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NUMERICAL METHODS.
FINITE-DIFFERENCE EQUATIONS

Stability of the Zero Solution of an Almost Periodic System of Finite-Difference Equations

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1. INTRODUCTION

Finite-difference equations have long been studied in numerous fields of mathematics. The first results in qualitative theory of such systems were obtained by Poincaré and Perron at the end of the nineteenth and the beginning of the twentieth century. A systematic exposition of the foundations of the theory of finite-difference equations can be found in [1–5]. Finite-difference equations prove to be a convenient model for discrete dynamical systems and for the mathematical modeling of sampled-data systems [6–8]. One of the directions arising in applications of finite-difference equations is related to the qualitative study of their solutions (stability, boundedness, controllability, observability, oscillation, robustness, etc.). There is a wide literature dealing with this direction (e.g., see [9–15]).

In the present paper, we consider the discrete system

$$x_{n+1} = f_n\left(x_n\right), \qquad f_n(0) = 0, \tag{1}$$

where $n = 0, 1, \ldots$ is the discrete time, $x_n = (x_n^1, x_n^2, \ldots, x_n^p) \in \mathbf{R}^p$, $f_n = (f_n^1, f_n^2, \ldots, f_n^p) \in \mathbf{R}^p$, and the functions f_n are assumed to satisfy the Lipschitz condition uniformly with respect to n: $\|f_n(x) - f_n(y)\| \le L_r \|x - y\|$ for $\|x\| \le r$ and $\|y\| \le r$. System (1) admits the trivial solution

$$x_n \equiv 0. \tag{2}$$

By $x_n\left(n_0, u\right)$ we denote the solution of system (1) coinciding with u for $n = n_0$. Next, we set $B_r = \{x \in \mathbf{R}^p : \|x\| \le r\}$. Throughout the following, we assume that the functions $f_n(x)$ are defined in B_H, where $H > 0$ is a given number.

2. MAIN DEFINITIONS AND SOME AUXILIARY ASSERTIONS

By analogy with ordinary differential equations [16–19], we introduce the following definitions.

Definition 1. The solution (2) of system (1) is said to be *stable* if, for arbitrary $\varepsilon > 0$ and $n_0 \in \mathbf{N}$, there exists a $\delta = \delta\left(\varepsilon, n_0\right) > 0$ such that if $\|x_{n_0}\| \le \delta$, then $\|x_n\| \le \varepsilon$ for all $n > n_0$. Here $\mathbf{N}$ is the set of nonnegative integers.

Definition 2. The solution (2) of system (1) is said to be *uniformly stable* if the number δ in Definition 1 can be chosen to be independent of n_0, $\delta = \delta(\varepsilon)$.

Definition 3. The solution (2) of system (1) is said to be *attracting* if, for each $n_0 \in \mathbf{N}$, there exists an $\eta = \eta\left(n_0\right) > 0$ such that, for any $\varepsilon > 0$ and $\|x_{n_0}\| < \eta$, there exists a $\sigma = \sigma\left(n_0, \varepsilon, \eta\right) \in \mathbf{N}$ with the property $\|x_n\| < \varepsilon$ for all $n \ge n_0 + \sigma$.

Definition 4. The solution (2) of system (1) is said to be *uniformly attracting* if, for some $\eta > 0$ and for each $\varepsilon > 0$, there exists a $\sigma = \sigma(\varepsilon) \in \mathbf{N}$ such that $\|x_n\| < \varepsilon$ whenever $n \ge n_0 + \sigma$, $n_0 \in \mathbf{N}$, and $\|x_{n_0}\| < \eta$.

Definition 5. The solution (2) of system (1) is said to be *asymptotically stable* if it is stable and attracting and *uniformly asymptotically stable* if it is uniformly stable and uniformly attracting.

Definition 6. A sequence $\{u_n\}$ defined for any integer n and ranging in $\mathbf{R}^p$ is said to be *almost periodic* if, for each $\varepsilon > 0$, there exists an $l = l(\varepsilon) \in \mathbf{N}$ with the following property: between two successive multiples of $l(\varepsilon)$, there exists an integer n such that $\|u_{n+m} - u_n\| < \varepsilon$ for all $n \in \mathbf{Z}$. Here $\mathbf{Z}$ is the set of integers. Numbers m with the above-mentioned property are referred to as ε-almost periods of the sequence.

Definition 7. A sequence $\{f_n(x)\}$ of functions is said to be *uniformly almost periodic* if, for each $\varepsilon > 0$, there exists an $l = l(\varepsilon, r) \in \mathbf{N}$ such that, between successive multiples of l, there exists an integer m such that $\|f_{n+m}(x) - f_n(x)\| < \varepsilon$ for any $n \in \mathbf{Z}$ and $\|x\| < r$.

Lemma 1 [2, p. 125 of the Russian translation]. *Let $\{u_n^1\}, \{u_n^2\}, \ldots, \{u_n^M\}$ be almost periodic sequences. Then for each $\varepsilon > 0$, there exists an $l = l(\varepsilon) \in \mathbf{N}$ such that, between two successive multiples of l, there exists at least one ε-almost period common for all these sequences.*

Lemma 2. *If a sequence $\{F_n(x)\}$ of functions is almost periodic for each $x \in B_H$ and each $F_n(x)$ satisfies the Lipschitz condition $\|F_n(x) - F_n(y)\| \le L_1\|x - y\|$ uniformly with respect to $n \in \mathbf{Z}$, then this sequence is uniformly almost periodic.*

Proof. The functions $F_n(x)$ satisfy the Lipschitz condition; consequently,

$$\|F_n(x) - F_n(y)\| \le L_1\|x - y\|, \tag{3}$$

where L_1 is the Lipschitz constant. Let $\varepsilon > 0$ be an arbitrary number. The set B_H is bounded and closed and hence compact. It follows that there exist finitely many points $z_1, \ldots, z_M$ such that $z_j \in B_H$ $(j = 1, \ldots, M)$ and, for each $x \in B_H$, there exists an index i $(1 \le i \le M)$ such that

$$\|x - z_i\| < \varepsilon / (3L_1). \tag{4}$$

It follows from Lemma 1 that there exists an $l = l(\varepsilon) \in \mathbf{N}$ such that, between two successive multiples of l, there exists a number $m \in \mathbf{Z}$ such that

$$\|F_n(z_i) - F_{n+m}(z_i)\| < \varepsilon/3 \tag{5}$$

for any $1 \le i \le M$ and $n \in \mathbf{Z}$.

Let us now show that, for an arbitrary $x \in B_H$, every integer m satisfying inequalities (5) is an ε-almost period of the sequence $\{F_n(x)\}$. Let z_k be an element of the set $z_1, \ldots, z_M$ such that $\|x - z_k\| < \varepsilon/(3L_1)$. Then, by (3)–(5), we have

$$\begin{aligned}\|F_{n+m}(x) - F_n(x)\| &\le \|F_{n+m}(x) - F_{n+m}(z_k)\| + \|F_{n+m}(z_k) - F_n(z_k)\| + \|F_n(z_k) - F_n(x)\| \\ &\le \varepsilon/3 + 2L_1\varepsilon/(3L_1) = \varepsilon.\end{aligned} \tag{6}$$

Inequality (6) completes the proof of Lemma 2.

By $\mathscr{K}$ we denote the Hahn function class. This is the class of scalar continuous strictly increasing functions $a : R_+ \to R_+$ satisfying the condition $a(0) = 0$. The second Lyapunov method proves to be efficient for the stability analysis of the zero solution of system (1). The following assertions were proved in [2, p. 27 of the Russian translation].

Theorem 1. *Suppose that there exists a sequence $\{V_n(x)\}$ of functions such that the following assertions are valid*:

(a) $a(\|x\|) \le V_n(x) \le b(\|x\|)$, *where $a \in \mathscr{K}$ and $b \in \mathscr{K}$*;

(b) $V_n(x_n) \ge V_{n+1}(x_{n+1})$ *for each solution x_n.*

Then the solution (2) *of system* (1) *is uniformly stable.*

Theorem 2. *Suppose that there exists a sequence* $\{V_n(x)\}$ *of functions which has property* (a) *and the following properties:*

(c) $V_{n+1}(x_{n+1}) - V_n(x_n) \le -c(\|x_n\|)$, *where* $c \in \mathscr{K}$;

(d) $|V_n(x) - V_n(y)| \le L\|x - y\|$, $n \in \mathbf{N}$, *for* $x, y \in B_H$, *where* $L > 0$.

Then the solution (2) *of system* (1) *is uniformly asymptotically stable.*

The following assertion [2, p. 34 of the Russian translation] is valid for the special case in which system (1) is autonomous [i.e., $f_n(x) = f(x)$].

Theorem 3. *If there exists a continuous function* $V(x)$ *such that* $a(\|x\|) \le V(x) \le b(\|x\|)$, *where* $a \in \mathscr{K}$ *and* $b \in \mathscr{K}$, *and* $V(x_{n+1}) - V_n(x_n) \le 0$ *for each solution of system* (1) *and if, moreover, the equality takes place only on some set that does not contain entire half-trajectories, then the solution* (2) *of system* (1) *is asymptotically stable.*

The aim of the present paper is to generalize Theorem 3 to the case in which the sequence $\{f_n(x)\}$ is almost periodic.

3. MAIN RESULTS

In this section, we consider a discrete system (1) satisfying the conditions imposed in Section 1. We also assume that each function $f_n(x)$ satisfies the Lipschitz condition with constant L and the sequence $\{f_n(x)\}$ is almost periodic for each $x \in B_H$.

Lemma 3. *Let the solution of system* (1) *satisfy the condition* $x_n(n_0, x_0) \in B_r$ $(0 < r < H)$ *for* $n \ge n_0$, *let* $\{\varepsilon_k\}$ *be a sequence of positive numbers monotone decreasing to zero, and let* $\{m_k\}$ *be a sequence of* ε_k*-almost periods of the sequence* $\{f_n(x)\}$. (*To each* ε_k, *there corresponds an* ε_k*-almost period* m_k.) *Then*

$$\lim_{k\to\infty} \|x_{n^*}(n_0, x_k) - x_{n^*+m_k}(n_0, x_0)\| = 0, \tag{7}$$

where $x_k = x_{n_0+m_k}(n_0, x_0)$ *and* n^* *is a given positive integer greater than* n_0.

Proof. Consider the solutions $x_n(n_0, x_k)$ and $x_n(n_0 + m_k, x_k)$ of system (1). In $\Delta n = n^* - n_0$ steps, the point x_k passes into the point $x_{n^*}(n_0, x_0)$ along the solution $x_n(n_0, x_k)$, and x_k passes into the point $x_{n^*+m_k}(n_0 + m_k, x_k) = x_{n^*+m_k}(n_0, x_0)$ along the solution $x_n(n_0 + m_k, x_k)$. Note that the solution $x_n(n_0 + m_k, x_k)$ of system (1) issuing from the point x_k for $n = n_0 + m_k$ can be treated as the solution of the system

$$x_{n+1} = f_{n+m_k}(x_n) \tag{8}$$

with the same initial point x_k and with initial index n_0. However, by the almost periodicity of the sequence $\{f_n(x)\}$ and the Lipschitz property of each function $f_n(x)$, the right-hand sides of Eqs. (1) and (8) differ arbitrarily little for sufficiently large k, which implies the limit relation (7). The proof of the lemma is complete.

Throughout the following, we consider system (1), where $\{f_n(x)\}$ is an almost periodic sequence of functions each of which satisfies the Lipschitz condition with constant L for $x \in B_H$. The following assertion is valid.

Theorem 4. *Suppose that there exists a sequence* $\{V_n(x)\}$ *of functions with the following properties: the sequence is almost periodic for each* $x \in B_H$; *each of these functions satisfies the Lipschitz condition uniformly with respect to* n; *the sequence has properties* (a) *and* (b) *for* $n \in \mathbf{N}$; *moreover, the equality in* (b) *is not identically valid along any nonzero solution of system* (1) *with* $n \ge n_1$, *where* n_1 *is an arbitrary positive integer. Then the solution* (2) *of system* (1) *is asymptotically stable.*

Proof. Assumptions (a) and (b), together with Theorem 1, imply that the solution (2) is uniformly stable. Let $\varepsilon \in (0, H)$ be arbitrary, and let n_0 be an arbitrary positive integer. Since the zero solution is uniformly stable, it follows that there exists a $\delta > 0$ such that if $x \in B_\delta$, then $x_n(n_0, x) \in B_\varepsilon$ for each $n \geq n_0$. We take such $\delta > 0$ and show that each solution $x_n(n_0, x)$ with $x \in B_\delta$ tends to zero as $n \to +\infty$. Suppose the contrary: there exist $\eta > 0$ and $x_0 \in B_\delta$ such that $\|x_n(n_0, x_0)\| > \eta > 0$ for $n \geq n_0$.

The sequence $\{V_n\}$, where $V_n = V_n(x_n(n_0, x_0))$, is monotone nonincreasing; therefore, there exists a limit

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} V_n(x_n(n_0, x_0)) = V_0 \geq a(\eta) > 0,$$

and moreover, $V_n(x_n(n_0, x_0)) \geq V_0$ for $n \geq n_0$.

Consider the sequence $\{\varepsilon_k\}$ of positive numbers monotonically tending to zero, where $\varepsilon_1 > 0$ is sufficiently small. By Lemmas 1 and 2, one can claim that, for each ε_i, there exists a sequence of ε_i-almost periods $m_{i,1}, m_{i,2}, \ldots, m_{i,k}, \ldots$ ($m_{i,k} < m_{i,k+1}$, $\lim_{k\to+\infty} m_{i,k} = +\infty$) of the sequences $\{f_n(x)\}$ and $\{V_n(x)\}$ such that $\left|V_{n+m_{i,k}}(x) - V_n(x)\right| < \varepsilon_i$ and $\left\|f_{n+m_{i,k}}(x) - f_n(x)\right\| < \varepsilon_i$ for any $n \in \mathbf{Z}$ and $x \in B_\varepsilon$. Without loss of generality, we assume that $m_{i,k} < m_{i+1,k}$ for any $i \in \mathbf{N}$ and $k \in \mathbf{N}$. We set $m_k = m_{k,k}$.

Consider the sequence $\{x_k\}$, where $x_k = x_{n_0+m_k}(n_0, x_0)$ $(k = 1, 2, \ldots)$. This sequence is bounded; consequently, there exists a subsequence converging to some point x^*. Without loss of generality, we assume that the sequence $\{x_k\}$ itself converges to x^*. The sequence $\{V_n\}$ is almost periodic for each $x \in B_H$, and every function $V_n(x)$ is continuous; consequently,

$$\begin{aligned} V_{n_0}(x_*) &= \lim_{n\to\infty} V_{n_0}(x_n) = \lim_{k\to\infty}\lim_{n\to\infty} V_{n_0+m_k}(x_n) = \lim_{n\to\infty} V_{n_0+m_n}(x_n) \\ &= \lim_{n\to\infty} V_{n_0+m_n}(x_{n_0+m_n}(n_0, x_0)) = V_0. \end{aligned}$$

Now consider the solution $x_n(n_0, x^*)$. By the assumptions of the theorem, there exists an $n^* > n_0$ ($n^* \in \mathbf{N}$) such that $V_{n^*}(x_{n^*}(n_0, x^*)) = V_1 < V_0$. The functions $f_n(x)$ satisfy the Lipschitz condition with respect to x; consequently,

$$\lim_{k\to\infty} \|x_{n^*}(n_0, x_k) - x_{n^*}(n_0, x^*)\| = 0,$$

since $\lim_{k\to\infty} \|x_k - x^*\| = 0$. Hence it follows that

$$\lim_{k\to\infty} V_{n^*}(x_{n^*}(n_0, x_k)) = V_1. \tag{9}$$

By using the almost periodicity of the sequence $\{f_n(x)\}$ and the limit relation (7), we obtain the inequality

$$\|x_{n^*}(n_0, x_k) - x_{n^*+m_k}(n_0, x_0)\| \leq \gamma_k, \tag{10}$$

where $\gamma_k \to 0$ as $k \to \infty$. Since $\{V_n\}$ is an almost periodic sequence, we have

$$|V_{n^*}(x) - V_{n^*+m_k}(x)| < \varepsilon_k \tag{11}$$

for each $x \in B_H$, and it follows from conditions (9) and (10) that

$$|V_{n^*}(x_{n^*+m_k}(n_0, x_0)) - V_1| < \eta_k, \tag{12}$$

where $\eta_k \to 0$ as $k \to \infty$. From (11), we have

$$|V_{n^*}(x_{n^*+m_k}(n_0, x_0)) - V_{n^*+m_k}(x_{n^*+m_k}(n_0, x_0))| < \varepsilon_k,$$

which, together with (12), implies that

$$|V_{n^*+m_k}(x_{n^*+m_k}(n_0, x_0)) - V_1| < \eta_k + \varepsilon_k, \tag{13}$$

where $\eta_k + \varepsilon_k \to 0$ as $k \to \infty$.

On the other hand,

$$\lim_{k\to\infty} V_{n^*+m_k}(x_{n^*+m_k}(n_0, x_0)) = V_0. \tag{14}$$

Relations (13) and (14) contradict the inequality $V_1 < V_0$. The resulting contradiction completes the proof of the theorem.

Theorem 5. *Suppose the following: for each $x \in B_H$, there exists an almost periodic sequence $\{V_n(x)\}$ of functions satisfying the Lipschitz condition uniformly with respect to n; $|V_n(x)| \le b(\|x\|)$ for $x \in B_H$, where $b \in \mathscr{K}$; for any $n \in \mathbf{N}$ and $\delta > 0$, there exists an $x \in B_\delta$ such that $V_n(x) > 0$; $V_{n+1}(x_{n+1}) \ge V_n(x_n)$ for any nonzero solution; moreover, the equality is not identically valid along any nonzero solution of system (1) for $n \ge n_1$, where n_1 is an arbitrary positive integer. Then the solution (2) of system (1) is unstable.*

Proof. Let $\varepsilon \in (0, H)$ be an arbitrary number. We take an arbitrary $n_0 \in \mathbf{N}$ and a sufficiently small $\delta > 0$. We take $x_0 \in B_\delta$ so as to ensure that $V_{n_0}(x_0) > 0$. It follows from the assumptions of the theorem that there exists an $\eta > 0$ such that $|V_n(x)| < V_{n_0}(x_0)$ for all $x \in B_\eta$. Consider the sequence $\{V_n\}$, where $V_n = V_n(x_n(n_0, x_0))$. This sequence is a nondecreasing; i.e., $V_n(x_n(n_0, x_0)) \ge V_{n_0}(x_0)$ for $n \ge n_0$. It follows that $\|x_n(n_0, x_0)\| \ge \eta$ for each $n \ge n_0$. Let us show that there exists an index $N_0 > n_0$ such that $\|x_{N_0}(n_0, x_0)\| > \varepsilon$. Suppose the contrary:

$$\eta \le \|x_n(n_0, x_0)\| \le \varepsilon \tag{15}$$

for each $n > n_0$.

By using the assumptions of the theorem and inequality (15), we arrive at a contradiction just as in the proof of Theorem 4. We omit the computations, which are the same word for word. The contradiction implies that the solution $x_n(n_0, x_0)$ leaves the set B_ε. The proof of the theorem is complete.

Example 1. Consider the two-dimensional almost periodic system

$$x_{n+1} = -y_n \sin\left(\sqrt{2}n\right), \qquad y_{n+1} = x_n \sin n. \tag{16}$$

For the function $V_n(x_n, y_n) = x_n^2 + y_n^2$, we obtain

$$V_{n+1}(x_{n+1}, y_{n+1}) - V_n(x_n, y_n) = -x_n^2\left(1 - \sin^2 n\right) - y_n^2\left(1 - \sin^2\left(\sqrt{2}n\right)\right). \tag{17}$$

By the Corduneanu theorem [18], for an arbitrarily small $\varepsilon > 0$, there exists a sequence

$$n_1, n_2, \ldots, n_k, \ldots \to \infty$$

such that $0 < 1 - \sin^2 n_k < \varepsilon$ and $0 < 1 - \sin^2\left(\sqrt{2}n_k\right) < \varepsilon$ $(k = 1, 2, \ldots)$. It follows that the expression (17) is not negative definite with respect to x_n and y_n, and one cannot use Theorem 2 for the stability analysis of the zero solution of system (16). System (16) is not autonomous; therefore, Theorem 3 cannot be applied to it. However, this system is almost periodic, and the expression (17) is negative for any nonzero solution; consequently, by Theorem 4, the zero solution of system (16) is asymptotically stable.

Example 2. Consider the system

$$\begin{aligned} x_{n+1} &= y_n - x_n^2 y_n\left(2 - \sin^2 n - \cos^2\sqrt{2}n\right), \\ y_{n+1} &= x_n + x_n y_n^2\left(2 - \sin^2 n - \cos^2\sqrt{2}n\right). \end{aligned} \tag{18}$$

By taking $V_n(x_n, y_n) = x_n^2 + y_n^2$, we obtain

$$V_{n+1}(x_{n+1}, y_{n+1}) - V_n(x_n, y_n) = x_n^2 y_n^2\left(x_n^2 + y_n^2\right)\left(2 - \sin^2 n - \cos^2\sqrt{2}n\right)^2,$$

which, together with Theorem 5, implies that the zero solution of system (18) is unstable.

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