ISRAEL JOURNAL OF MATHEMATICS **168** (2008), 101–118
DOI: 10.1007/s11856-008-1058-2

TOWARD THE THEORY OF RING Q–HOMEOMOPHISMS

BY

V. Ryazanov and E. Sevost'yanov

Institute of Applied Mathematics and Mechanics,
National Academy of Sciences of Ukraine
74 Roze Luxemburg Str., Donetsk, 83114, Ukraine
e-mail: ryazanov@iamm.ac.donetsk.ua; ryazanov@www.math.helsinki.fi;
vlryazanov1@rambler.ru; sevostyanov@skif.net; sevostyanov@iamm.ac.donetsk.ua

ABSTRACT

We investigate classes of the so-called ring Q-homeomorphisms including, in particular, Q-homeomorphisms, various classes of homeomorphisms with finite length distortion, Sobolev's classes etc. In terms of the majorant $Q(x)$, we give a series of criteria for normality based on estimates of the distortion of the spherical distance under ring Q-homeomorphisms. In particular, it is shown that the class $\Re_{Q,\Delta}$ of all ring Q-homeomorphisms f of a domain $D \subset \mathbb{R}^n$ into $\overline{\mathbb{R}^n}$, $n \geq 2$, with $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \Delta > 0$, forms a normal family, if $Q(x)$ has finite mean oscillation in D. We also prove normality of $\Re_{Q,\Delta}$, for instance, if $Q(x)$ has singularities of logarithmic type whose degrees are not greater than $n-1$ at every point $x \in D$. The results are applicable, in particular, to mappings with finite length distortion and Sobolev's classes.

1. Introduction

The study of ring Q-homeomorphisms began in the plane, see [38]–[35], cf. also [32] and [40]. Ring Q-homeomorphisms induce Q-homeomorphisms arisen earlier in [24]–[26] and [13]–[14]. The theory of ring Q-homeomorphisms is applicable to various classes of mappings with finite distortion, intensively investigated in many recent works, see, e.g., [3], [8], [12], [15], [16], [19], [20], [21], [25], [28] and [29]. In particular, the normality theorems can be applied

Received November 22, 2006

to the theory of the local and boundary behavior of mappings, see, e.g., [9] and [10]–[11].

Given a family of paths Γ in $\mathbb{R}^n$, $n \geq 2$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called **admissible** for Γ, $\rho \in adm\,\Gamma$, if

$$\int_\gamma \rho(x)\,|dx| \geq 1 \tag{1.1}$$

for each $\gamma \in \Gamma$. The **modulus** of Γ is the quantity

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int_D \rho^n(x)\,dm(x)\,. \tag{1.2}$$

Recall the following concept, see [24]–[26]. Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to [1, \infty]$ be a measurable function. A homeomorphism $f : D \to \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ is said to be a **Q-homeomorphism** if

$$M(f\Gamma) \leq \int_D Q(x) \cdot \rho^n(x)\; dm(x) \tag{1.3}$$

for every family of paths Γ in D and every admissible function ρ for Γ.

Given a domain D and two sets E and F in $\overline{\mathbb{R}^n}$, $n \geq 2$, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Gamma(E, F) = \Gamma(E, F, \overline{\mathbb{R}^n})$ if $D = \overline{\mathbb{R}^n}$. A **ring domain**, or shortly a **ring** in $\overline{\mathbb{R}^n}$, is a domain R in $\overline{\mathbb{R}^n}$ whose complement has two connected components. Let R be a ring in $\overline{\mathbb{R}^n}$. If C_1 and C_2 are the connected components of $\overline{\mathbb{R}^n} \setminus R$, we write $R = R(C_1, C_2)$. The **capacity** of R can be defined by the equality

$$cap\; R(C_1, C_2) = M(\Gamma(C_1, C_2, R)), \tag{1.4}$$

see, e.g., [5]. Note that

$$M(\Gamma(C_1, C_2, R)) = M(\Gamma(C_1, C_2)), \tag{1.5}$$

see, e.g., [41, Theorem 11.3].

The following notion is motivated by the ring definition of quasiconformality in [6]. It extends the above notion of a Q-homeomorphism, cf. [38]. Let D be a domain in $\mathbb{R}^n$, $Q : D \to [0, \infty]$ be a (Lebesgue) measurable function. Set

$$A(r_1, r_2, x_0) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}, \tag{1.6}$$

$$S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}\,, \quad i = 1, 2. \tag{1.7}$$

We say that a homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is a **ring Q-homeomorphism** at a point $x_0 \in D$ if

$$M\left(\Gamma\left(fS_1,\, fS_2\right)\right) \;\le \int_A Q(x)\cdot \eta^n(|x-x_0|)\; dm(x) \tag{1.8}$$

for every ring $A = A(r_1, r_2, x_0)$, $0 < r_1 < r_2 < r_0 = dist(x_0, \partial D)$, and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r)\; dr \;=\; 1. \tag{1.9}$$

Note that every Q-homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is a ring Q-homeomorphism at every point $x_0 \in D$, but the inverse conclusion, generally speaking, is not true. Examples of ring Q-homeomorphisms at x_0 with $Q(x) \in (0,\, 1)$ on a set for which x_0 is a density point can be found in [38]–[35] .

2. On normal families of maps in metric spaces

First, we give some general facts about normal families of functions in metric spaces. Let (X, d) and (X', d') be metric spaces with distances d and d', respectively. A family $\mathfrak{F}$ of continuous mappings $f : X \to X'$ is said to be **normal** if every sequence of mappings $f_m \in \mathfrak{F}$ has a subsequence f_{m_k} converging uniformly on each compact set $C \subset X$ to a continuous mapping. Normality is closely related to the following. A family $\mathfrak{F}$ of mappings $f : X \to X'$ is said to be **equicontinuous at a point** $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathfrak{F}$ and $x \in X$ with $d(x, x_0) < \delta$. The family $\mathfrak{F}$ is **equicontinuous** if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$.

2.1 Proposition: *Let (X, d) and (X', d') be arbitrary metric spaces and let $\mathfrak{F}$ be a normal family of mappings $f : X \to X'$. Then $\mathfrak{F}$ is equicontinuous.*

Proof. Indeed, let us assume that there exist $x_0 \in X$, $\varepsilon_0 > 0$ and sequences of mappings $f_m \in \mathfrak{F}$ and points $x_m \in X$ such that $x_m \to x_0$ and

$$d'(f_m(x_m), f_m(x_0)) \ge \varepsilon_0.$$

Without loss of generality, we may consider that $f_m \to f$ uniformly on each compact set $C \subset X$ where f is a continuous mapping. However, $\bigcup\{x_m\}$ is a compact set. Hence $d'(f_m(x_0), f(x_0)) < \varepsilon_0/3$ and $d'(f_m(x_m), f(x_m)) < \varepsilon_0/3$ for all m large enough. Moreover, $d'(f(x_m), f(x_0)) < \varepsilon_0/3$ by the continuity of the mapping f and, consequently, $d'(f_m(x_m), f_m(x_0)) < \varepsilon_0$ by the triangle inequality. The latter estimate contradicts the assumption. ∎

A family $\mathfrak{F}$ of mappings $f : X \to X'$ is said to be **uniformly equicontinuous on a set** $E \subset X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d'(f(x), f(x')) < \varepsilon$ for all $f \in \mathfrak{F}$ and for all x and $x' \in E$ with $d(x, x') < \delta$. The following results are also well-known, cf., e.g., [23].

2.2 Lemma: *Let (X, d) and (X', d') be metric spaces and let $\mathfrak{F}$ be a family of equicontinuous mappings $f : X \to X'$. Then $\mathfrak{F}$ is uniformly equicontinuous on every compact set $C \subset X$.*

2.3 Corollary: *Normal families of mappings between metric spaces are uniformly equicontinuous on compacts.*

Proof of Lemma 2.2. Let us assume that there exist a compact set $C \subseteq X$, a number $\varepsilon_0 > 0$ and sequences of mappings $f_m \in \mathfrak{F}$ and points x_m, $x'_m \in C$ such that $d(x_m, x'_m) \to 0$ as $m \to \infty$ and $d'(f_m(x_m), f_m(x'_m)) \geq \varepsilon_0$. Without loss of generality, we may consider that $x_m \to x_0$ and $x'_m \to x_0 \in C$ since C is compact. Then $d'(f_m(x_m), f_m(x_0)) < \varepsilon_0/2$ and $d'(f_m(x_0), f_m(x'_m)) < \varepsilon_0/2$ for m large enough and hence $d'(f_m(x_m), f_m(x'_m)) < \varepsilon_0$ that contradicts the assumption. ∎

The function

$$\omega_E(t) = \omega_E^{\mathfrak{F}}(t) = \sup d'(f(x), f(z)), \tag{2.4}$$

where the supremum is taken over all $x, z \in E$ such that $d(x, z) \leq t$ and all $f \in \mathfrak{F}$ is called the **modulus of continuity of the family $\mathfrak{F}$ on the set E.**

Similarly, the function

$$\omega_{x_0}(t) = \omega_{x_0}^{\mathfrak{F}}(t) = \sup d'(f(x_0), f(x)), \tag{2.5}$$

where the supremum is taken over all $x \in X$ such that $d(x, x_0) \leq t$ and all $f \in \mathfrak{F}$ is called the **modulus of continuity of $\mathfrak{F}$ at the point** $x_0 \in X$.

Note that by definition ω_E and ω_{x_0} are nonnegative, nondecreasing and continuous from the right. Note also that $\omega_{x_0}(t) \to 0$ as $t \to 0$ for every $x_0 \in X$ if the family $\mathfrak{F}$ is equicontinuous. Moreover, the following statement follows from Lemma 2.2.

2.6 Corollary: *If a family* $\mathfrak{F}$ *of mappings* $f : X \to X'$ *is equicontinuous, then* $\omega_C(t) \to 0$ *as* $t \to 0$ *for every compact set* $C \subset X$.

The next statement is also obvious.

2.7 Proposition: *Let* (X, d) *and* (X', d') *be metric spaces and let* $\overline{\mathfrak{F}}$ *be a closure of a family* $\mathfrak{F}$ *of mappings* $f : X \to X'$ *with respect to the pointwise convergence in X. Then the moduli of continuity* (2.4) *and* (2.5) *of* $\overline{\mathfrak{F}}$ *and* $\mathfrak{F}$ *coincide.*

2.8 Corollary: *If a sequence of mappings* $f_m : X \to X'$, $m = 1, 2 \ldots$, *is equicontinuous and* $f_m(x) \to f(x)$ *as* $m \to \infty$ *for every* $x \in X$, *then the limit function* $f : X \to X'$ *is continuous.*

A sequence of mappings $f_m : X \to X'$, $m = 1, 2 \ldots$, is called **continuously convergent** to $f : X \to X'$, if $f_m(x_m) \to f(x_0)$ as $m \to \infty$ for every convergent sequence of points $x_m \to x_0$ in X.

2.9 Remark: The uniform convergence of continuous mappings on compact sets always implies the continuous convergence, because $\bigcup_{m=0}^{\infty}\{x_m\}$ is a compact set as $x_m \to x_0$ and because, by the triangle inequality,

$$d'\left(f_m(x_m), f(x_0)\right) \le d'\left(f_m(x_m), f(x_m)\right) + d'\left(f(x_m), f(x_0)\right). \tag{2.10}$$

If the second space X' has a countable basis at each point, for example, if X' is separable, then the convergences are equivalent, see, e.g., [4, p. 268]. It is also obvious that the continuous convergence implies pointwise convergence. The converse conclusion is, generaly speaking, not true as it is shown by the example $f_m(x) = x^m$, $x \in [0, 1] : f_m(x) \to 0$ for $x < 1$ and $f_k(1) \to 1$, but $f_m(x_m) \equiv 1/2$ for $x_m = 2^{-1/m} \to 1$ as $m \to \infty$.

The following theorem shows that all three convergences are equivalent for equicontinuous families of mappings in arbitrary metric spaces.

2.11 Theorem: *Let* (X, d) *and* (X', d') *be metric spaces and let* $\mathfrak{F}$ *be an equicontinuous family of mappings* $f : X \to X'$. *Then the following statements are equivalent for all sequences* $f_m \in \mathfrak{F}$:

1) f_m *converges uniformly on every compact set;*
2) f_m *converges continuonsly;*
3) f_m *converges at every point* $x \in X$.

2.12 Corollary: *The closeres $\overline{\mathfrak{F}}$ of equicontinuous families $\mathfrak{F}$ with respect to the pointwise convergence and the uniform convergence on compact sets coincide in arbitrary metric spaces.*

Proof of Theorem 2.11. The implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are obvious, see Remark 2.9. Thus, it remains to prove the implication 3) $\Rightarrow$ 1). Indeed, let us assume there is a sequence $f_m \in \mathfrak{F}$ such that $f_m(x) \to f(x)$ as $m \to \infty$ for every $x \in X$ and, simultaneously, for a compact set $C \subset X$, there is a number $\varepsilon_0 > 0$ such that $d'(f_m(x_m), f(x_m)) \geq \varepsilon_0$ for some sequence of points $x_m \in C$. Without loss of generality, we may consider that $x_m \to x_0 \in C$ as $m \to \infty$. However, by the triangle inequality $d'(f_m(x_m), f(x_m)) \leq d'(f_m(x_m), f_m(x_0)) + d'(f_m(x_0), f(x_0)) + d'(f(x_0), f(x_m))$ and by Corollaries 2.6 and 2.8 we come to a contradiction. ∎

2.13 Lemma: *Let (X, d) be a metric space and a set $E \subset X$ be dense everywhere in X and (X', d') be a complete metric space. If an equicontinuous sequence of mappings $f_m : X \to X'$ is pointwise convergent on the set E, then f_m converges uniformly on every compact set $C \subset X$.*

Proof. In view of Theorem 2.11 it is sufficient to prove that the pointwise convergence of f_m on E implies the pointwise convergence of f_m on X. Indeed, for every $x_0 \in X \backslash E$, there exists a sequence $x_k \in E$ such that $x_k \to x_0$ as $k \to \infty$. By the equicontinuity of f_m, for every $\varepsilon > 0$, there exists $K = K(\varepsilon)$ such that $d'(f_m(x_k), f_m(x_0)) < \varepsilon/3$ for all $k \geq K$ and all $m = 1, 2 \ldots$. Let us fix $k_0 \geq K$. Then by the Cauchy Criterion we have that $d'(f_n(x_{k_0}), f_m(x_{k_0})) < \varepsilon/3$ for all n and $m \geq N = N(\varepsilon, k_0)$. Finally, by the triangle inequality $d'(f_n(x_0), f_m(x_0)) \leq d'(f_n(x_0), f_n(x_{k_0})) + d'(f_n(x_{k_0}), f_m(x_{k_0})) + d'(f_m(x_{k_0}), f_m(x_0)) < \varepsilon$ for all n and $m \geq N$, i.e., the sequence $f_m(x_0)$ is fundamental and hence it is convergent by the completeness of the space X'. ∎

As is well-known, every compact metric space is complete, see, e.g., Theorem 3 in Section 33, II, [22]. Thus, applying the diagonal process, we obtain the following consequence of Proposition 2.1 and Lemma 2.13, cf. Section 20.4 in [41] .

2.14 Corollary: *If (X, d) is a separable metric space and (X', d') is compact metric space, then a family $\mathfrak{F}$ of mappings $f : X \to X'$ is normal if and only if $\mathfrak{F}$ is equicontinuous.*

In what follows, we use in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ the **spherical (chordal) metric** $h(x,y) = |\pi(x) - \pi(y)|$ where π is the stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$:

$$\begin{aligned} h(x,y) &= \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \quad x, y \neq \infty \\ h(x,\infty) &= \frac{1}{\sqrt{1+|x|^2}} \,. \end{aligned} \tag{2.15}$$

Thus, by definition $h(x,y) \leq 1$ for all x and $y \in \overline{\mathbb{R}^n}$. The **spherical (chordal) diameter** of a set $E \subset \overline{\mathbb{R}^n}$ is

$$h(E) = \sup_{x,y \in E} h(x,y). \tag{2.16}$$

Note that

$$h(x,y) \leq |x-y| \tag{2.17}$$

for all x,$y \in \mathbb{R}^n$ and

$$h(x,y) \geq \frac{1}{2}|x-y| \tag{2.18}$$

for all x and y in the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ with the equality in (2.18) on $\partial\mathbb{B}^n$.

3. Characterization of ring Q-homeomorphisms.

Below we use the standard conventions $a/\infty = 0$ for $a \neq \infty$ and $a/0 = \infty$ if $a > 0$ and $0 \cdot \infty = 0$, see, e.g., [39, p. 6].

3.1 Lemma: *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, $Q : D \to [0,\infty]$ a measurable function and $q_{x_0}(r)$ the average of $Q(x)$ over the sphere $|x - x_0| = r$. Set*

$$I \;=\; I(r_1,r_2) \;=\; \int_{r_1}^{r_2} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} \tag{3.2}$$

and $S_j = \{x \in \mathbb{R}^n : |x - x_0| = r_j\}$, $j = 1,2$ where $x_0 \in D$ and $0 < r_1 < r_2 < r_0 = \mathrm{dist}\,(x_0, \partial D)$. Then

$$M\left(\Gamma\left(fS_1,\, fS_2\right)\right) \;\leq\; \frac{\omega_{n-1}}{I^{n-1}} \tag{3.3}$$

whenever $f : D \to \overline{\mathbb{R}^n}$ is a ring Q-homeomorphism where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Proof. With no loss of generality, we may assume that $I \neq 0$ because otherwise (3.3) is trivial and that $I \neq \infty$ because otherwise we can replace $Q(z)$ by $Q(z)+\delta$ with arbitrarily small $\delta > 0$ and then take the limit as $\delta \to 0$ in (3.3). The condition $I \neq \infty$ implies, in particular, that $q(r) \neq 0$ almost everywhere in (r_1, r_2). Set

$$\psi(t) = \begin{cases} 1/[tq_{x_0}^{\frac{1}{n-1}}(t)], & t \in (r_1, r_2), \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Then

$$\int_A Q(x) \cdot \psi^n(|x - x_0|) \, dm(x) \; = \; \omega_{n-1} I \tag{3.5}$$

where

$$A \; = \; A(r_1, r_2, x_0) \; = \; \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\} \; . \tag{3.6}$$

Let Γ be a family of all paths joining the circles S_1 and S_2 in A. The function

$$\eta(t) \; = \psi(t)/I$$

is measurable and satisfies to condition (1.9). Since f is a ring Q-homeomorphism, we get by (3.5) that

$$M(f\Gamma) \; \leq \; \int_A Q(x) \cdot \eta^n(|x - x_0|) \, dm(x) \; = \frac{\omega_{n-1}}{I^{n-1}}. \qquad \blacksquare$$

The following lemma shows that the estimate (3.3) cannot be improved for ring Q-homeomorphisms.

3.7 Lemma: *Fix $x_0 \in \mathbb{R}^n, 0 < r_1 < r_2 < r_0$, $A = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$, $B = B(x_0, r_0) = \{x \in \mathbb{R}^n : |x - x_0| < r_0\}$, and suppose that $Q : D \to [0, \infty]$ is a measurable function. Set*

$$\eta_0(r) = \frac{1}{Irq_{x_0}^{\frac{1}{n-1}}(r)} \tag{3.8}$$

where $q(r)$ is the average of $Q(x)$ over the sphere $|x - x_0| = r$ and I as in Lemma 3.1. Then

$$\frac{\omega_{n-1}}{I^{n-1}} \; = \; \int_A Q(x) \cdot \eta_0^n(|x - x_0|) \, dm(x) \; \leq \; \int_A Q(x) \cdot \eta^n(|x - x_0|) \, dm(x) \tag{3.9}$$

whenever $\eta : (r_1, r_2) \to [0, \infty]$ is such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \; = 1. \tag{3.10}$$

Proof. If $I = \infty$, then the left side in (3.9) is equal to zero and the inequality is obvious. If $I = 0$, then $q_{x_0}(r) = \infty$ for almost every $r \in (r_1, r_2)$ and the both sides in (3.9) are equal to ∞. Hence we may assume below that $0 < I < \infty$.

Now, by (3.2) and (3.10) $q_{x_0}(r) \neq 0$ and $\eta(r) \neq \infty$ almost everywhere in (r_1, r_2). Set $\alpha(r) = r q_{x_0}^{\frac{1}{n-1}}(r)\eta(r)$ and $w(r) = 1/r q_{x_0}^{\frac{1}{n-1}}(r)$. Then by the standard conventions $\eta(r) = \alpha(r)w(r)$ almost everywhere in (r_1, r_2) and

$$(3.11)\qquad C :\, = \int_A Q(x)\cdot \eta^n(|x-x_0|)\; dm(x) = \omega_{n-1}\int_{r_1}^{r_2} \alpha^n(r)\cdot w(r)\; dr.$$

By Jensen's inequality with weights, see, e.g., [30, Theorem 2.6.2], applied to the convex function $\varphi(t) = t^n$ in the interval $\Omega = (r_1, r_2)$ with the probability measure

$$(3.12)\qquad \nu(E) \;=\; \frac{1}{I}\int_E w(r)\; dr$$

we obtain that

$$(3.13)\qquad \left(\fint \alpha^n(r)w(r)\; dr\right)^{1/n} \;\geq\; \fint \alpha(r)w(r)\; dr \;=\; \frac{1}{I}$$

where we also used the fact that $\eta(r) = \alpha(r)w(r)$ satisfies (3.10). Thus,

$$(3.14)\qquad C \;\geq\; \omega_{n-1}/I^{n-1}$$

and the proof is complete. ∎

3.15 Theorem: *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : D \to [0,\infty]$ a measurable function. A homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is a ring Q-homeomorphism at a point x_0 if and only if for every $0 < r_1 < r_2 < r_0 = dist\,(x_0, \partial D)$,*

$$(3.16)\qquad M\left(\Gamma\left(fS_1,\; fS_2\right)\right) \;\leq\; \omega_{n-1}/I^{n-1}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $q_{x_0}(r)$ is the average of $Q(x)$ over the sphere $|x-x_0| = r$, $S_j = \{x \in \mathbb{R}^n : |x-x_0| = r_j\}$, $j = 1, 2$, and

$$I \;=\; I(r_1, r_2) \;=\; \int_{r_1}^{r_2} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)}.$$

Moreover, the infimum from the right hand side in (1.8) *holds for the function*

$$(3.17)\qquad \eta_0(r) = \frac{1}{I r q_{x_0}^{\frac{1}{n-1}}(r)}\;.$$

4. Estimates of distortion

4.1 Lemma: *Let D be a domain in $\mathbb{R}^n$, $f : D \to \overline{\mathbb{R}^n}$ be a homeomorphism with $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \Delta > 0$ and let x_0 be a point in D, $y \in B(x_0, r_0)$, $r_0 < dist\,(x_0, \partial D)$,*

$$S_0 = \{x \in \mathbb{R}^n : |x - x_0| = r_0\} \quad \textit{and} \quad S = \{x \in \mathbb{R}^n : |x - x_0| = |y - x_0|\}.$$

Then

$$h(f(y), f(x_0)) \ \leq \ \frac{\alpha_n}{\Delta} \cdot \exp\left(- \left\{ \frac{\omega_{n-1}}{M\,(\Gamma\,(fS,\, fS_0))} \right\}^{\frac{1}{n-1}} \right) \tag{4.2}$$

where ω_{n-1} is the area of the unit sphere S^{n-1} in $\mathbb{R}^n$, $\alpha_n = 2\lambda_n^2$, $\lambda_n \in [4, e^n)$, $\lambda_2 = 4$ and $\lambda_n^{\frac{1}{n}} \to e$ as $n \to \infty$.

Proof. Let E denote the component of $\overline{\mathbb{R}^n} \setminus fA$ containing $f(x_0)$ and F the component containing ∞ where $A = \{x \in \mathbb{R}^n : |y - x_0| < |x - x_0| < r_0\}$. By the known Gehring lemma

$$cap\ R(E, F) \ \geq \ cap\ R_T\left(\frac{1}{h(E)h(F)}\right) \tag{4.3}$$

where $h(E)$ and $h(F)$ denote the spherical diameters of the continua E and F, correspondingly, and $R_T(t)$ is the Teichmüller ring

$$R_T(t) \ = \ \mathbb{R}^n \setminus ([-1, 0] \cup [t, \infty]) \ , \quad \text{for } t > 1, \tag{4.4}$$

see, e.g., [42, Section 7.37] or [7]. It is also known that

$$cap\ R_T(t) = \frac{\omega_{n-1}}{\{\log \Phi(t)\}^{n-1}} \tag{4.5}$$

where the function Φ admits the good estimates:

$$t + 1 \leq \Phi(t) \leq \lambda_n^2 \cdot (t + 1) < 2\lambda_n^2 \cdot t, \quad \text{for } t > 1, \tag{4.6}$$

see, e.g., [7, pp. 225–226] and [42, (7.19) and (7.22)]. Hence the inequality (4.3) implies that

$$cap\ R(E, F) \ \geq \ \frac{\omega_{n-1}}{\left[\log \frac{2\lambda_n^2}{h(E)h(F)}\right]^{n-1}}\ . \tag{4.7}$$

Thus,

$$h(E) \ \leq \frac{2\lambda_n^2}{h(F)} \ \exp\left(- \left\{ \frac{\omega_{n-1}}{cap\ R(E, F)} \right\}^{\frac{1}{n-1}} \right) \tag{4.8}$$

that implies the desired statement. ∎

Now, combining Lemmas 3.1, 3.7 and 4.1, we have the following lemma.

4.9 Lemma: *Let $f : D \to \overline{\mathbb{R}^n}$, $n \geq 2$, be a ring Q-homeomorphism with $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \Delta > 0$. If*

$$\int_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x) \cdot \psi_\varepsilon^n(|x-x_0|)\ dm(x) \leq c \cdot I^p(\varepsilon), \quad \varepsilon \in (0,\varepsilon_0), \tag{4.10}$$

for $x_0 \in D$ and $0 < \varepsilon_0 \leq dist(x_0, \partial D)$ where $p \leq n$ and $\psi_\varepsilon(t)$ is a nonnegative function on $(0,\infty)$ such that

$$0 < I(\varepsilon) = \int_\varepsilon^{\varepsilon_0} \psi_\varepsilon(t)\ dt < \infty, \quad \varepsilon \in (0,\varepsilon_0), \tag{4.11}$$

then

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \exp\{-\beta_n I^{\gamma_{n,p}}(|x-x_0|)\} \tag{4.12}$$

for all $x \in B(x_0, \varepsilon_0)$ where

$$\alpha_n = 2\lambda_n^2, \quad \beta_n = \left(\frac{\omega_{n-1}}{c}\right)^{\frac{1}{n-1}}, \quad \gamma_{n,p} = 1 - \frac{p-1}{n-1}, \tag{4.13}$$

$\lambda_n \in [4, e^n)$, $\lambda_2 = 4$ and $\lambda_n^{1/n} \to e$ as $n \to \infty$.

4.14 Corollary: *Under the conditions of Lemma 4.9 and p=1*

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \exp\{-\beta_n I(|x-x_0|)\}. \tag{4.15}$$

From Lemmas 3.1 and 4.1 we also obtain the following estimate.

4.16 Theorem: *Let $f : D \to \overline{\mathbb{R}^n}$, $n \geq 2$, be a ring Q-homeomorphism with $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \Delta > 0$. Then, for every $x_0 \in D$ and $x \in B(x_0, \varepsilon(x_0))$, $\varepsilon(x_0) < d(x_0) = dist(x_0, \partial D)$,*

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \exp\left\{-\int_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)}\right\}, \tag{4.17}$$

where α_n is given by (4.13) and $q_{x_0}(r)$ is the average of $Q(x)$ over the sphere $|x-x_0| = r$.

4.18 Remark: Of course, the average $q_{x_0}(r)$ of $Q(x)$ over some spheres $|x-x_0| = r$ can be infinite. However, $q_{x_0}(r)$ is measurable in the parameter r because $Q(x)$ is measurable in x. Moreover, at every point $x \neq x_0$

$$\int_{|x-x_0|}^{\varepsilon(x_0)} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} < \infty \tag{4.19}$$

for any ring Q-homeomorphism because in the contrary case we would have from (4.17) that $f(x) = f(x_0)$. The integral in (4.19) can be equal to 0 if $q_{x_0}(r) = \infty$ almost everywhere but then the inequality (4.17) is obvious, because $\alpha_n \geq 32$ and δ as well as $h(f(x), f(x_0))$ is less or equal to 1.

4.20 Corollary: *If*

$$q_{x_0}(r) \leq \left[\log \frac{1}{r}\right]^{n-1} \tag{4.21}$$

for $r < \varepsilon(x_0) < \min\{1, r(x_0)\}$, *then*

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \frac{\log \frac{1}{\varepsilon(x_0)}}{\log \frac{1}{|x-x_0|}} \tag{4.22}$$

for all $x \in B(x_0, \varepsilon(x_0))$.

4.23 Corollary: *If*

$$Q(x) \leq \left[\log \frac{1}{|x-x_0|}\right]^{n-1}, \quad x \in B(x_0, \varepsilon(x_0)), \tag{4.24}$$

then (4.22) *holds in the ball* $B(x_0, \varepsilon(x_0))$.

4.25 Remark: If instead of (4.21) and (4.24) we have the conditions

$$q_{x_0}(r) \leq c \cdot \left[\log \frac{1}{r}\right]^{n-1} \tag{4.26}$$

and, correspondingly,

$$Q(x) \leq c \cdot \left[\log \frac{1}{|x-x_0|}\right]^{n-1}, \tag{4.27}$$

then

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta} \left[\frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x-x_0|}}\right]^{1/c^{1/(n-1)}}. \tag{4.28}$$

Choosing in Lemma 4.9 $\psi(t) = 1/t$ and $p = 1$, we also have the following conclusion.

4.29 Corollary: *Let* $f : \mathbb{B}^n \to \mathbb{B}^n$ *be a ring Q-homeomorphism such that* $f(0) = 0$ *and*

$$\int_{\varepsilon<|x|<1} Q(x) \, \frac{dm(x)}{|x|^n} \leq c \log \frac{1}{\varepsilon}, \quad \varepsilon \in (0, 1). \tag{4.30}$$

Then

$$|f(x)| \leq 2\alpha_n \cdot |x|^{\beta_n}, \tag{4.31}$$

where the constants α_n *and* β_n *are defined by* (4.13).

4.32 Remark: Note that, if $Q(x) \geq 1$ or at least $q_{x_0}(r) \geq 1$ almost everywhere, then one may use any degree $\beta \geq 1/(n-1)$ and, in particular, $\beta = 1$ instead of $1/(n-1)$ in inequalities (4.17) and (4.19). Indeed, for the function

$$\psi(t) = \begin{cases} \frac{1}{t q_{x_0}^{\beta}(t)}, & t \in (0, \varepsilon_0), \\ 0, & \mathrm{t} \in [\varepsilon_0, \infty), \end{cases} \tag{4.33}$$

we have that

$$\begin{aligned} \int_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x) \cdot \psi^n(|x-x_0|) dm(x) &= \omega_{n-1} \int_{\varepsilon}^{\varepsilon_0} \frac{dr}{r q_{x_0}^{\beta n-1}(r)} \\ &\leq \omega_{n-1} \int_{\varepsilon}^{\varepsilon_0} \frac{dr}{r q_{x_0}^{\beta}(r)} \end{aligned} \tag{4.34}$$

and, thus, the statement follows immediately from Corollary 4.14.

5. Finite mean oscillation

Recall that a real valued function $\varphi \in \mathrm{L}^1_{loc}(D)$ is said to be of **bounded mean oscillation** in a domain $D \subset \mathbb{R}^n$, denoted $\varphi \in \mathrm{BMO(D)}$ or simply $\varphi \in$ **BMO**, if

$$\|\varphi\|_* = \sup_{B \subset D} \fint_B |\varphi(x) - \varphi_B| \, dm(x) < \infty, \tag{5.1}$$

where the supremum is taken over all balls B in D and

$$\varphi_B = \fint_B \varphi(x) \, dm(x) = \frac{1}{|B|} \int_B \varphi(x) \, dm(x) \tag{5.2}$$

is the average of the function φ over B. It is well-known that $\mathrm{L}^\infty(\mathrm{D}) \subset \mathrm{BMO(D)} \subset \mathrm{L}^p_{loc}(\mathrm{D})$ for all $1 \leq p < \infty$, see, e.g., [17] and [31]. For connections of BMO functions and quasiconformal and quasiregular mappings, see, e.g., [1], [2], [6], [18], [27] and [31].

Following [13]–[14], we say that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation** at a point $x_0 \in D$ if

$$\overline{\lim_{\varepsilon \to 0}} \fint_{B(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi_\varepsilon}| \, dm(x) < \infty, \tag{5.3}$$

where

$$\overline{\varphi_\varepsilon} = \fint_{B(x_0,\varepsilon)} \varphi(x)\, dm(x) \tag{5.4}$$

is the average of the function $\varphi(x)$ over $B(x_0,\varepsilon)$. Note that under (5.3) it is possible that $\overline{\varphi_\varepsilon} \to \infty$ as $\varepsilon \to 0$. We also say that a function $\varphi : D \to \mathbb{R}$ is of finite mean oscillation in the domain D, and write $\varphi \in$ FMO(D) or simply $\varphi \in$ **FMO**, if φ has finite mean oscillation at every point $x \in D$. Note that FMO is not BMO_{loc}, see examples in [34]–[37].

Recall some facts on finite mean oscillation from [13].

5.5 Corollary: *If, for some numbers $\varphi_\varepsilon \in \mathbb{R}, \varepsilon \in (0,\varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{B(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon|\, dm(x) < \infty\,, \tag{5.6}$$

then φ has finite mean osillation at x_0.

5.7 Corollary: *If, for a point $x_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{B(x_0,\varepsilon)} |\varphi(x)|\, dm(x) < \infty\,, \tag{5.8}$$

then φ has finite mean oscillation at x_0.

5.9 Lemma: *Let $\varphi : D \to \mathbb{R}, n \geq 2$, be a nonnegative function with a finite mean oscillation at $0 \in D$. Then*

$$\int_{\varepsilon<|x|<\varepsilon_0} \frac{\varphi(x)\, dm(x)}{(|x| log \frac{1}{|x|})^n} = O\left(\log\log\frac{1}{\varepsilon}\right) \tag{5.10}$$

as $\varepsilon \to 0$, for some $\varepsilon_0 \leq dist\,(0,\partial D)$.

Thus, from Lemma 4.9 we obtain the following.

5.11 Theorem: *Let $f : D \to \overline{\mathbb{R}^n}$, $n \geq 2$, be a ring Q-homeomorphism with $h(\overline{\mathbb{R}^n} \setminus f(D)) \geq \Delta > 0$. If $Q(x)$ has finite mean oscillation at a point $x_0 \in D$, then, for some $\varepsilon_0 < dist(x_0, \partial D)$,*

$$h(f(x), f(x_0)) \leq \frac{\alpha_n}{\Delta}\left\{\frac{\log\frac{1}{\varepsilon_0}}{\log\frac{1}{|x-x_0|}}\right\}^{\beta_0} \tag{5.12}$$

for $x \in B(x_0,\varepsilon_0)$, where α_n depends only on n and $\beta_0 > 0$ depends only n and the function Q.

6. On normal families of Q-homeomorphisms

Let D be a domain in $\mathbb{R}^n$ and $Q : D \to [0,\infty]$ be a measurable function. Let $\mathfrak{R}_{Q,\Delta}(D)$ be the class of all ring Q-homeomorphisms $f : D \to \overline{\mathbb{R}^n}$, $n \geq 2$, such that $h(\overline{\mathbb{R}^n} \backslash f(D)) \geq \Delta > 0$. The above results now yield:

6.1 Theorem: *If* $Q \in FMO$, *then* $\mathfrak{R}_{Q,\Delta}(D)$ *is a normal family.*

6.2 Corollary: *The class* $\mathfrak{R}_{Q,\Delta}(D)$ *is normal if*

$$\overline{\lim_{\varepsilon\to 0}} \;⨍_{B(x_0,\varepsilon)} Q(x) \;\; dm(x) < \infty \quad \text{for every } x_0 \in D. \tag{6.3}$$

6.4 Corollary: *The class* $\mathfrak{R}_{Q,\Delta}(D)$ *is normal if every* $x_0 \in D$ *is a Lebesgue point of* $Q(x)$.

6.5 Theorem: *Let* $\Delta > 0$ *and* $Q : D \to [0,\infty]$ *be a measurable function such that*

$$\int_0^{\varepsilon(x_0)} \frac{dr}{r q_{x_0}^{\frac{1}{n-1}}(r)} = \infty \tag{6.6}$$

holds at every point $x_0 \in D$ *where* $\varepsilon(x_0) < dist(x_0, \partial D)$ *and* $q_{x_0}(r)$ *denotes the average of* $Q(x)$ *over the sphere* $|x - x_0| = r$. *Then* $\mathfrak{R}_{Q,\Delta}$ *is a normal family.*

6.7 Corollary: *The class* $\mathfrak{R}_{Q,\Delta}(D)$ *is normal if* $Q(x)$ *has singularities of the logarithmic type of the order which is not more than* $n-1$ *at every point* $x \in D$.

6.8 Remark: In view of Remark 4.32, if $Q(x) \geq 1$ almost everywhere in D, then one may use any degree $\beta \geq 1/(n-1)$ and, in particular, $\beta = 1$ instead of $1/(n-1)$ in condition (6.6). In particular, all the above results hold for homeomorphisms f of the Sobolev class $W^{1,n}_{loc}$ with $f^{-1} \in W^{1,n}_{loc}$ under the condition that almost everywhere

$$K_I(x,f) \leq Q(x) \tag{6.9}$$

where $K_I(x,f)$ is the so-called inner dilatation of the mapping f at the point $x \in D$ because such f are Q-homeomorphisms, see, e.g., [25, Theorems 4.6 and 7.10]. Note that, for homeomorphisms $f \in W^{1,n}_{loc}$ with $K_I \in L^1_{loc}$, f^{-1} belongs to $W^{1,n}_{loc}$, see [21, Corollary 2.3]. Moreover, the results can be applied to the classes of homeomorphisms f with finite length distortion which are also Q-homeomorphisms with $Q(x) = K_I\,(x,\,f)$, see [25, Theorem 6.10].

POSTSCRIPT: As in the classical case of Q-quasiconformal mappings, cf., e.g., [23, Theorem II.5.1], a family $\mathfrak{R}$ of ring Q-homeomorphisms $f : D \to \overline{\mathbb{R}^n}$ in a domain $D \subset \mathbb{R}^n$, $n \geq 2$, is normal under every condition given above for Q if there is a number $\Delta > 0$ such that one of the following conditions holds:

1) Every mapping $f \in \mathfrak{R}$ omits two values whose spherical distance is greater than Δ.
2) Every mapping $f \in \mathfrak{R}$ omits one value w_0 and $h(w(x_i)), w_0) > \Delta$, $i = 1, 2$, at two fixed points x_1 and $x_2 \in D$.
3) At three fixed points x_1, x_2 and $x_3 \in D$, $h(w(x_i), w(x_j) > \Delta$, $i \neq j$, $i, j = 1, 2, 3$.

In particular, $\mathfrak{R}$ is normal if all mappings $f \in \mathfrak{R}$ omit two fixed values in $\overline{\mathbb{R}^n}$.

ACKNOWLEDGMENTS. The research was partially supported by grants from the University of Helsinki, from Technion – Israel Institute of Technology and by Grant 01.07/00241 of State Fund of Fundamental Investigations of Ukraine.

References

[1] K. Astala, *A remark on quasiconformal mappings and BMO–functions*, The Michigan Mathematical Journal **30** (1983), 209–212.
[2] K. Astala and F. W. Gehring, *Injectivity, the BMO norm and the universal Teichmuller space*, Journal d'Analyse Mathématique **46** (1986), 16–57.
[3] K. Astala, T. Iwaniec, P. Koskela and G. Martin, *Mappings of BMO-bounded distortion*, Mathematische Annalen **317** (2000), 703–726.
[4] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, MA, 1966.
[5] F. W. Gehring, *Extremal length definitions for the conformal capacity of ring in space*, Michigan Math. J. **9** (1962), 137–150.
[6] F. W. Gehring, *Rings and quasiconformal mappings in space*, Transactions of the American Mathematical Society **103** (1962), 353–393.
[7] F. W. Gehring, *Quasiconformal mappings in Complex Analysis and its Applications*, V. 2, International Atomic Energy Agency, Vienna, 1976.
[8] F. W. Gehring and T. Iwaniec, *The limit of mappings with finite distortion*, Annales Academiæ Scientiarium Fennicæ **24** (1999), 253–264.
[9] V. Gutlyanskii, O. Martio and V. Ryazanov, *Infinitesimal geometry of quasiregular mappings,* Annales Academiæ Scientiarium Fennicæ **25** (2000), 101–130.
[10] V. Ya. Gutlyanskii and V. I. Ryazanov, *On the theory of the local behavior for quasiconformal mappings*, Rossiĭskaya Akademiya Nauk. Izvestiya. Seriya Matematicheskaya **59** (1995), no. 3, 31–58.

[11] V. Ya. Gutlyanskii and V. I. Ryazanov, *On the boundary correspondence under quasiconformal mappings,* Academiæ Scientiarum Fennicæ Annales. Mathematicæ **21** (1996), 168–178.

[12] J. Heinonen and P. Koskela, *Sobolev mappings with integrable dilatations*, Archive for Rational Mechanics and Analysis **125** (1993), 81–97.

[13] A. Ignat'ev and V. Ryazanov, *Finite mean oscillation in the mapping theory*, Ukrainian Mathematical Bulletin **2** (2005), 395–417.

[14] A. Ignat'ev and V. Ryazanov, *On the boundary behavior of space mappings*, Ukrainian Mathematical Bulletin **3** (2006), 199–211.

[15] T. Iwaniec and G. Martin, *Geometrical Function Theory and Non-Linear Analysis*, Clarendon Press, Oxford, 2001.

[16] T. Iwaniec and V. Sverak, *On mappings with integrable dilatation*, Proceedings of the American Mathematical Society **118** (1993), 181–188.

[17] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Communications on Pure and Applied Mathematics **14** (1961), 415–426.

[18] P. M. Jones, *Extension theorems for BMO*, Indiana University Mathematics Journal **29** (1980), 41–66.

[19] J. Kauhanen, P. Koskela and J. Maly, *Mappings of finite distortion: discreteness and openness*, Archive for Rational Mechanics and Analysis **160** (2001), 135–151.

[20] J. Kauhanen, P. Koskela and J. Maly, *Mappings of finite distortion: condition N*, The Michigan Mathematical Journal **49** (2001), 169–181.

[21] P. Koskela and J. Onninen, *Mappings of finite distortion: the sharp modulus of continuity*, Transactions of the American Mathematical Society **355** (2003), 1905–1920

[22] K. Kuratowski, *Topology*, Vol. 1, Academic Press, New York–London, 1966.

[23] O. Lehto and K. Virtanen, *Quasiconformal Mappings in the Plane*, Springer, New York–Heidelberg, 1973.

[24] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, *Q-homeomorphisms,* Contemporary Mathematics **364** (2004), 193–203.

[25] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, *Mappings of finite length distortion*, Journal d'Analyse Mathématique **93** (2004), 215–236.

[26] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, *On Q-homeomorphisms,* Annales Academiæ Scientiarium Fennicæ Annales. Mathematicæ **30** (2005), 49–69.

[27] O. Martio, V. Ryazanov and M. Vuorinen, *BMO and Injectivity of Space Quasiregular Mappings*, Mathematische Nachrichten **205** (1999), 149–161.

[28] J. J. Manfredi and E. Villamor, *Mappings with integrable dilatation in higher dimensions*, American Mathematical Society. Bulletin **32** (1995), 235–240.

[29] J. J. Manfredi and E. Villamor, *An extension of Reshetnyak's theorem*, Indiana University Mathematics Journal **47** (1998), 1131–1145.

[30] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

[31] H. M. Reimann and T. Rychener, *Funktionen Beschränkter Mittlerer Oscillation*, Springer-Verlag, Berlin–New York, 1975.

[32] V. Ryazanov and E. Sevost'yanov, *On normal families of Q−homeomorphisms*, Proceedings of the Institute of Applied Mathematics and Mechanics of National Academy of Sciences of Ukraine **9** (2004), 161–176.

[33] V. Ryazanov, U. Srebro and E. Yakubov, *On ring solutions of Beltrami equations*, Journal d'Analyse Mathématique **96** (2005), 117–150.

[34] V. Ryazanov, U. Srebro and E. Yakubov, *The Beltrami equation and FMO functions*, Contemporary Mathematics **382** (2005), 357–364.

[35] V. Ryazanov, U. Srebro and E. Yakubov, *To the theory of the Beltrami equations,* Ukrainian Mathematical Journal **58** (2006), no. 11, 1571–1583.

[36] V. Ryazanov, U. Srebro and E. Yakubov, *Finite mean oscillation and the Beltrami equation*, Israel Journal of Mathematics **153** (2006), 247–266.

[37] V. Ryazanov, U. Srebro and E. Yakubov, *The Beltrami equation and ring homeomorphisms,* Ukrainian Mathematical Bulletin **4** (2007), 79–115.

[38] V. Ryazanov, U. Srebro and E. Yakubov, *Degenerate Beltrami equation and radial Q−homeomorphisms*, 2003; Preprint of Department of Mathematics, University of Helsinki, 2003.369.

[39] S. Saks, *Theory of the Integral*, New York, Dover Publ. Inc., 1964.

[40] E. A. Sevost'yanov, *Convergence theory and quasilinear Beltrami equations*, Proceedings of the Institute of Applied Mathematics and Mechanics of National Academy of Sciences of Ukraine **12** (2006), 128–139.

[41] J. Väisälä, *Lectures on n−Dimensional Quasiconformal Mappings*, Lecture Notes in Mathematics, vol. 229, Springer-Verlag, Berlin–New York, 1971.

[42] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Mathematics, vol. 1319, Springer–Verlag, Berlin, 1988.