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REMOVAL OF SINGULARITIES AND ANALOGS OF THE SOKHOTSKII–WEIERSTRASS THEOREM FOR Q-MAPPINGS

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We prove that an open discrete Q-mapping $f\colon D \to \overline{\mathbb{R}^n}$ has a continuous extension to an isolated boundary point if the function $Q(x)$ has finite mean oscillation or logarithmic singularities of order at most $n-1$ at this point. Moreover, the extended mapping is open and discrete and is a Q-mapping. As a corollary, we obtain an analog of the well-known Sokhotskii–Weierstrass theorem on Q-mappings. In particular, we prove that an open discrete Q-mapping takes any value infinitely many times in the neighborhood of an essential singularity, except, possibly, for a certain set of capacity zero.

1. Introduction

As is known, the definition of quasiconformal mappings given in a domain D from $\mathbb{R}^n$, $n \ge 2$, is based on the following inequality for an arbitrary family Γ of curves γ in the domain D:

$$M(f\Gamma) \le K M(\Gamma), \tag{1}$$

where M is the modulus of a family of curves (an external measure defined on families of curves in $\mathbb{R}^n$) and $K \ge 1$ is a certain constant. In other words, the modulus of any family of curves is distorted at most K times. In capacity language , relation (1) means that the mapping f distorts the capacity of any condenser in D at most K times. Now let the definition of the considered class of mappings be based not on relation (1) but on the inequality

$$M(f\Gamma) \le \int\limits_D Q(x)\rho^n(x)\,dm(x), \tag{2}$$

where $m(x)$ is the n-dimensional Lebesgue measure, ρ is an arbitrary nonnegative Borel function such that the length of an arbitrary curve γ of the family Γ is at least 1 in the metric ρ, and $Q\colon D \to [1,\infty]$ is a fixed real-valued function. In the case where $Q(x) \le K$ almost everywhere, we again arrive at inequality (1). In the general case, the last inequality means that the modulus of the original family Γ is distorted with certain weight $Q(x)$ (see [1]):

$$M(f\Gamma) \le M_{Q(x)}(\Gamma).$$

In the present paper, we consider the problem of finding conditions for the function $Q(x)$ in the definition of the mapping f [see relation (2)] under which f can be extended by continuity to an isolated boundary point.

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The mapping f is only assumed to be open and discrete. For homeomorphisms, analogous theorems were proved in [2]. The procedure of investigation of mappings with branching differs significantly from that for homeomorphisms. We show that the theorems obtained below yield several interesting corollaries, in particular, the Sokhotskii–Weierstrass theorem. Rigorous definitions are given in what follows.

The theory of *Q-homeomorphisms,* i.e., homeomorphisms for which inequality (2) is true and related classes, was developed (see, e.g., [4]) mainly in the case where a majorant belongs to the known BMO-space (the space of John–Nirenberg functions of bounded mean oscillation [3]). The possibility of continuous extension to an isolated boundary point for quasiregular mappings was shown by Martio, Rickman, and Väisälä (see, e.g., [5, 6]).

2. Preliminary Results

We introduce the main definitions and notation used in this paper. In what follows, D is a domain in $\mathbb{R}^n$, $n \geq 2$. A mapping $f\colon D \to \mathbb{R}^n$ is called *discrete* if the preimage $f^{-1}(y)$ of every point $y \in \mathbb{R}^n$ consists of isolated points and *open* if the image of any open set $U \subseteq D$ is an open set in $\mathbb{R}^n$. In what follows, the notation $f\colon D \to \mathbb{R}^n$ means that the mapping f is continuous in the domain of definition. We write $G \Subset D$ if $\overline{G}$ is a compact subset of the domain D. We say that a mapping f preserves (antipreserves) orientation if the topological index satisfies the inequality $\mu(y, f, G) > 0$ ($\mu(y, f, G) < 0$) for an arbitrary domain $G \Subset D$ and any $y \in f(G) \setminus f(\partial G)$. In what follows,

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}, \qquad B(r) = \{x \in \mathbb{R}^n : |x| < r\},$$

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\},$$

and $m(x)$ is the n-dimensional Lebesgue measure. The notions presented above can naturally be extended to the mappings $f\colon D \to \overline{\mathbb{R}^n}$, where $D \subset \mathbb{R}^n$ and $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ is a one-point compactification of $\mathbb{R}^n$.

Recall that a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for a family Γ of curves γ in $\mathbb{R}^n$ if

$$\int_\gamma \rho(x)|dx| \geq 1$$

for all curves $\gamma \in \Gamma$. In this case, we write $\rho \in \operatorname{adm}\Gamma$. The quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int_D \rho^n(x)\, dm(x)$$

is called the *modulus* of the family of curves Γ. Let $Q\colon D \to [1, \infty]$ be a Lebesgue-measurable function. A mapping $f\colon D \to \overline{\mathbb{R}^n}$ is called a *Q-mapping* if

$$M(f\Gamma) \leq \int_D Q(x)\rho^n(x)\, dm(x)$$

for any family Γ of curves γ in D and for every admissible function $\rho \in \operatorname{adm}\Gamma$. This definition differs insignificantly from the definition presented in Sec. 1 in [4] (see also [7]), where the modulus inequality was studied for a special class of mappings. Nevertheless, this difference does not affect applications to other known classes (see the last section).

Following [6] (Sec. 10, Chap. II), we call a pair $E = (A, C)$, where A is an open set in $\mathbb{R}^n$, $n \geq 2$, and C is a compact subset of A, a *condenser* in $\mathbb{R}^n$. The quantity

$$\operatorname{cap} E = \operatorname{cap}(A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^n \, dm(x), \tag{3}$$

is called the capacity of the condenser E. Here, $W_0(E) = W_0(A, C)$ is a family of nonnegative continuous functions $u: A \to \mathbb{R}$ with compact support in A such that $u(x) \geq 1$ for $x \in C$ and $u \in ACL$. In relation (3), we have also used the standard notation

$$|\nabla u| = \left(\sum_{i=1}^{n} (\partial_i u)^2 \right)^{1/2}.$$

Recall that a mapping $f: D \to \mathbb{R}^n$ is called *absolutely continuous on lines* $(f \in ACL)$ if, in any n-dimensional parallelepiped P having edges parallel to the coordinate axes and such that $\overline{P} \subset D$, all coordinate functions $f = (f_1, \ldots, f_n)$ are absolutely continuous on almost all straight lines parallel to the coordinate axes. For details of application of moduli (capacities) in the theory of mappings, see, e.g., [8] and [9]. Some useful information on moduli, capacities, various characteristics of condensers, and relations between them can be found in [10]. In the present paper, we do not consider these relations in detail.

In what follows, in the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ we use the *spherical (chordal) metric* $h(x, y) = |\pi(x) - \pi(y)|$, where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(e_{n+1}/2, 1/2)$ in $\mathbb{R}^{n+1}$:

$$h(x, \infty) = \frac{1}{\sqrt{1+|x|^2}}, \qquad h(x, y) = \frac{|x-y|}{\sqrt{1+|x|^2}\,\sqrt{1+|y|^2}}, \qquad x \neq \infty \neq y.$$

Let $f: D \to \mathbb{R}^n$, $n \geq 2$, be an open discrete mapping, let $\beta: [a, b) \to \mathbb{R}^n$ be a certain curve, and let $x \in f^{-1}(\beta(a))$. A curve $\alpha: [a, c) \to D$ is called the *maximum lifting* of the curve β under the mapping f with origin at the point x if the following conditions are satisfied:

(i) $\alpha(a) = x$;

(ii) $f \circ \alpha = \beta|_{[a,c)}$;

(iii) if $c < c' \leq b$, then there does not exist a curve $\alpha': [a, c') \to D$ such that $\alpha = \alpha'|_{[a,c)}$ and $f \circ \alpha' = \beta|_{[a,c')}$.

Let f be an open discrete mapping and let $x \in f^{-1}(\beta(a))$. Then the curve β has a maximum lifting under the mapping f with origin at the point x (see Corollary 3.3 in Chap. II in [6]).

Lemma 1. *Let* $E = (A, C)$ *be an arbitrary condenser in* $\mathbb{R}^n$ *and let* Γ_E *be the family of all curves of the form*

$$\gamma : [a, b) \to A, \qquad \gamma(a) \in C, \qquad \text{and} \qquad |\gamma| \bigcap (A \setminus F) \neq \varnothing$$

for an arbitrary compact set $F \subset A$. *Then* $\operatorname{cap} E = M(\Gamma_E)$ *(see Proposition 10.2 in Chap. II in [6]).*

Remark 1. The notions of condenser and condenser capacity in $\mathbb{R}^n$ can be extended to $\overline{\mathbb{R}^n}$ (see Sec. 2.1 in [5]). Lemma 1 remains true for condensers from $\overline{\mathbb{R}^n}$ (see Remark 10.8(1) in Chap. II in [6]).

We say that a compact set C in $\mathbb{R}^n$, $n \geq 2$, has capacity zero ($\operatorname{cap} C = 0$) if there exists a bounded open set A with $C \subset A$ such that $\operatorname{cap}(A, C) = 0$. It is known (see Lemma 3.4 in Chap. II in [4]) that, in the last case and for any other bounded open set A in $\mathbb{R}^n$ containing C, one has $\operatorname{cap}(A, C) = 0$. Otherwise we set $\operatorname{cap}(A, C) > 0$. It is easy to see that an arbitrary one-point set $C = \{a\}$ has capacity zero. By analogy with the definition of a set of capacity zero in $\mathbb{R}^n$, we can define a set of capacity zero in $\overline{\mathbb{R}^n}$ (see, e.g., Sec. 2.12 in [5]).

Lemma 2. *Let* E *be a compact proper subset of* $\overline{\mathbb{R}^n}$ *such that* $\operatorname{cap} E > 0$. *Then, for every* $a > 0$, *there exists a positive number* $\delta > 0$ *such that* $\operatorname{cap}\left(\overline{\mathbb{R}^n} \setminus E, C\right) \geq \delta$ *for an arbitrary continuum* $C \subset \overline{\mathbb{R}^n} \setminus E$ *that satisfies the condition* $h(C) \geq a$ *(see Lemma 3.11 in [5] or Lemma 2.6 in Chap. III in [6]).*

Remark 2. If a compact set F in $\mathbb{R}^n$, $F \subset D$, has capacity zero, then, for every $\alpha > 0$, the α-dimensional Hausdorff measure $\Lambda_\alpha(F)$ of the set F is equal to zero (see Lemma 2.13 in [5]). Therefore, $\operatorname{mes} F = 0$, $\operatorname{Int} F = \varnothing$, and $D \setminus F$ is a domain according to the Menger–Uryson theorem (see, e.g., Theorem IV.4 in [12]).

3. Main Lemma on Extension

Lemma 3. *Let* $f : \mathbb{B}^n \setminus \{0\} \to \overline{\mathbb{R}^n}$, $n \geq 2$, *be an open discrete* Q*-mapping such that*

$$\operatorname{cap}\left(\overline{\mathbb{R}^n} \setminus f(\mathbb{B}^n \setminus \{0\})\right) > 0.$$

Suppose that there exists $\varepsilon_0 > 0$, $\varepsilon_0 < 1$, *such that, as* $\varepsilon \to 0$, *one has*

$$\int\limits_{\varepsilon < |x| < \varepsilon_0} Q(x)\psi^n(|x|)\,dm(x) = o(I^n(\varepsilon, \varepsilon_0)), \tag{4}$$

where $\psi(t)$ *is a function nonnegative on* $(0, \infty)$ *and such that* $\psi(t) > 0$ *for almost all* t *and*

$$0 < I(\varepsilon, \varepsilon') = \int\limits_{\varepsilon}^{\varepsilon'} \psi(t)\,dt < \infty$$

for all (fixed) ε' *from* $(0, \varepsilon_0]$ *and* $\varepsilon \in (0, \varepsilon')$. *Then* f *has a continuous extension* $f : \mathbb{B}^n \to \overline{\mathbb{R}^n}$ *into* $\mathbb{B}^n$. *The continuity is understood in the sense of the space* $\overline{\mathbb{R}^n}$ *with respect to the chordal metric* h.

Proof. Assume that the converse statement is true, namely, assume that the mapping f cannot be extended by continuity to the point $x_0 = 0$. Then there exist two sequences x_j and x_j' that belong to $\mathbb{B}^n \setminus \{0\}$ and are such that $x_j \to 0$, $x_j' \to 0$, and $h(f(x_j), f(x_j')) \geq a > 0$ for all $j \in \mathbb{N}$. Without loss of generality, we can assume that x_j and x_j' lie inside the ball $B(\varepsilon_0)$. We set $r_j = \max\{|x_j|, |x_j'|\}$ and connect the points x_j and x_j' by a closed curve lying in $\overline{B(r_j)} \setminus \{0\}$. Denote this curve by C_j and consider the condenser $E_j = (\mathbb{B}^n \setminus \{0\}, C_j)$. Since the mapping f is open and continuous, fE_j is also a condenser. Consider the family of curves Γ_{E_j} and Γ_{fE_j} (see the notation introduced in Lemma 1). Let Γ_j^* be the family of maximum liftings of Γ_{fE_j} under the mapping f with origin at C_j lying in $\mathbb{B}^n \setminus \{0\}$. Let us show that $\Gamma_j^* \subset \Gamma_{E_j}$.

Assume that the converse statement is true. Then there exists a curve $\beta : [a, b) \to \mathbb{R}^n$ of the family Γ_{fE_j} for which the corresponding maximum lifting $\alpha : [a, c) \to \mathbb{B}^n \setminus \{0\}$ lies, together with its closure $\overline{\alpha}$, in a certain compact set inside $\mathbb{B}^n \setminus \{0\}$. Therefore, $\overline{\alpha}$ is a compact set in $\mathbb{B}^n \setminus \{0\}$. First, note that $c \neq b$ because otherwise $\overline{\beta}$ is a compact set in fA, which contradicts the condition $\beta \in \Gamma_{fE_j}$. Consider the set

$$G = \left\{x \in \mathbb{R}^n : x = \lim_{k \to \infty} \alpha(t_k)\right\},$$

where $t_k \in [a, c)$ are such that

$$\lim_{k \to \infty} t_k = c : \lim_{k \to \infty} \alpha(t_k) = x.$$

Note that, passing to subsequences, we can restrict ourselves here to monotone sequences t_k. For $x \in G$, by virtue of the continuity of f in $\mathbb{B}^n \setminus \{0\}$, we have $f(\alpha(t_k)) \to f(x)$ as $k \to \infty$, where $t_k \in [a, c)$ and $t_k \to c$ as $k \to \infty$. However, $f(\alpha(t_k)) = \beta(t_k) \to \beta(c)$ as $k \to \infty$. This implies that f is constant on G in $\mathbb{B}^n \setminus \{0\}$. On the other hand, according to the Cantor condition in a compact set $\overline{\alpha}$ (see [13, pp. 8, 9]), we have

$$G = \bigcap_{k=1}^{\infty} \overline{\alpha([t_k, c))} = \limsup_{k \to \infty} \alpha([t_k, c)) = \liminf_{k \to \infty} \alpha([t_k, c)) \neq \varnothing$$

by virtue of the monotonicity of the sequence of connected sets $\alpha([t_k, c))$. Therefore, G is connected with respect to I [(9.12) in [14]]. Thus, by virtue of the discreteness of f, the set G cannot consist of more than one point, and the curve $\alpha : [a, c) \to \mathbb{B}^n \setminus \{0\}$ can be extended to a closed curve $\alpha : [a, c] \to \mathbb{B}^n \setminus \{0\}$. Then $f(\alpha(c)) = \beta(c)$, i.e., $\alpha(c) \in f^{-1}(\beta(c))$. On the other hand, we can construct (see Corollary 3.3 in Chap. II in [6]) the maximum lifting α' of the curve $\beta|_{[c,b)}$ with origin at the point $\alpha(c)$. Combining the liftings α and α', we obtain a new lifting α'' of the curve β defined on $[a, c')$, which contradicts the maximality of the lifting α.

Thus, $\Gamma_j^* \subset \Gamma_{E_j}$. Note that $\Gamma_{fE_j} > f\Gamma_j^*$, whence

$$M(\Gamma_{fE_j}) \le M(f\Gamma_j^*) \le M(f\Gamma_{E_j}). \tag{5}$$

Note that the family Γ_{E_j} can be divided into two subfamilies:

$$\Gamma_{E_j} = \Gamma_{E_{j_1}} \cup \Gamma_{E_{j_2}}, \tag{6}$$

where $\Gamma_{E_{j_1}}$ is the family of all curves $\alpha(t)\colon [a, c) \to \mathbb{B}^n \setminus \{0\}$ with origin at C_j such that there exists $t_k \in [a, c)$ with $\alpha(t_k) \to 0$ as $t_k \to c-0$, and $\Gamma_{E_{j_2}}$ is the family of all curves $\alpha(t)\colon [a, c) \to \mathbb{B}^n \setminus \{0\}$ with origin at C_j such that there exists $t_k \in [a, c)$ with $\operatorname{dist}(\alpha(t_k), \partial\mathbb{B}^n) \to 0$ as $t_k \to c-0$.

It follows from (5) and (6) that

$$M(\Gamma_{fE_j}) \le M(f\Gamma_{E_{j_1}}) + M(f\Gamma_{E_{j_2}}). \tag{7}$$

Let us show that $M(f\Gamma_{E_{j_1}}) = 0$ for any fixed $j \in \mathbb{N}$. We fix an integer $j \ge 1$ and set $l_j = \min\{|x_j|, |x'_j|\}$. Consider the ring $A_{\varepsilon,j} = \{x \in \mathbb{R}^n\colon \varepsilon < |x| < l_j\}$. According to the Luzin theorem, there exists a Borel function $\psi_*(t) = \psi(t)$ for almost all t. Therefore, the function

$$\rho_\varepsilon(x) = \begin{cases} \psi_*(|x|)/I(\varepsilon, l_j), & x \in A_{\varepsilon,j}, \\ 0, & x \in \mathbb{R}^n \setminus A_{\varepsilon,j}, \end{cases}$$

is well defined and is a Borel function. Furthermore, for any $\gamma \in \Gamma_{E_{j_1}}$, we have (see Theorem 5.7 in [9])

$$\int_\gamma \rho_\varepsilon |dx| \ge \frac{1}{I(\varepsilon, l_j)} \int_\varepsilon^{l_j} \psi_*(t)\,dt = 1.$$

Therefore, $\rho_\varepsilon \in \operatorname{adm} \Gamma_{E_{j_1}}$ and, by the definition of Q-mapping,

$$M(f\Gamma_{E_{j_1}}) \le \mathcal{F}(\varepsilon) := \frac{1}{I(\varepsilon, l_j)^n} \int_{\varepsilon<|x|<\varepsilon_0} Q(x)\psi_*^n(|x|)\,dm(x). \tag{8}$$

Let us show that $\mathcal{F}(\varepsilon) \to 0$. Taking (4) into account, we obtain

$$\int_{\varepsilon<|x|<\varepsilon_0} Q(x)\psi_*^n(|x|)\,dm(x) = G(\varepsilon)\left(\int_\varepsilon^{\varepsilon_0} \psi_*(t)\,dt\right)^n,$$

where $G(\varepsilon) \to 0$ as $\varepsilon \to 0$. Note that

$$\mathcal{F}(\varepsilon) = G(\varepsilon)\left(1+\frac{\int_{l_j}^{\varepsilon_0}\psi_*(t)\,dt}{\int_{\varepsilon}^{l_j}\psi_*(t)\,dt}\right)^n,$$

where

$$\int_{l_j}^{\varepsilon_0}\psi_*(t)\,dt < \infty$$

is a fixed number and

$$\int_{\varepsilon}^{l_j}\psi_*(t)\,dt \to \infty \quad \text{as} \quad \varepsilon \to 0$$

because the value of the integral on the left-hand side of (4) increases as ε decreases. Thus, $\mathcal{F}(\varepsilon) \to 0$. Note that the left-hand side of inequality (8) does not depend on ε, and j is fixed. This yields $M(f\Gamma_{E_{j_1}}) = 0$. By analogy with the scheme presented above, we consider the ring $A_j = \left\{x \in \mathbb{R}^n \colon r_j < |x| < \varepsilon_0\right\}$. According to the Luzin theorem, there exists a Borel function $\psi_*(t) = \psi(t)$ for almost all t. Then the function

$$\rho_j(x) = \begin{cases}\psi_*(|x|)/I(r_j,\varepsilon_0), & x \in A_j,\\ 0, & x \in \mathbb{R}^n \setminus A_j,\end{cases}$$

is also a Borel function and

$$\int_\gamma \rho_j |dx| \geq \frac{1}{I(r_j,\varepsilon_0)}\int_{r_j}^{\varepsilon_0}\psi_*(t)\,dt = 1$$

for an arbitrary curve $\gamma \in \Gamma_{E_{j_2}}$. Thus, by the definition of Q-mapping, according to conditions (4) and (7) we get

$$M(f\Gamma_{E_j}) \leq \mathcal{S}(r_j) := \frac{1}{I(r_j,\varepsilon_0)^n}\int_{r_j<|x|<\varepsilon_0} Q(x)\psi_*^n(|x|)\,dm(x),$$

where $\mathcal{S}(r_j) \to 0$ as $j \to \infty$. Finally, by virtue of Lemma 1 and relation (5), $\operatorname{cap} fE_j \to 0$ as $j \to \infty$. On the other hand, according to Lemma 2, $\operatorname{cap} fE_j \geq \delta > 0$ for all $j \in \mathbb{N}$. The contradiction obtained disproves the assumption that f does not have a limit as $x \to 0$ in $\overline{\mathbb{R}^n}$.

Theorem 1. *Suppose that* $x_0 \in D$ *and* $f\colon D\setminus\{x_0\} \to \overline{\mathbb{R}^n}$ *is an open discrete* Q*-mapping such that* $\operatorname{cap}\left(\overline{\mathbb{R}^n}\setminus f(D\setminus\{x_0\})\right) > 0$. *If*

$$q_{x_0}(r) = O\left(\left[\log\frac{1}{r}\right]^{n-1}\right) \quad as \quad r \to 0, \tag{9}$$

where $q_{x_0}(r)$ *is the mean integral value of* $Q(x)$ *over the sphere* $|x - x_0| = r$, *then* f *has a continuous extension* $f: D \to \overline{\mathbb{R}^n}$. *In particular, if, for a certain* $\varepsilon(x_0)$, *one has*

$$Q(x) \le \left[\log\frac{1}{|x - x_0|}\right]^{n-1} \qquad \forall x \in B(x_0, \varepsilon(x_0)),$$

then relation (9) is true, and, hence, the statement of the theorem is true.

Proof. Without loss of generality, we can assume that $x_0 = 0$ and $\mathbb{B}^n \subset D$. We fix $\varepsilon_0 < 1$ and set

$$\psi(t) = \frac{1}{t \log t^{-1}}.$$

Note that

$$\int\limits_{\varepsilon < |x| < \varepsilon_0} \frac{Q(x)\,dm(x)}{\left(|x|\log|x|^{-1}\right)^n} = \int\limits_{\varepsilon}^{\varepsilon_0}\left(\int\limits_{|x| = r} \frac{Q(x)\,dm(x)}{\left(|x|\log|x|^{-1}\right)^n}\,dS\right)dr$$

$$\le C\omega_{n-1}\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dr}{r\log r^{-1}} = C\omega_{n-1}\log\frac{\log\varepsilon^{-1}}{\log(\varepsilon_0)^{-1}} = C\omega_{n-1} I(\varepsilon, \varepsilon_0),$$

where

$$I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt$$

and C is a certain constant. The required conclusion follows directly from Lemma 3.

4. Finite Mean Oscillation

We say that a function $\varphi: D \to R$, $\varphi \in L^1_{\mathrm{loc}}(D)$, is of *bounded mean oscillation* in a domain D $(\varphi \in BMO)$ if

$$\|\varphi\|_* = \sup_{B \subset D}\frac{1}{|B|}\int\limits_B |\varphi(x) - \varphi_B|\,dm(x) < \infty,$$

where the least upper bound is taken over all balls $B \subset D$ and

$$\varphi_B = \frac{1}{|B|}\int_B \varphi(x)\,dm(x)$$

is the mean value of the function φ over the ball B (see [3]). In what follows, for simplicity, we denote

$$\fint_A f(x)dm(x) := \frac{1}{|A|}\int_A f(x)dm(x),$$

where, as usual, $|A|$ is the Lebesgue measure of a set $A \subseteq \mathbb{R}^n$. It is known that $L^\infty(D) \subset BMO(D) \subset L^p_{\text{loc}}(D)$ (see, e.g., [3]). Following [2], we introduce the following definitions: We say that a function $\varphi : D \to \mathbb{R}$ is of *finite mean oscillation* at a point $x_0 \in D$ $(\varphi \in FMO$ at $x_0)$ if

$$\limsup_{\varepsilon\to 0} \fint_{B(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon|\,dm(x) < \infty, \tag{10}$$

where

$$\overline{\varphi}_\varepsilon = \fint_{B(x_0,\varepsilon)} \varphi(x)\,dm(x).$$

Note that, under condition (10), it is possible that $\overline{\varphi}_\varepsilon \to \infty$ as $\varepsilon \to 0$. For example, φ is a function of finite mean oscillation at a point x_0 if the following relation holds at the point $x_0 \in D$:

$$\limsup_{\varepsilon\to 0} \fint_{B(x_0,\varepsilon)} |\varphi(x)|\,dm(x) < \infty.$$

Theorem 2. *Suppose that $x_0 \in D$ and $f : D\setminus\{x_0\} \to \overline{\mathbb{R}^n}$ is an open discrete Q-mapping such that* $\operatorname{cap}\left(\overline{\mathbb{R}^n}\setminus f(D\setminus\{x_0\})\right) > 0$. *If $Q(x)$ is a function of finite mean oscillation at the point x_0, then f has a continuous extension $f : D \to \overline{\mathbb{R}^n}$.*

Proof. Without loss of generality, we can assume that $x_0 = 0$ and $\mathbb{B}^n \subset D$. Let $\varepsilon_0 < e^{-1}$. Using Corollary 2.3 from [2], for the function $0 < \psi(t) = (t \log t^{-1})^{-1}$ we obtain

$$\int_{\varepsilon<|x|<\varepsilon_0} Q(x)\psi^n(|x|)dm(x) = O\left(\log\log\frac{1}{\varepsilon}\right).$$

Also note that

$$I(\varepsilon, \varepsilon_0) := \int_\varepsilon^{\varepsilon_0} \psi(t)dt = \log\frac{\log(1/\varepsilon)}{\log(1/\varepsilon_0)}.$$

The remaining part of the statement follows from Lemma 3.

Corollary 1. *In particular, if*

$$\int\limits_{|x-x_0|<\varepsilon} Q(x)\,dm(x) = O(\varepsilon^n) \tag{11}$$

as $\varepsilon \to 0$, *then* f *has a continuous extension in* D.

5. Corollaries. Analog of the Sokhotskii–Weierstrass Theorem

Recall that an isolated point x_0 of the boundary ∂D of a domain D in $\mathbb{R}^n$ is called *removable* if there exists the finite limit

$$\lim_{x\to x_0} f(x).$$

If $f(x) \to \infty$ as $x \to x_0$, then the point x_0 is called a *pole*. An isolated point x_0 of the boundary ∂D is called an *essential singularity* of a mapping $f\colon D \to \overline{\mathbb{R}^n}$ if the limit

$$\lim_{x\to x_0} f(x)$$

does not exist.

Theorem 3. *Suppose that* x_0 *is an isolated point of the boundary* D, $f\colon D \to \overline{\mathbb{R}^n}$ *is an open discrete* Q*-mapping, and* $Q(x)$ *is a function of finite mean oscillation at a point* x_0 *or satisfies at least one of conditions (9) and (11). If* x_0 *is an essential singularity of the mapping* f, *then* $\operatorname{cap}\left(\overline{\mathbb{R}^n} \setminus f(U \setminus \{x_0\})\right) = 0$ *for any neighborhood* U *of the point* x_0.

Proof. The required statement follows directly from Theorems 1 and 2 and Corollary 1.

Theorem 4. *Suppose that* x_0 *is an isolated point of the boundary* D, $f\colon D \to \overline{\mathbb{R}^n}$ *is an open discrete* Q*-mapping, and* $Q(x)$ *is a function of finite mean oscillation at a point* x_0 *or satisfies at least one of conditions (9) and (11). The point* x_0 *is removable for the mapping* f *if and only if* f *is bounded in a certain neighborhood* U *of the point* x_0.

Proof. Assume that the point x_0 is removable, i.e., there exists the limit

$$\lim_{x\to x_0} f(x) = A < \infty.$$

Then $|f(x)| \le |A|+1$ in a sufficiently small neighborhood U of the point x_0. Conversely, assume that there exists a neighborhood U of the point x_0 such that $|f(x)| \le M$ for a certain $M \in (0,\infty)$ and all $x \in U \setminus \{x_0\}$. Then $\operatorname{cap}\left(\overline{\mathbb{R}^n} \setminus f(U \setminus \{x_0\})\right) > 0$, and the required statement follows from Theorem 3.

Theorem 5. *Suppose that* x_0 *is an isolated point of the boundary* D, $f: D \to \overline{\mathbb{R}^n}$ *is an open discrete Q-mapping, and* $Q(x)$ *is a function of finite mean oscillation at a point* x_0 *or satisfies at least one of conditions (9) and (11). If* $\mathrm{cap}\left(\overline{\mathbb{R}^n} \setminus f(U \setminus \{x_0\})\right) > 0$ *for a certain neighborhood* U *of the point* x_0, *then* f *can be continuously extended to an open discrete* Q*-mapping* $f\colon D \bigcup \{x_0\} \to \overline{\mathbb{R}^n}$.

Proof. Indeed, by virtue of Theorems 1 and 2 and Corollary 1, f can be extended to a continuous mapping $f\colon D \bigcup \{x_0\} \to \overline{\mathbb{R}^n}$. The modulus of a family of curves in $\mathbb{R}^n$ that pass through a point is equal to zero (see 7.9 in [9]), which implies that the extended mapping $f\colon D \bigcup \{x_0\} \to \overline{\mathbb{R}^n}$ is a Q-mapping.

It is known that discrete open mappings in $\mathbb{R}^n$, $n \geq 2$, either preserve or do not preserve orientation (see, e.g., [6], Chap. I, Sec. 4). Assume, for definiteness, that f preserves orientation. Let us show that the extended mapping preserves orientation and is open and discrete. As usual, $B_f(D)$ denotes the set of branch points of a mapping f in a domain D, and $B_f(D')$ denotes the set of branch points of the mapping f in the domain $D' = D \bigcup \{x_0\}$. If x_0 is a point of local homeomorphism of the mapping f, then there is nothing to prove.

Let $x_0 \in B_f(D')$. By virtue of the Chernavskii theorem, we have $\dim B_f(D) = \dim f(B_f(D)) \leq n-2$ (see, e.g., Theorem 4.6 in Chap. I in [6]), where dim denotes the topological dimension of a set (see [12]). Then

$$\dim f(B_f(D')) \leq n - 2 \tag{12}$$

because $f(B_f(D')) = f(B_f(D)) \bigcup \{f(x_0)\}$, the set $\{f(x_0)\}$ is closed, and the topological dimension of each of the sets $f(B_f(D))$ and $\{f(x_0)\}$ does not exceed $n-2$ (see Corollary 1 in [12], Chap. III, Sec. 3).

Let G be a domain in D' such that $G \Subset D'$ and let $y \in f(G) \setminus f(\partial G)$. Then, by virtue of (12), there exists a point $y_0 \notin f(B_f(D'))$ that belongs to the same connected component of the set $\overline{\mathbb{R}^n} \setminus f(\partial G)$ as y. Since the topological index is constant on each connected component of the set $\overline{\mathbb{R}^n} \setminus f(\partial G)$ (see [11], Chap. I, Sec. 2), we have

$$\mu(y, f, G) = \mu(y_0, f, G) = \sum_{x \in G \bigcap f^{-1}(y_0)} i(x, f) > 0.$$

Thus, the mapping f preserves orientation in D'.

Finally, for any $y \in f(D')$, by virtue of the discreteness of the mapping f in the domain D, the set $\{f^{-1}(y)\}$ is at most countable, and, hence, $\dim\{f^{-1}(y)\} = 0$. Therefore [15, p. 333], the mapping f is open and discrete, which was to be proved.

Theorem 6 (analog of the Sokhotskii–Weierstrass theorem)**.** *Suppose that* x_0 *is an isolated point of the boundary* D, $f: D \to \overline{\mathbb{R}^n}$ *is an open discrete Q-mapping, and* $Q(x)$ *is a function of finite mean oscillation at a point* x_0 *or satisfies at least one of conditions (9) and (11). If* x_0 *is an essential singularity of the mapping f, then, for any* $a \in \overline{\mathbb{R}^n}$, *there exists a sequence* $x_k \to x_0$ *as* $k \to \infty$ *such that* $f(x_k) \to a$ *as* $k \to \infty$.

Proof. Assume that the theorem is not true for a certain $a \in \overline{\mathbb{R}^n}$. Then there exist a neighborhood U of the point x_0 and $\varepsilon_0 > 0$ such that

$$h(f(x), a) \geq \varepsilon_0 \quad \forall x \in U \setminus \{x_0\},$$

and, by the triangle inequality, $d_0 = h(B(a, \varepsilon_0/2), f(U \setminus \{x_0\})) \geq \varepsilon_0/2$. Therefore, $\mathrm{cap}\left(\overline{\mathbb{R}^n} \setminus f(U \setminus \{x_0\})\right) > 0$. Hence, by virtue of Theorem 3, there exists a (finite or infinite) limit of the mapping f at the point x_0, which contradicts the assumption of impossibility of removal.

Theorem 7. *Suppose that x_0 is an isolated point of the boundary D, $f: D \to \overline{\mathbb{R}^n}$ is an open discrete Q-mapping, and $Q(x)$ is a function of finite mean oscillation at a point x_0 or satisfies at least one of conditions (9) and (11). If x_0 is an essential singularity of the mapping f, then there exists a set $C \subset \overline{\mathbb{R}^n}$ of capacity zero of the type F_σ in $\overline{\mathbb{R}^n}$ such that*

$$N(y, f, U \setminus \{x_0\}) = \infty \tag{13}$$

for any neighborhood U of the point x_0 and for all $y \in \overline{\mathbb{R}^n} \setminus C$.

Proof. Let U be an arbitrary neighborhood of the point x_0. Without loss of generality, we can assume that $x_0 = 0$ and $U = \mathbb{B}^n$. Consider the sets $V_k = B(1/k) \setminus \{0\}$, $k = 1, 2, \ldots$. We set

$$C = \bigcup_{k=1}^{\infty} \overline{\mathbb{R}^n} \setminus f(V_k). \tag{14}$$

According to Theorem 3, each set $B_k := \overline{\mathbb{R}^n} \setminus f(V_k)$ in the union on the right-hand side of (14) has capacity zero. Then C also has capacity zero (see, e.g., [8, p. 126]).

It remains to prove relation (13). We fix $y \in \overline{\mathbb{R}^n} \setminus C$. Then

$$y \in \bigcap_{k=1}^{\infty} f(V_k). \tag{15}$$

It follows from (15) that there exists a subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ such that $x_{k_i} \to 0$ as $i \to \infty$ and $f(x_{k_i}) = y$, $i = 1, 2, \ldots$.

Theorem 7 is proved.

6. Analog of the Liouville Theorem

Let D be a domain in $\overline{\mathbb{R}^n}$, $n \geq 2$. We say that $\varphi: D \to \mathbb{R}$ is a function of finite mean oscillation at the point ∞ if

$$\varphi^*(x) = \varphi\left(\frac{x}{|x|^2}\right)$$

is a function of finite mean oscillation at the point 0.

Note that the mapping

$$\psi(x) = \frac{x}{|x|^2}$$

maps the sphere $S(0, r)$ similarly onto the sphere $S(0, 1/r)$, whence $|J(x, \psi)| = (|x|^{-1})^{2n}$. According to the results presented above, changing the variables in the integral we can reformulate the definition of finite mean oscillation at a point ∞ as follows:

We say that $\varphi : D \to \mathbb{R}$ is a function of finite mean oscillation at the point ∞ $(\varphi \in FMO(\infty))$ if

$$\int\limits_{|x|>R} |\varphi(x) - \varphi_R| \frac{dm(x)}{|x|^{2n}} = O\left(\frac{1}{R^n}\right) \quad \text{as} \quad R \to \infty,$$

where

$$\varphi_R = \frac{R^n}{\Omega_n} \int\limits_{|x|>R} \varphi(x) \frac{dm(x)}{|x|^{2n}}$$

and Ω_n is the volume of the unit ball in $\mathbb{R}^n$.

By analogy, we can reformulate conditions (9) and (11) for infinity as follows:

$$\fint\limits_{S(0,R)} Q(x)\, dS = O([\log R]^{n-1}), \tag{16}$$

$$\int\limits_{|x|>R} Q(x) \frac{dm(x)}{|x|^{2n}} = O\left(\frac{1}{R^n}\right). \tag{17}$$

Using Theorems 1 and 2 and Corollary 1, we obtain the following statement:

Theorem 8 (analog of the Liouville theorem)**.** *Suppose that* $f : \mathbb{R}^n \to \overline{\mathbb{R}^n}$ *is an open discrete Q-mapping and* $Q(x)$ *is a function of finite mean oscillation at* ∞ *or satisfies at least one of conditions (16) and (17). Then* $\operatorname{cap}\left(\overline{\mathbb{R}^n} \setminus f(\mathbb{R}^n)\right) = 0$. *In particular, f cannot map the entire space* $\mathbb{R}^n$ *onto a bounded domain.*

7. Remarks on Applications to Other Known Classes

Estimates of the form (2) are established for practically all known classes of mappings. The results obtained in the present paper can be applied, e.g., to mappings with finite distortion of length (see, e.g., Theorem 6.10 in [4]). Moreover, these results are also true for so-called mappings with finite distortion. A mapping $f : D \to \mathbb{R}^n$ is called a *mapping with finite distortion* if $f \in W^{1,n}_{\text{loc}}(D)$ and, almost everywhere, one has $\| f'(x) \|^n \le K(x)\, J(x, f)$ for a certain function $K(x) : D \to [1, \infty)$ (see, e.g., [16]).

Theorem 9. *Every open discrete mapping* $f: D \to \mathbb{R}^n$ *with finite distortion such that* $K(x) \in L^{n-1}_{\mathrm{loc}}$ *and the measure of the set* B_f *of branch points of the mapping* f *is equal to zero is a* Q*-mapping with* $Q = K^{n-1}(x)$ *[see Remark 4.10, Theorem 6.10, and inequality (4.14) in [4]].*

The results obtained can also be applied to mappings of the type Q on surfaces (see, e.g., [17]).

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