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SOLIDS

Magnetic Flux Penetration into High-Temperature Superconductors in the Vortex Liquid State under the Blow-Up Conditions

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Abstract—The problem of magnetic field penetration into a type-II high-temperature superconductor that is in the weakly pinned vortex-liquid phase is considered. A magnetic field on the superconductor boundary rises with time in the blow-up regime. A model hydrodynamic equation describing the magnetic induction distribution in the vortex-liquid phase for thermomagnetic motion of the flux is derived. Analytical expressions for the depth and rate of magnetic field penetration into the superconductor are found. It is demonstrated that these quantities depend on parameters of the problem: index of power n in the boundary regime characterizing the penetration rate of vortices into the superconducting half-space and a parameter describing the effect of random pinning forces and thermal fluctuations on the magnetic flux distribution.

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INTRODUCTION

Thermal fluctuations influence the properties of pinned vortex lines and vortex lattices: for example, they lead to smoothing the random pinning potential, thereby decreasing the critical current density. The type of interplay between thermal fluctuations responsible for a vortex line displacement and a random pinning force depends on the phase in which the superconductor is. It is known that random perturbations of a vortex lattice under the action of a pinning force transform it into the vortex-glass phase [1, p. 1139]. The influence of random perturbations of vortex lines due to pinning in the vortex-liquid phase can be displayed by the *vortex lattice* $\longrightarrow$ *vortex glass* or *vortex liquid* $\longrightarrow$ *vortex liquid* transitions [1]. In other words, random fluctuations may convert a vortex lattice to a vortex glass. In this case, random pinning of the vortex lattice changes the state of the high-temperature superconductor. On the other hand, random fluctuations of the vortex liquid due to the random pinning potential do not change the state of the superconductor: the pinned liquid remains in the same phase.

This means that the response of a superconductor to small perturbations of the transport current density ($j \longrightarrow 0$) differs in the vortex-glass phase and vortex-liquid phase. The former is characterized by the divergence of pinning barrier $U(j)$; i.e., $U(j) \longrightarrow \infty$ as $j \longrightarrow 0$ [1, 2]. This property of the activation barrier persists in the presence of thermal fluctuations. Therefore, the corresponding *I–V* characteristic remains strongly nonlinear in the vortex-glass phase at low transport current densities.

A random pinning potential present in the vortex-liquid phase significantly decreases linear resistivity $\rho_{lin} = dE/dj|_{j \to 0}$ compared with its value in the viscous flow regime, when $\rho_{lin} > 0$. Pinning is a result of vortex structure inhomogeneity. Although thermal fluctuations smooth the vortex core, the vortex lattice remains inhomogeneous, and the interaction of such a periodic inhomogeneous structure with a random pinning potential retains pinning at all temperatures $T > T_m(B)$, where T_m is the melting point of the vortex lattice (Fig. 1).

Consider the response of the superconductor to perturbations that act on the vortex structure presenting one or another phase. As an external action, one can use a magnetic field applied to the boundary of a high-temperature superconductor. Its amplitude increases with time as follows:

$$b(0, t) = b_0(1 + t/t_{sc})^m, \quad m > 0, \tag{1}$$

where $b = B/H_{c_2}$ (H_{c2} is the second critical field), t_{sc} is the time it takes for the magnetic induction distribution to be described by the scaling law, and m is the parameter characterizing the rate of increase of the magnetic field amplitude. At a constant boundary magnetic field

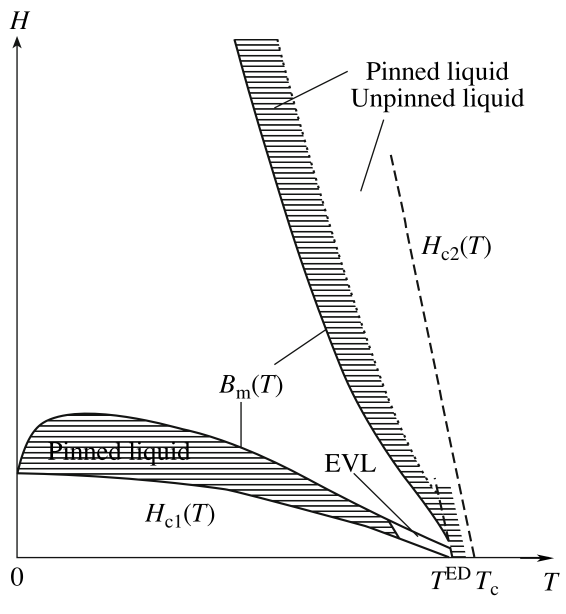

Fig. 1. Magnetic field intensity H vs. temperature T. $B_m(T)$ is the melting line of the vortex lattice, H_{c_1} is the first critical field, H_{c_2} is the second critical field, T_c is the critical temperature, and T^{ED} is the temperature of the entangled liquid–disentangled liquid phase transition (the phase of entangled vortex lines changes to the phase of disentangled vortex lines in the vortex-liquid phase [1]). Near the melting line, the inequality $T \ll U_0 < \infty$ is fulfilled, where U_0 is the barrier of plastic deformation. Near the critical temperature, $U_0 \ll T$ in the phase of entangled vortex lines (entangled vortex liquid) [1, Fig. 23].

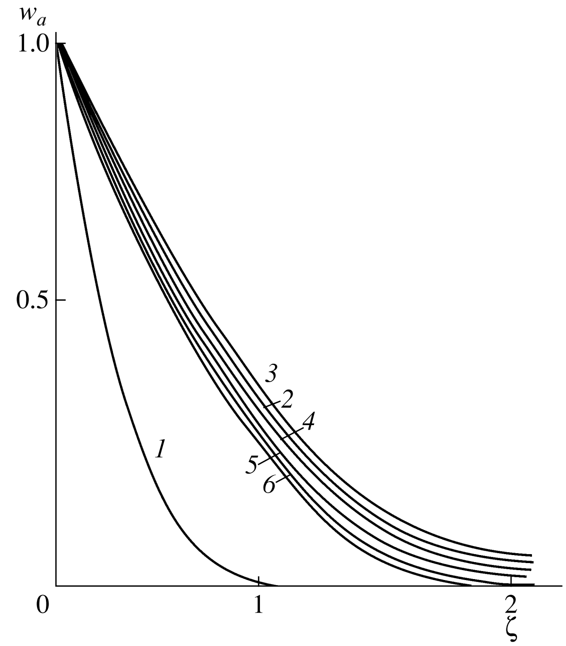

Fig. 2. Dependence of solution w_a to the boundary-value problem on self-similar variable $\zeta > 0$. The solution is calculated at time instant t_0 = (*1*) 0.95, (*2*) 0.47, (*3*) 0.90, (*4*) 5×10^{-2}, (*5*) 2.5×10^{-2}, and (*6*) 1.2×10^{-2} [5]. The function $w_a(\zeta, t)$ is defined by relationship (36) and is calculated for at $n = -1$, $k_0 = 1$, $A_0 = R_0 = 1$, and $x_s = 2$. Here, $w_a(\zeta)$ is the self-similar representation of the solution at $t = t_0$, where t_0 is the blow-up time of the boundary regime.

($m = 0$), the same boundary-value problem was considered in [1] for the vortex-glass phase and in [3] for the heavily viscous vortex liquid. For $m = 1$, this problem was investigated under the conditions of classical thermally activated flux creep [4].

The results of [3] with the boundary conditions $b(0, t) = b_0$ ($t > 0$) cannot be extended to a model problem with boundary conditions (1), since the corresponding model equation for the magnetic induction distribution is significantly nonlinear (see Eq. (23) below). However, if we consider more rapid (compared with condition (1)) pumping of vortices into the superconductor, i.e., use the boundary condition in the blow-up regime [5],

$$b(0, t) = b_0(1 - t/t_{sc})^m, \quad m < 0, \tag{2}$$

the corresponding physical problem for the vortex-liquid phase can be treated mathematically. In particular, it is possible to construct the magnetic field amplitude profile in the superconducting half-space, $x > 0$, that meets boundary condition (2) and derive accurate analytical formulas for the depth and rate of magnetic flux penetration into the superconductor.

To study the effect of the random pinning potential on the dynamic properties of the disordered vortex liquid, we will use the general dynamic method described in [1, p. 1248]. This method makes it possible to find the analytical form of resistivity $\rho(b)$ (i) for thermally activated motion of the magnetic flux ($j < j_{cr}$, where j_{cr} is the critical current density) and (ii) for viscous flow of the liquid ($j > j_{cr}$).

With such an approach, one can study the behavior of the vortex lattice at high current densities and calculate correction (perturbation) δv to velocity v of the main flux (without pinning) that is due to a random pinning potential. Based on the results of [1], one can find the analytical form of the resistivity for the vortex-liquid phase in the regime of the thermally activated motion of the flux (see formulas (6) and (7) below). In the general case, resistivity $\rho(b)$ depends on thermal fluctuations and the pinning amplitude.

Next, from the shape of the *I–V* characteristics, $E = \rho_{flow}(b)j$ at $j \ll j_{cr}$ ([1, Fig. 27]) and Maxwell equations, we derive a model hydrodynamic equation (Eq. (23)). This equation describes the magnetic induction distribution in the blow-up regime. The blow-up regime like boundary condition (2) or, in other words, superfast pulsed application of an external magnetic field was considered in [6]. In particular, the pulsed application

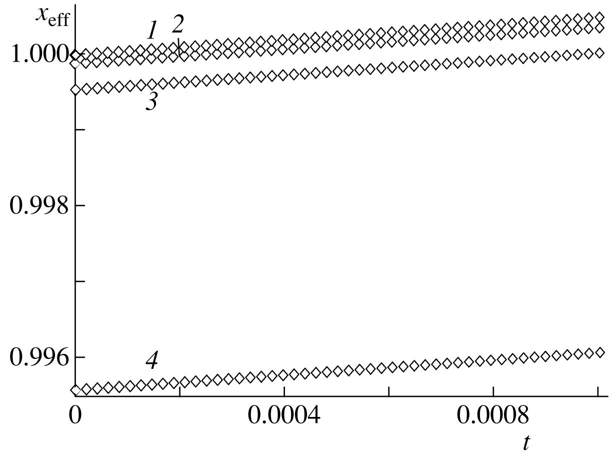

Fig. 3. Time dependence of penetration depth x_{eff} of the magnetic flux. Here, $t \longrightarrow t/t_{sc}$, where $t_{sc} \approx 1$ is the time to scaling behavior.

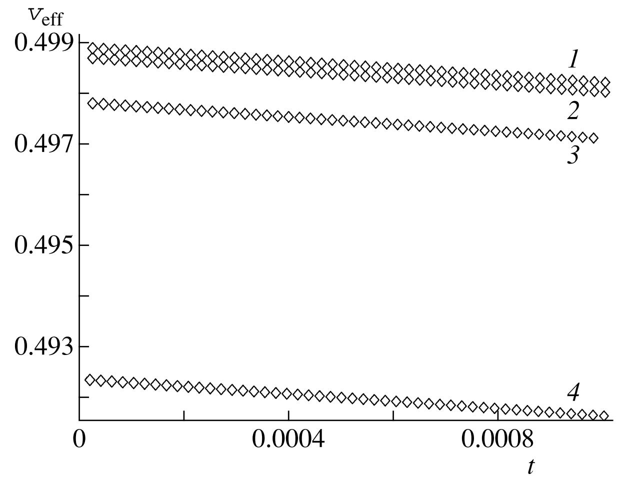

Fig. 4. Time dependence of penetration velocity v_{eff} calculated at the same values of the parameters as in Fig. 3.

of an external magnetic field gives rise to so-called thermomagnetic dendrite structures [7, 8].

The main result of this study is determination of effective coordinate $x_{eff}(t, A, T)$ and effective rate $v_{eff}(t, A, T)$ of magnetic field penetration into the superconductor (see relationships (37) and (38)). Here, parameter $A = A(f_{pin}^2, n, T)$ depends on thermal fluctuations T, random pinning force f_{pin}, and defect density $n \sim 10^5$. The basic trend in the magnetic field amplitude evolution is that, when the pinning force increases (see relationship (9)), the depth and rate of magnetic flux penetration decrease. As thermal fluctuations grow, the depth and rate of magnetic field penetration increase according to relationships (37) and (38) and the random pinning potential smoothes out along the coordinate, so that the averaged pinning force decreases. This, in turn, leads to a further increase in the depth and rate of magnetic flux penetration as the temperature rises. As a result, we have the following relationship for the depth of magnetic flux penetration: $x_{eff}(\theta) \propto 1 - \theta^{-1}$, where $\theta = T/T_m$ and $T > T_m$. Here, $T_m(B)$ is the melting line of the vortex lattice and B is the magnetic field induction.

STATEMENT OF THE PROBLEM

Consider a type-II high-temperature superconductor occupying the half-space $x \geq 0$ in the parallel geometry: $B \parallel e_z$; E; $j \parallel e_y$; and $v \parallel e_x$. Here, e is the unit vector directed along the respective axis, E is the electric field, j is the transport current density, and v is the vortex velocity.

Let the superconductor be in the vortex-liquid phase. Then, two dissipative regimes may exist: (i) viscous flow of the magnetic flux at $T > T_p$, when the liquid is unpinned, and (ii) thermally activated motion of the magnetic flux at $T_m < T < T_p$, when the vortex liquid is pinned [2]. Here, T_p is the depinning temperature [1].

If pinning is absent, the vortex lattice and vortex liquid move under the action of an applied current the resistivity is a linear function of the magnetic induction,

$$\rho_{flow}(B) \approx \rho_n B/H_{c_2}, \tag{3}$$

where ρ_n is the resistivity in the normal state.

Consider the motion of a vortex structure under the action of the Lorentz force with allowance for a random pinning potential [2],

$$U_{pin} = \sum_i U(r_\perp) p(r_\perp - \tilde{r}_{\perp_i})(z, t). \tag{4}$$

Here, z is the coordinate of the vortex line along the c axis. In what follows, this coordinate is taken as a parameter or as a quantity averaged over characteristic scale $\mathbf{z}_{ch}$.

Potential U_{pin} will be viewed as a small perturbation; potential $U(r_\perp)$, as δ-normalized so that

$$\langle U(r_\perp) U(r'_\perp) \rangle = \nu (2\pi)^3 \delta(r_\perp - r'_\perp). \tag{5}$$

Here, δ is the Dirac delta and symbol ⟨ ⟩ denotes averaging over all random trajectories. Vector $r_\perp$ is located in an (a–b) plane of the superconductor. The liquid is assumed to be isotropic in each of these planes; i.e., $r_\perp = \sqrt{x^2 + y^2}$. We put $r = x$, since the direction of the pinning force in an (a–b) plane is of no significance.

For so-called δT_c pinning, coupling parameter ν in relationship (5) is given by the formula $\nu = f_{pin}^2 n \zeta^2$, where ζ is the correlation length [1, p. 1145]. Then, quantity $p(r_\perp)$ reflects the interaction of the vortex core with randomly distributed defects. Note that $p(r_\perp) \longrightarrow 0$ at $r_\perp > \zeta$; otherwise, function $p_\perp$ is arbitrary [2]. Summation in expression (4) is performed over all vortices with index i.

Vector $\tilde{r}_i$ defines the coordinate of an ith vortex by the formula $\tilde{r}_i = ri + vt + U_{\mathrm{pin},\,i}$. Here, r_i is the unperturbed coordinate of the ith vortex in the vortex structure that moves as a whole with velocity v. Quantity $U_{\mathrm{pin},\,i}(z, t)$ defines a small displacement of the ith vortex under the action of a force generated by the random pinning potential. Velocity v is defined by the formula $v = v_0 + \delta v$, where $v_0 = j \times B/\eta c$ is the unpinned liquid velocity due to the Lorentz force (η is the viscosity). Quantity δv is a small perturbation of velocity v_0 due to a random pinning force. Pinning becomes substantial when the condition $\delta v \sim v$ is met. Then, the condition $\delta v(v_{\mathrm{cr}}) \sim v_{\mathrm{cr}}$ defines critical current density $j_{\mathrm{cr}} = c\eta v_{\mathrm{cr}}/B$, where v_{cr} is the critical velocity.

Pinning is considered weak if $j_{\mathrm{cr}} \ll j_0$, where j_0 is the depairing current [1, p. 1250]. The characteristic pinning time is given by the formula $t_{\mathrm{pin}} = r_{\mathrm{p}}/v_{\mathrm{cr}}$. Here, the typical range of action of the random pinning force is defined by the formula $r_{\mathrm{p}} \approx (\zeta^2 + v_{\mathrm{th}}^2)^2$. The critical current (and, accordingly, pinning) decreases with an increase in temperature. Quantity

$$u_{\mathrm{th}} = \langle u^2 \rangle_T^{1/2} \approx \zeta (T/T_{\mathrm{p}})^{1/2}$$

means the averaged displacement of a vortex under the action of thermal fluctuations. The depinning temperature is given by

$$T_{\mathrm{p}} \sim (\Phi_0^3 B m/M)^{1/2}/(2\pi k)^2.$$

Here, Φ_0 is a fluxon; m and M are the masses in (a–b) planes and along the c axis, respectively; $k = \lambda/\zeta$ is the Ginzburg–Landau constant; and λ is the London penetration depth.

If the inequality $t_{\mathrm{th}} \ll t_{\mathrm{pin}}$ is fulfilled (t_{th} is the characteristic time of variation of thermal fluctuations), averaging of thermal fluctuations over times on the order of t_{pin}, yields a homogeneous vortex structure. In this case, pinning can be ignored. Such a situation is typical of a "normal" liquid, in which case all characteristic times are of the same order as t_{th}. For a very viscous liquid, the inequality $t_{\mathrm{pl}} \gg t_{\mathrm{pin}}$ is valid; consequently, averaging over times on the order of t_{pin} is incomplete. For this reason, the vortex structure after averaging is inhomogeneous; i.e., it is effectively pinned under the action of the random pinning force [1]. Exponentially long characteristic times t_{pl} of plastic deformation for the vortex-liquid phase are provided by high plastic deformation barrier U_{pl}, which is produced by the thermally activated motion of the vortex structure.

Activation barrier U_{pl} is estimated by the formula [3]

$$U_{\mathrm{pl}} \sim \sqrt{\varepsilon}\Phi_0^2 a/8\pi^2\lambda^2 \propto (T_{\mathrm{c}} - T)/\sqrt{H} \; (H \parallel c). \qquad (6)$$

Formula (6) is valid at $H \gg H_{\mathrm{c}_1}$, where H_{c_1} is the first critical field. Here, $\varepsilon = m/M$ is the anisotropy parameter; $m_x = m_y = m$ and $m_z = M$, where M is the mass; and T_{c} is the critical temperature of switching into the regime of vortex viscous flow. In fields $H > H_{\mathrm{c}_1}$, a similar formula is valid that coincides with expression (6) up to the normalizing factor. The energy defined by expression (4) has the same order of magnitude as the energy of a vortex segment of length $\sim a$. Consequently, relationship (6) works for any vortex deformation on a special scale of the order of a.

The equation of motion of a pinned vortex liquid can be written in the form

$$\eta v = f_{\mathrm{L}} + f_{\mathrm{pin}}, \qquad (7)$$

where f_{L} is the Lorentz force and $f_{\mathrm{pin}} = -\eta_{\mathrm{pin}}(v)v$. Here, viscosity coefficient $\eta_{\mathrm{pin}}(v)$ can be expressed via relative velocity correction $\delta v/v$ defined above, $\eta_{\mathrm{pin}} = \eta(\delta v/v)$. Then, using (7), relationship $E = (v/c)B$, and the definition of the resistivity $\rho = E/j$, one easily obtains the relationship [1, formula (6.35)]

$$\rho = \frac{\rho_{\mathrm{flow}}}{1 + \delta v/v}. \qquad (8)$$

The relationship $E = \rho j$ for the thermally activated motion of the magnetic field in the case of the vortex-liquid phase holds only for sufficiently low current densities (see the I–V characteristics in [1, Fig. 27]). Ratio $\delta v/v$ for the very viscous liquid is known at temperatures close to melting point $T_{\mathrm{m}}(B)$, when $t_{\mathrm{pl}} \gg t_{\mathrm{th}}$ [1, formula (6.19)]. It has the form

$$\left.\frac{\delta v}{v}\right|_{v \to 0} \simeq \frac{\nu\zeta^4}{\eta a_0^4}\frac{K_0^6}{4\pi}\frac{\langle u^2(t_{\mathrm{pl}})\rangle_{\mathrm{th}}}{T} t_{\mathrm{pl}}, \qquad (9)$$

where $K_0 = 2\pi/a_0$ and a_0 is the period of the Abrikosov vortex lattice.

Supposing that $\langle u^2(t_{\mathrm{pl}})\rangle_{\mathrm{th}} \sim a_0^2$, we can recast formula (9) in the form

$$\left.\frac{\delta v}{v}\right|_{v \to 0} \simeq \frac{\nu\zeta^4}{\eta T}\frac{K_0^6}{4\pi} t_{\mathrm{pl}}. \qquad (10)$$

Next, for the thermally activated motion of the magnetic flux, formula [1]

$$\rho = \rho_0 e^{-U_{\mathrm{pl}}/T} \propto \frac{1}{t_{\mathrm{pl}}} \qquad (11)$$

is valid. Here, one can estimate factor ρ_0 from (11) and (9), bearing in mind that the regime of viscous flow changes to the regime of thermally activated motion of the magnetic flux when the value of $\delta v/v$ in (8) tends to unity.

From (10) and (11), the simplest interpolation formula for the resistivity can be derived,

$$\rho = \rho_{\text{flow}}\left[1 + \frac{\nu\zeta^3}{\eta T}\frac{K_0^6}{4\pi}t_{\text{pl}}\right]^{-1} \text{ if } \delta v/v \longrightarrow 0 \quad (12)$$

and

$$\rho = \frac{\rho_{\text{flow}}}{1 + Ae^{U_{\text{pl}}/T}} \quad (12')$$

if the correction to the velocity is finite [1, formula (6.31)].

In relationship (12'), parameter A governs the amount of δT_c pinning [1, formula (2.31)]. In the case of weak pinning, it is given by

$$A = 4(2\pi)^6\frac{\gamma\zeta^4}{\Phi_0^2 T}\frac{\lambda}{a_0}; \quad (13)$$

in the case of strong pinning, by

$$A = \sqrt{2}G_i\left(\frac{j_{\text{cr}}(0)}{j_0(0)G_i}\right)^{3/2}\frac{B}{H_{c2}(T)}. \quad (13')$$

Here, the Ginzburg parameter is defined by the formula $G_i = (T_c/H_c^2(0)\varepsilon\zeta^3(0))^2/2$, where H_c is the thermodynamic critical field. Parameter G_i is the ratio of the energy of thermal fluctuations, $k_B T_c$, at critical temperature T_c (Boltzmann constant $k_B = 1$ here is expressed in energy units) to condensation energy $H_c^2(0)\varepsilon\zeta^3(0)$ in a coherent volume of radius $\zeta(0)$. Here, $\varepsilon^2 < 1$ is the anisotropy parameter of the type-II high-temperature superconductor.

From (6), it follows that

$$\frac{U_{\text{pl}}}{T} \ll \frac{T_c}{T} - 1 \text{ at } T \approx T_c. \quad (14)$$

Below, we will consider only the situation when estimator (14) is valid. This estimator is in disagreement with the inequality $t_{\text{pl}} \ll t_{\text{pin}}$ implying a high pinning force for a very viscous liquid (see the Introduction). This readily follows from the relationship

$$t_{\text{pl}} = t_{\text{th}}e^{U_{\text{pl}}/T}, \quad (14')$$

which shows that inequality (14) is fulfilled for the low-viscosity vortex liquid with weak pinning. Indeed, the viscosity obeys the formula $\eta_{\text{pin}} = \eta(\delta v/v)$, where $\eta_{\text{pin}}(0) = 0$.

If the above conditions are fulfilled, the I–V characteristic of the pinned liquid is a superposition of two ohmic regimes,

$$\rho(j \longrightarrow 0) \simeq \rho_0 e^{-U_{\text{pl}}/T} \propto \frac{1}{t_{\text{pl}}} \text{ at } j < j_{\text{cr}} \quad (15)$$

and

$$\rho(j > j_{\text{cr}}) \simeq \rho_{\text{flow}}. \quad (16)$$

Relationship (15) reflects the thermally activated magnetic motion of the flux; relationship (16), the classical viscous motion of the flux according to the Bardeen–Stephen formula.

In (15), the plastic deformation barrier can be written in the form

$$U_{\text{pl}} \sim \epsilon\epsilon_0 a_0 \propto \frac{T_c - T}{\sqrt{H}}. \quad (16')$$

Here, $\epsilon < 1$ is the anisotropy parameter and $\epsilon_0 = (\Phi_0/4\pi\lambda)^2$ is the characteristic energy scale.

Pinning is significant at $T < T_m$; therefore, it is natural to perform the normalization to the maximum of T_m in (16'). The melting point is approximated as

$$T_m \approx 2c_.^2 U_{\text{pl}}. \quad (17)$$

Relationship (17) shows that a high barrier impeding the plastic motion of vortices in the vortex-liquid phase is achieved if Lindemann number c_{Lin} is small, i.e., if the inequality $U_{\text{pl}}(T_m) \gg T_m$ is fulfilled. Below, we consider only the case when this inequality is valid. Then, from (16) and (17), it follows that

$$\frac{U_{\text{pl}}}{T} \approx \frac{1}{2c_L^2}\frac{T_m}{T}\left(\frac{\theta^{-1} - 1}{\sqrt{b}}\right), \quad (18)$$

where $\theta = T/T_c$. Temperature T_c is determined from the relationship $t_{\text{pl}}(T_c) \simeq t_{\text{pin}}(T_c)$, and, at $T > T_c$, the pinned liquid phase transforms into the unpinned liquid phase (i.e., to the viscous flow) if the relationship $\rho(j \longrightarrow 0) \simeq \rho_{\text{flow}}$ is fulfilled.

Expression (18) will be analyzed only at temperatures $T \approx T_c$, i.e., at temperatures of δT_c pinning due to Ginsburg–Landau random distribution $\alpha = \alpha(0)(1 - T/T_c)$ [1, p. 1141]. As a result, in the case of weak pinning, relationship (12) in the neighborhood of the transition point, $T \approx T_c'$, can be represented in the form

$$\rho(b) = \rho_{\text{flow}}(b)[(1 - A) - \beta(\theta^1 - 1)b^{-1/2}], \quad (19)$$

where $\beta = \frac{A}{2c_L^2}\frac{T_m}{T}$. Relationship (19) follows from (12') using the approximation

$$(1 + Ae^{U_{\text{pl}}/T})^{-1} \simeq (1 - Ae^{U_{\text{pl}}/T}) \simeq \left[1 - A\left(1 + \frac{U_{\text{pl}}}{T}\right)\right],$$

where U_{pl}/T is found from expression (18). This expansion takes place at $0 < A \ll 1$ and $U_{\text{pl}}/T \ll 1$.

Consider now the one-dimensional Maxwell system

$$c^{-1}B_t = -E_x, \quad (20)$$

$$j = -c(4\pi)^{-1}B_x \tag{21}$$

and the functional relationship

$$E = \rho j. \tag{22}$$

Substituting the value of E from (22) into Eq. (20) and using Eq. (21), we arrive at the equation

$$b_t = \kappa(\rho(b)b_x)_x, \tag{23}$$

where $k = \rho_n t_{ch} c^2/(4\pi\lambda^2)$ is the diffusion coefficient of the magnetic flux and t_{ch} is the characteristic time scale on which the magnetic field amplitude varies. Below, model equation (23) will be the main object of mathematical investigation.

Let us set a boundary condition for solutions to Eq. (23). In a number of experiments, the subject of consideration is the response of a superconductor to a perturbation of an external magnetic field, the perturbation being an increasing function of time. Specifically, in the study of the flux creep [6], the magnetic field amplitude was assumed to rise linearly (or sublinearly) with time [6]. Thermomagnetic dendrite structures mentioned above arise only at superfast application of a magnetic flux to the superconductor boundary [7, 8]. Formally, this may correspond to the following boundary condition in the blow-up regime [9, 10]:

$$b(0, t) = b_0(t_0 - t)^n, \quad n < 0, \quad 0 < t < t_0, \tag{24}$$

where $b_0 = B_0/B_m$ ($B_0 > B_{c1}$). In the next section, we construct self-similar solutions to the problem given by (23) and (24).

MAGNETIC FIELD SELF-SIMILAR DISTRIBUTIONS IN THE BLOW-UP REGIME

According to the phase diagram in Fig. 1, we will consider the problem at $H_{c_1} < B < B_m$ (the lower branch of the diagram). After the change of variables $t \longrightarrow \kappa t$, Eq. (23) can be rewritten in the form

$$b_t = \rho(b)b_{xx} + \rho'(b)b_x^2. \tag{23'}$$

Let us find a so-called subsolution and supersolution to Eq. (23'). By virtue of the respective comparison theorems [5], the remaining solutions to Eq. (23') are between the subsolution and supersolution. We introduce the following notation:

$$k_1^- = \rho(b^-), \quad k_2^- = \rho'(b^-), \quad b^- = B_{c1}/B_m, \tag{25}$$

$$k_1^+ = \rho(b^+), \quad k_2^+ = \rho(b^+), \quad b^+ = B/B_m \simeq 1. \tag{26}$$

By virtue of (25) and (26), the coefficients in Eq. (23') meet the inequalities

$$k_1^- \le \rho(b) < k_1^+ \text{ and } k_2 \le \rho'(b) < k_2^+.$$

Let the inequality $b_{xx} \ge 0$ be valid (see Fig. 2). Then, it follows from the inequalities written above and comparison theorems that a solution to Eq. (23') lies between the minimal and maximal solutions to the corresponding equations with constant coefficients.

First, we find a subsolution from the equation

$$b_t = k_1^- b_{xx} + k_2^- b_x^2 \tag{27}$$

which is more convenient to write in the form

$$b_t = k_0 b_{xx} + k_0 A_0^{-1} b_x^2, \tag{28}$$

where $k_0 = k_1^-$ and $A_0 = k_2/k_1$.

The boundary conditions may be expressed as

$$b(0, t) = A_0 R_0 (t_0 - t)^n, \quad 0 < t < t_0 \ (b_0 = A_0 R_0), \quad n < 0. \tag{29}$$

With such a representation, one can directly apply the mathematical results obtained in [5, p. 166] to problem (28)–(29).

Indeed, discarding the term with the higher derivative in Eq. (28), we come to the degenerate problem,

$$b_t^A = k_0 A_0 (b_x^A)^2, \tag{30}$$

with the boundary condition

$$b^A(x, 0) = A_0 R_0 (t_0 - t)^n, \quad 0 < t < t_0 \ (n < 0). \tag{31}$$

Boundary-value problem (30)–(31) has the self-similar solution

$$b^A(x, t) = A_0 R_0 (t_0 - t)^n w_a(\zeta), \tag{32}$$

where

$$\zeta = \frac{x}{(k_0 R_0)^{1/2} (t_0 - t)^{(1+n)/2}}.$$

Function $w_a(\zeta) \ge 0$ satisfies the equation

$$(w_a')^2 - \frac{1+n}{2} w_a' \zeta + n w_a = 0, \quad \zeta > 0, \quad w_a(0) = 1, \tag{33}$$

and is implicitly defined from the equality

$$\left[\sqrt{\left(\frac{1+n}{4}\right)^2 - n w_a \zeta^{-2}} - \frac{1+n}{4}\right]^{(1+n)/2} \times \left[\sqrt{\left(\frac{1+n}{4}\right)^2 - n w_a \zeta^{-2}} + \frac{1-n}{4}\right]^{(1-n)/2} = \frac{(-n)^{1/2}}{\zeta}$$

in the domain where it is positive. In the remaining points, we put $w_a(\zeta) = 0$.

The curves $w_a(\zeta)$ of the solution to boundary-value problem (30)–(31) are presented in Fig. 2. Function $w_a(\zeta)$ is convex downward; consequently, $b_{\zeta\zeta} > 0$, which makes it possible to compare the solutions to Eq. (23') by varying the positive values of the coefficients multiplying the respective (always positive) terms of the equation.

The properties of monotonically decreasing function $w_a(\zeta)$ depend on parameter n.

(i) If $-1 < n < 0$, $w_a(\zeta) > 0$ for all $\zeta > 0$ and $w_a(\zeta) = C(n)\zeta^{2n/(1+n)} + \ldots + \ldots$ at $\zeta \longrightarrow \infty$, where

$$C(n) = -\frac{1+n}{2n}2^{-2n/(1+n)}(-n)^{1/(1+n)}.$$

(ii) When $n = -1$, the solution has the form $w_a(\zeta) = (1 - \zeta/2)^2$ at $0 < \zeta < 2$ and $w_a(\zeta) = 0$ at $\zeta \ge 2$.

(iii) For $n < -1$, function $w_a(\zeta)$ is finite. That is, $w_a(\zeta) > 0$ at $0 \le \zeta \le \zeta_f$, where $\zeta_f = 2(-n)^{n/2}(-1-n)^{-(1+n)/2}$, and $w_a(\zeta) = 0$ for all $\zeta \ge \zeta_f$, with

$$w_a(\zeta) = -[(1+n)/2]\zeta_f(\zeta_f - \zeta) + O(\zeta_f - \zeta) \quad \text{as} \quad \zeta \longrightarrow \zeta_f^-.$$

In all the cases,

$$w_a'(0) = -(-n)^{-1/2}, \quad w_a''(\zeta) > 0 \tag{34}$$

in the domain where $w_a > 0$ and

$$w_a''(\zeta) \ge w_a''(0) = (1-n)/4 \ \text{ at } \ 0 < \zeta < \zeta_f.$$

From the properties of self-similar solutions and the maximum principle [5], we have the estimates

$$-A_0R_0t_0 \le |b(x,t) - b^A(x,t)| \le A_0|w_a''(\zeta)|\ln\left(\frac{t_0}{t_0 - t}\right) \tag{35}$$

which are valid in the interval $0 < \zeta < \zeta_f$. Here, $b(x, t)$ is the solution to Eq. (28) with higher derivative $b_{xx}(x, t)$.

For the self-similar representation of the solution,

$$w(\zeta, t) = (A_0R_0)^{-1}(t_0 - t)^{-n}b(\zeta(k_0R_0)^{1/2}(t_0 - t)^{(1+n)/2}),$$

inequalities (35) give the following estimate for the rate of convergence to the approximate self-similar solution:

$$|w(\cdot, t) - w_a(\cdot)| = O((t_0 - t)^{-n}|\ln(t_0 - t)|) \longrightarrow 0 \quad \text{at} \quad t \longrightarrow t_0. \tag{36}$$

Estimate (36) shows that, as $t \longrightarrow t_0$, one actually can take into account only self-similar solution $w_a(\zeta)$ to the equation without diffusion. Formally, result (36) can be obtained by simple substitution of representation (31) into Eq. (28) to yield

$$-A_0R_0(t_0 - t)^{n-1}\left[nw_a(\zeta) + \frac{1+n}{2}w_a'(\zeta)\zeta\right] = A_0R_0(t_0 - t)^{n-1}(w_a'(\zeta))^2 + A_0(t_0 - t)^{-1}w_a''(\zeta).$$

Having multiplied both parts of the equation by factor $(t - t_0)^{1-n}$, one can easily see that this equation consists of Eq. (33) giving the self-similar solution and a regular perturbation produced by the term $A_0^{-1}(t_0 - t)^{-n}w_a''(\zeta)$. It is evident that, if the second derivative $w_a''(\zeta)$ is limited, this term can be neglected in the limit $t \longrightarrow t_0$. This fact is embodied in estimates (35) and (36).

Figure 2 shows the numerical solutions to problem (23)–(24) for $n = -1$, which indicate that (36) converges at $t \longrightarrow t_0$.

From the self-similar representation of variable ζ (see relationship (32)), it follows that the coordinate of the magnetic wave front (i.e., the point where the magnetic field amplitude vanishes) is found from the equality

$$x_{\text{eff}}^-(t) = \zeta_{\text{eff}}(k_2^-b_0)^{1/2}(t_0 - t)^{(1+n)/2}. \tag{37}$$

The penetration rate of the magnetic flux into the superconductor is calculated from the equality

$$v_{\text{eff}}(t) = -\frac{1+n}{2}\zeta_{\text{eff}}(k_2^-b_0)^{1/2}(t_0 - t)^{(n-1)/2}, \tag{38}$$

where ζ_{eff} is found from condition $b^A(\zeta_{\text{eff}}) = 1/2$.

From the definition of the resistivity,

$$\rho(b) = (1 - A)b - \beta(\theta^{-1} - 1)b^{1/2},$$

we find that

$$\rho'(b) = 1 - A - \frac{\beta}{2}(\theta^{-1} - 1)b^{-1/2}$$

and, consequently,

$$k_2^- = 1 - A - \frac{\beta}{2}(\theta^{-1} - 1)(b^-)^{-1/2}, \tag{39}$$

where $\beta = \dfrac{A}{2c_L^2}\dfrac{T_m}{T}$.

From (38), we have that the magnetic flux penetrates into the superconductor only if $1 + n < 0$. Thus, property (i) of function $w_a(\zeta)$ does not meet the given situation. Accordingly, only properties (ii) and (iii) hold. This means that the magnetic flux localizes at all $n \ll -1$. For $n = -1$, we obtain a self-similar solution,

$$b^A(x, t) = A_0R_0(t_0 - t)^{-1}\left(1 - \frac{x}{x_0}\right), \tag{40}$$

where $x_0 = (k_0 R_0)^{1/2}$. At $t \longrightarrow t_0$, solution (40) represents a stopped (stationary) magnetic wave. The position of point $x_f(t) \equiv x_0$ of the wave front ($x_0 = 2(k_2^{-1} b_0)^{1/2}$ does not change throughout the time of blow-up, $0 < t < t_0$. Magnetic flux perturbations concentrating in the region of localization $0 < x < x_0$ do not penetrate deeper into the superconductor, although the magnetic field amplitude indefinitely increases in this region as $t \longrightarrow t_0^-$.

It should be noted that the boundary condition under consideration is an idealization of the real physical situation observed in the experiment. Putting $b_0 = H_{c_1}/B_m$, we note that the condition $b < 1$ is always fulfilled in the lower branch of the phase diagram by virtue of normalization $b = B/B_m$ (Fig. 1). Then, from boundary condition (24), we have the inequality $t < t_0^- + H_{c_1}/B_m$, which is fulfilled if the observation period obeys the inequality

$$t < t_{\exp} < t_0^- + H_{c_1}/B_m$$

(see, for example, [6]).

Next, from (39) and (40), it follows that the penetration depth of the magnetic field varies as $x_{\text{eff}}(\cdot) \propto (1 - A)^{1/2}$; i.e., as the pinning force grows, the penetration depth decreases by the square-root law at $n = -1$. Similarly, from (37) and (39), it follows that, as the temperature rises, the approximate dependence

$$x_{\text{eff}}(T) \propto \left[1 - A - \frac{A}{2c_L^2}\frac{T_m}{T}\left(\frac{T_m}{T} - 1\right)^{1/2}\right]$$

holds; i.e., when the temperature rises at $T < T_m$, the penetration depth of the flux increases in the regime of the stopped magnetic wave.

The same conclusions concerning the penetration depth follow from (38) at $n + 1 < 0$. At $n = -1$, the penetration rate of the magnetic field into the superconductor equals zero. At $n < -1$, the penetration rate behaves in the same way as the penetration depth. The corresponding curves are presented in Figs. 3 and 4.

Similarly, assuming that $k_2^+ = \rho'(1)$, i.e., at

$$k_2^+ = 1 - A - \frac{\beta}{2}(\theta^{-1} - 1)(k_1^+ = \rho(1)),$$

we derive the upper limits of the penetration depth and penetration rate of the magnetic flux in the range $H_{c_1} < B < B_m$.

The real values of $x_{\text{eff}}(t)$ and $v_{\text{eff}}(t)$ lie between these limiting values.

MAGNETIC FLUX PENETRATION IN THE CASE OF STRONG PINNING

The results of the previous section were obtained in the zero approximation in relative velocity perturbations $\delta v/v \ll 1$, so that resistivity $\rho(b, A)$ could be described by formula (12) and constant A in (12) was independent of the magnetic field induction. Now, consider the situation when pinning is weak but not to an extent that one can apply approximation (12), which was obtained as a velocity correction in the case of weak pinning.

When pinning is strong enough, we use approximation (12') for resistivity $\rho(b, j_{cr}/j_0)$ provided that the inequality

$$\mu = \sqrt{2}\pi^2 G_i\left(\frac{j_{cr}(0)}{j_0(0)G_i}\right)^{3/2} \ll 1 \qquad (41)$$

is valid. For $G_i = 10^{-2}$, which is typical of high-temperature superconductors, we have from (41)

$$\frac{j_{cr}(0)}{j_0(0)} \ll 10^{-4/3}.$$

In high-temperature superconductors, such as $YBa_2Cu_3O_{7-y}$, pinning is usually weak: $j_{cr}/j_0 \simeq 10^{-3}$–$10^{-2}$. At the same time, thermal fluctuations are significant: $G_i \simeq 10^{-2}$ (for normal superconductors, $G_i \sim 10^{-7}$).

Thus, condition (41) is fulfilled and, consequently, expression (12') can be expanded in small parameter μ, since the plastic deformation barrier is small compared with thermal fluctuations at $T \approx T_c$. Eventually, we arrive at the expression

$$\rho(b) = \rho_{\text{flow}}\left[1 - A\left(1 + \frac{1}{2c_L^2}\frac{T_m}{T}\left(\frac{\theta^{-1} - 1}{\sqrt{b}}\right)\right)\right]. \qquad (42)$$

Here, $A = \mu b$, where μ is determined from (41) at $T > T_m$.

Let us write expression (42) in the form

$$\rho_\mu(b) = b - \mu b^2 - \mu q b^{3/2}, \qquad (42')$$

where

$$q = 2^{-1}c_L^2(T_m/T)(T_c/T - 1).$$

Then, we obtain, as before, an equation for the magnetic induction distribution in the case of strong pinning,

$$b_t = (\rho(b)b_x), \qquad (43)$$

where $\rho(b)$ is defined by equality (42').

Since the function $\rho_0(b)$ is monotonic, formula (42') makes sense for those values of $\mu > 0$ at which the function $\rho_\mu(b)$ is also monotonic. The monotonicity condition for function (42') can be written in the form

$$\rho'_\mu(b) = 1 - 2\mu b - \frac{2\mu q}{2} b^{1/2} > 0. \quad (44)$$

Then, Eq. (43) can be recast as

$$b_t = \rho'_\mu(b) b_x^2 + \rho_\mu(b) b_{xx}. \quad (45)$$

The behavior of solutions to Eq. (45) with the boundary condition in the blow-up regime is investigated in the same way as in the previous case. We introduce the notation

$$\kappa_1^- = \rho'_\mu(b^-), \quad \kappa_2^- = \rho_\mu(b^-)$$

and

$$\kappa_1^+ = \rho'_\mu(1), \quad \kappa_2^+ = \rho_\mu(1);$$

then, a subsolution is derived from the equation

$$b_t = \kappa_1^- b_x^2 + \kappa_2^- b_{xx}, \quad (46)$$

where

$$\kappa_1^- = 1 - 2\mu b^- - \frac{3\mu q}{2}(b^-)^{1/2},$$

$$\kappa_2^- = b^- - \mu(b^-)^2 - \mu q (b^-)^{3/2} \; (b^{\text{áå}} = H_{c_1}/B_m).$$

A supersolution is derived from the equation

$$b_t = \kappa_1^+ b_x^2 + \kappa_2^+ b_{xx}, \quad (47)$$

where

$$\kappa_1^+ = 1 - 2\mu - \frac{3q}{2},$$

$$\kappa_2^+ = 1 - \mu - \mu q (b^+ \simeq 1).$$

We put $k_0 = k_1^-$ and $k_2^- = k_0/A_0$; i.e., $A_0 = k_1^-/k_2^-$. Then, Eq. (46) is recast as

$$b_t = k_0 b_x^2 + \frac{k_0}{A_0} b_{xx}. \quad (48)$$

Consider boundary condition (31). Then, proceeding in the same way as in the previous section, one can demonstrate that, as $t \longrightarrow t_0$, the diffusion term in Eq. (46) can be neglected. Then, we can restrict our analysis to only the corresponding Hamilton–Jacobi equation a solution to which at $n = -1$ has the form

$$b^A(x, t) = A_0 R_0 (t_0 - t)^{-1} w_a(\zeta), \quad (49)$$

where

$$\zeta = x/(k_0 R_0)^{1/2}$$

and

$$w_a(\zeta) = (1 - \zeta/2)^2, \quad 0 < \zeta < 2$$

or $w_a(\zeta) = 0$ if $\zeta \geq 2$. General solution $b(x, t)$ converges to self-similar solution $b_A(x, t)$ at $t \longrightarrow t_0^-$. From (49), it follows that the depth of stopped magnetic flux localization is given by

$$x_L = 2R_0^{1/2}\left(1 - 2\mu b^- - \frac{3\mu q}{2}(b^-)^{1/2}\right)^{1/2}.$$

At $n \leq -1$, the penetration depth of the magnetic flux appears as

$$x_{\text{eff}}^-(t) = \zeta_{\text{eff}}(\kappa_1^- R_0)^{1/2}(t_0 - \kappa t)^{(1+n)/2} \quad (50)$$

and the velocity of the magnetic wave front as

$$v_{\text{eff}}^-(t) = -\zeta_{\text{eff}}\kappa\left(\frac{1+n}{2}\right)(\kappa_1^- R_0)^{1/2}(t_0 - \kappa t)^{(n-1)/2}. \quad (51)$$

Formulas (50) and (51) describe the characteristics of the magnetic wave for the subsolution.

For the supersolution, the analogues of these formulas are

$$x_{\text{eff}}^+(t) = \zeta_{\text{eff}}(\kappa_1^+ R_0)^{1/2}(t_0 - \kappa t)^{(1+n)/2}, \quad (52)$$

$$v_{\text{eff}}^+(t) = -\zeta_{\text{eff}}\left(\frac{1+n}{2}\right)(\kappa_1^+)^{3/2} R_0^{1/2}(t_0 - \kappa t)^{(1-n)/2}. \quad (53)$$

From (50)–(53), one can conclude that the real coordinate and real velocity of the magnetic wave fall into the intervals

$$v_{\text{eff}}^-(t) \leq v_{\text{eff}}(t) \leq v_{\text{eff}}^+(t) \;\text{ and }\; x_{\text{eff}}(t) \leq x_{\text{eff}}(t) \leq x_{\text{eff}}^+(t)$$

at $H_{c_1} < B < B_m$ (Fig. 2).

Now, we turn back to Figs. 3 and 4, which plot the depth and rate of magnetic flux penetration into the high-temperature superconductor as functions of problem parameter j_{cr}/j_0. Here, j_{cr} is the critical depinning current, which is always limited from above by depairing current $j_0 = cH_c/3\sqrt{6}\pi\lambda$. Then, by virtue of the formula $j_{cr} = cF_{pin}/B$, parameter j_{cr}/j_0 is a measure of the pinning force and is a fundamental parameter in the phenomenological theory of type-II superconductors. It characterizes inhomogeneities in superconducting systems.

The curves in Fig. 3 are constructed from formulas (52) and (37) for $\kappa_1^+ = 1 - 2\mu$ ($q \approx 0$ at $T \approx T_c$), $R_0 = t_0 = 1$, and $n = -2$ at $\mu = 10\sqrt{2}\pi^2(j_{cr}/j_0)^{3/2}$. For curve *3*, $j_{cr}/j_0 = 10^{-4}$; for curve *4*, $j_{cr}/j_0 = 10^{-3}$. Calculations performed for $j_{cr}/j_0 = 10^{-2}$ indicate $x_{\text{eff}}(t)$ becomes negative. This means that interpolation formula (3) is valid only at $j_{cr}/j_0 \geq 10^{-2}$, which agrees with experimental data [1]. Curves *3* and *4* reflect a simple tendency: as the pinning force grows, depth $x_{eff}(t)$ of magnetic flux penetration decreases. Curves *1* and *2* are calculated at $A = 10^{-4}$ and 10^{-3}, respectively. Consequently, we again can conclude that penetration depth $x_{eff}(t)$ of the magnetic field decreases with increasing pinning force.

The same curves in Fig. 4 are calculated at $j_{cr}/j_0 = 10^{-3}$ for strong pinning and at $A = 10^{-3}$ and 10^{-4} for weak pinning. Here, the tendency is the same: the penetration rate of the flux decreases as the pinning force grows.

The graphs in Figs. 3 and 4 show the initial stage of magnetic flux penetration. As $t \longrightarrow t_0$, i.e., as soon as the scaling behavior is observed, the graphs coincide (with the accuracy determined by relationship (36)). This is because a very high magnetic field smoothes out the difference between the influence of weak pinning and thermal fluctuations on the motion of the main magnetic flux. As follows from the definition of parameter $q = q(T, T_m, T_c)$, the temperature dependence at $T \approx T_c$ is such that, as thermal fluctuations build up, the depth and rate of magnetic flux penetration increase.

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