}{b} + \frac{j}{N} - \frac{1}{d}} \in [i/j, 1).$$

Lemma A.2 [28]. *Let $\Omega(s) := \{x = (x_1, x') : x_1 > d(s + |x'|),\ 0 < d < \infty\}$, $\Omega(s) \subset \mathbb{R}^N$, $p > 1$, $r > 0$, $r \leq p$, $q > 0$, and $1 - b = pq(pq + N(p-r))^{-1}$. Then the following are valid for all $v(t,x) \in L^p\big(0,T; W^1_{p,\mathrm{loc}}(\Omega(s))\big)$:*

$$\int_0^T \int_{\Omega(s)} |v|^p \leq d_1 \Big(\int_0^T \int_{\Omega(s)} |\nabla v|^p \Big)^b \Big(\int_0^T \int_{\Omega(s)} |v|^r \Big)^{1-b} \sup_{t \in [0,T]} \Big(\int_{\Omega(s)} |v|^q \Big)^{\frac{(p-r)(1-b)}{q}} \quad \forall T > 0,$$

where $0 < d_1 = d_1(N,p,q,r)$ is independent of T.

Lemma A.3 [14]. *Let $f : [0,\infty) \to [0,\infty)$ be a nonnegative nonincreasing function such that $f(s + f(s)) \leq \gamma f(s)$ $\forall s \geq s_0 \geq 0$, $0 < \gamma < 1$. Then*

$$f(s) \equiv 0 \quad \forall s \geq s_0 + (1-\gamma)^{-1} f(s_0).$$

Lemma A.4. *Let $F(s) = s + ds^{-a}$, $a > 0$, and $d > 0$ ($F(s) = s + bd_1^{-a_1 s}$, $a_1 > 0$, $b > 0$, and $d_1 > 1$) $\forall s > 0$. Then*

$$\min F(s) = F(s_{\mathrm{opt}}) = (1 + a^{-1}) s_{\mathrm{opt}}$$

$$(\min F(s) = F(s_{\mathrm{opt}}) = (a_1 \log d_1)^{-1}(1 + \log(a_1 b \log d))),$$

where $s_{\mathrm{opt}} = (ad)^{\frac{1}{a+1}}$ $\big(s_{\mathrm{opt}} = a_1^{-1} \log_{d_1}(a_1 b \log d_1)\big)$.

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